THE MINIMAL FIBERING DEGREE OF A TORIC VARIETY EQUALS THE LATTICE WIDTH OF ITS POLYTOPE

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The purpose of this paper is to compute the minimal fibering degree of a pair (X, L) when X is a projective toric variety with a globally generated line bundle L – in particular, we show it equals the lattice width of the polytope associated to (X, L).

For any dominant, rational fibration of an n-dimensional projective variety X:

$$\Phi\colon X \dashrightarrow \mathbb{P}^{n-1}$$

and any effective line bundle L on X, the *degree of* Φ *with respect to* L is

$$\deg_L(\Phi) = [C_\Phi] \cdot L$$

where $C_\Phi \subset X$ is the closure of a general fiber of Φ. The *minimal fibering degree* of a pair (X, L) is

$$\mathrm{mfd}(X, L) = \min\{\deg_L(\Phi) \mid \Phi\colon X \dashrightarrow \mathbb{P}^{n-1} \text{ is dominant}\}$$

The minimal fibering degree was introduced recently ([7]) in order to calculate the degree of irrationality – a higher dimensional analogue of the gonality of a curve – of (very positive) divisors in an ambient variety. Only a few elementary examples of the minimal fibering degree have been computed. In this paper we compute the minimal fibering degree of an arbitrary projective toric variety.

Theorem A. *Let* (X, L) *be a projective toric variety with* L *an ample toric line bundle (or more generally,* L *may be taken to be simply globally generated). If* $P = P(X, L)$ *is the lattice polytope associated to* (X, L) *then*

$$\mathrm{mfd}(X, L) = \ell\mathrm{w}(P)$$

(where $\ell\mathrm{w}(P)$ *is the lattice width of the polytope* P*, described below).*

This gives a complete answer to [7, Ques. 1.13(3)]. In the case L is sufficiently ample, Theorem A (together with [7, Thm. A]) gives a new proof of the computation of the gonality of a curve in a smooth toric surface – originally proved by Kawaguchi ([6, Thm. 1.3]).

Given a lattice polytope $P \subset \mathbb{R}^n$ — i.e. the convex hull of finitely many points $x_1, \ldots, x_e \in \mathbb{Z}^n$ — the *lattice width of* P (Defn 2.8) computes the minimal width of the image of P under nonzero linear projections

$$P \longrightarrow \mathbb{R}$$

that sends lattice points to lattice points. This is an invariant of lattice polytopes of independent interest (see [2, 3, 4]). The lattice width has made an appearance in algebraic geometry in several places as well ([1, 6, 8]). One feature of our proof is that we show how to compute the lattice width of a polytope $P(X, L)$ explicitly in terms of a toric resolution of singularities of X.

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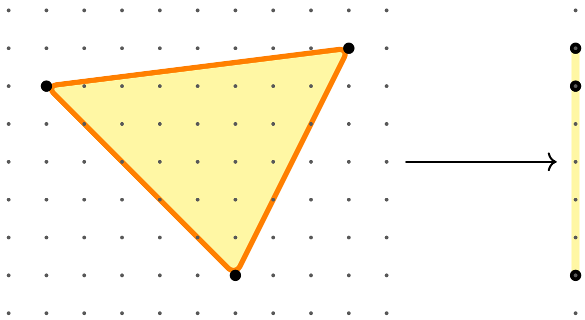

FIGURE 0.1. A lattice projection that computes the lattice width of a polytope corresponding to a singular toric surface.

To prove the theorem we first show that any lattice linear projection

$$P \longrightarrow \mathbb{R}$$

gives rise to an (equivariant) fibration

$$X \dashrightarrow \mathbb{P}^{n-1},$$

with fiber class a 1-parameter curve, such that the degree of the fibration equals the width of the image of P. This shows $\operatorname{mfd}(X, L) \leq \ell\mathrm{w}(P)$. In the other direction (Lemma 2.5), we show that for any fibration

$$\Phi \colon X \dashrightarrow \mathbb{P}^{n-1}$$

there is a 1-parameter curve $C \subset X$ such that $[C] \cdot L \leq [C_\Phi] \cdot L$, and the degree of this one parameter curve is computed by the width of a lattice projection.

Throughout we work with varieties over an arbitrary algebraically closed field.

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1. PRELIMINARIES ON TORIC VARIETIES

The purpose of this section is to recall basic facts about toric varieties including their curves, invariant divisors, and intersection numbers. We also recall the definition of a polytope of a toric pair. Throughout this section, let X denote a projective toric variety of dimension n.

First we set up the standard notation. Let $N \cong \mathbb{Z}^n$ be a lattice and $M = \operatorname{Hom}(N, \mathbb{Z})$ the dual lattice (we use the notation $N_\mathbb{R} = N \otimes \mathbb{R}$ and $M_\mathbb{R} = M \otimes \mathbb{R}$). Then from a fan Σ in N one constructs a toric variety X with torus $T \cong \mathbb{G}_m^n$ ([5, ch. 1]). There is a correspondence between cones and torus orbits: for each cone $\sigma \in \Sigma$ there is a associated distinguished point $x_\sigma \in X(k)$ ([5, p. 28]). Let O_σ denote the torus orbit of x_σ. For each cone σ, let $V(\sigma)$ denote the closure of O_σ in X. There are bijections:

$$(1) \quad \left\{ \begin{matrix} \text{orbits of the action} \\ T \circlearrowright X \end{matrix} \right\} \leftrightarrow \left\{ O_\sigma \,\middle|\, \begin{matrix} \sigma \in \Sigma \text{ is} \\ \text{a cone} \end{matrix} \right\}, \text{ and}$$

$$\text{(2)}\quad \left\{ \begin{matrix} T-\text{invariant, closed} \\ \text{subvarieties of } X \end{matrix} \right\} \leftrightarrow \left\{ V(\sigma) \middle| \begin{matrix} \sigma \in \Sigma \text{ is} \\ \text{a cone} \end{matrix} \right\}.$$

We have that $\operatorname{codim}(V(\sigma)) = \dim(\sigma)$.

Let $\tau_1, \dots, \tau_r$ be the one-dimensional cones of Σ. Let $D_i = V(\tau_i)$ for each i. Then $D_1, \dots, D_r$ are the irreducible torus-invariant divisors of X.

1.1. **Cartier divisors on toric varieties**. Let $D_1, \dots D_r$ be the torus-invariant irreducible divisors of X. We have a map ([5, p. 61])

$$M \longrightarrow \oplus \mathbb{Z} D_i \tag{1.1}$$

$$m \mapsto \operatorname{div}(\chi^m) = \sum_{i=1}^{r} \langle m, v_i \rangle D_i. \tag{1.2}$$

There are standard exact sequences:

$$0 \to M \xrightarrow{\text{div}} \oplus \mathbb{Z} D_i \to \operatorname{Cl}(X) \to 0,$$

and

$$0 \to M \xrightarrow{\text{div}} \operatorname{Div}_T(X) \to \operatorname{Pic}(X) \to 0$$

(where $\operatorname{Div}_T(X)$ is the subgroup of Cartier divisors in $\oplus \mathbb{Z} D_i$ [5, p. 63]).

A convenient representation of the data of a torus-invariant Cartier divisor is that of its support function.

Definition 1.1 ([5, p. 66]). Let D be a torus-invariant Cartier divisor. The *support function of D* is a piecewise linear function:

$$\psi_D : N_{\mathbb{R}} \to \mathbb{R}$$

defined as follows: for each cone $\sigma \in \Sigma$, $D|_{U_\sigma} = \operatorname{div}(\chi^{-m_\sigma})$ for some $m_\sigma \in M$ and set

$$\psi_D|_\sigma = \langle m_\sigma, \cdot \rangle.$$

By the definitions, $\psi_D|_N$ is integer valued. It can also be checked that ψ_D is continuous.

Remark 1.2. In fact, any continuous function

$$\psi : N_{\mathbb{R}} \to \mathbb{R}$$

which is linear on each cone and integral on N arises uniquely as the support function of some torus-invariant Cartier divisor $D = a_1 D_1 + \cdots + a_r D_r$. Specifically, if $v_i \in N$ is the primitive lattice generator of the ray associated to the divisor D_i then

$$a_i = -\psi(v_i).$$

Under this correspondence, the globally linear functions correspond to the globally principal T-invariant divisors.

Example 1.3. When $X = \mathbb{P}^1$, the support function associated to the divisor $D = a_1[0] + a_2[\infty]$ is the function

$$\psi_D \colon \mathbb{R} \to \mathbb{R}$$

$$v \mapsto \begin{cases} -a_1 v & v \geq 0 \\ a_2 v & v \leq 0 \end{cases}$$

and ψ_D is linear if and only if $a_1 + a_2 = 0$.

Definition 1.4. If $L = \mathcal{O}(a_1D_1 + \cdots a_rD_r)$ is a globally generated line T-equivariant bundle on X then *the polytope associated to* (X,L) is by definition

$$P(X,L) := \{m \in M_{\mathbb{R}} \mid \langle m, v_i \rangle \geq -a_i\}$$

where $v_i \in N$ is the integral generator of the ray in $N_{\mathbb{R}}$ associated to D_i. Alternatively, one can define the polytope as follows. For each torus invariant point

$$x_1, \ldots, x_m \in X^T$$

the restriction $L|_{x_i}$ gives a 1-dimensional T-representation, to which we can associate a character $m_i \in M$. Then

$$P(X,L) = \begin{pmatrix} \text{the convex hull} \\ \text{of } m_i \in M \end{pmatrix} \subset M_{\mathbb{R}}.$$

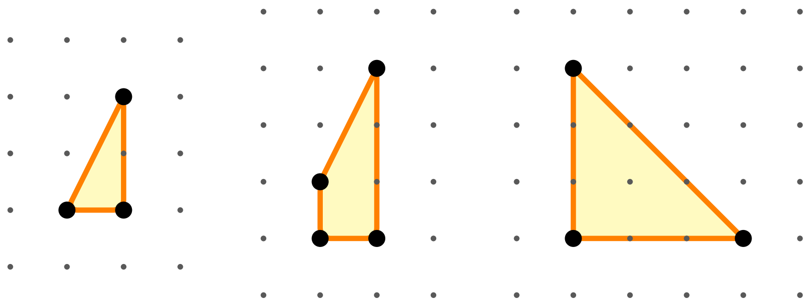


Figure: The polytopes of a cone over a conic, its resolution $\mathbb{F}_2$, and $(\mathbb{P}^2, \mathcal{O}(3))$.

Remark 1.5. The lattice points of $P(X,L)$ correspond to a basis for the global sections of L. More precisely, each $m \in P(X,L) \cap M$ is a character which gives rise a rational function χ^m on X. We have

$$\operatorname{div}_L(\chi^m) = \operatorname{div}_{\mathcal{O}_X}(\chi^m) + D = \sum_{i=1}^{r} (\langle m, v_i \rangle + a_i) D_i$$

is effective since $m \in P(X,L)$ and so χ^m is a global section of L. Furthermore, these sections χ^m form a basis for $H^0(X,L)$. In the opposite direction, given a T-equivariant, globally generated line bundle L, the characters that appear in the T-representation $H^0(X,L)$ give the integral points of the polytope $P(X,L) \subset M_{\mathbb{R}}$.

Remark 1.6. There is a surjective map

$$f\colon \left\{ \begin{array}{c} T\text{-invariant, closed} \\ \text{subvarieties of } X \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{faces of} \\ P(X,L) \end{array} \right\} \tag{1.3}$$

that respects the partial ordering given by inclusion. It sends a divisor D_i to the face:

$$P(X,L) \cap (\langle \cdot, v_i \rangle = 0)$$

and this determines the map as every T-invariant, closed subvariety is an intersection of divisors. In the case L is very ample this map is a bijection. Finallly, if $V(\sigma)$ is a closed T-invariant subvariety and $m \in P(X,L)$ is a lattice point then $\operatorname{div}(\chi^m) \in H^0(X,L)$ and we have

$$V(\sigma) \subset \operatorname{Supp}(\operatorname{div}(\chi^m)) \iff m \notin f(V(\sigma)). \tag{1.4}$$

Remark 1.7 ([5, §3.4]). If D is a T-invariant divisor with a base point free linear system one can recover the support function ψ_D from the polytope $P = P(X, \mathcal{O}(D))$ and visa versa:

$$P = \{m \in M_{\mathbb{R}} \mid \langle m, \cdot \rangle \geq \psi_D\}, \text{ and } \psi_D(v) = \min_{m \in P} \langle m, v \rangle.$$

1.2. **Toric rational maps**.

Let X and Y be toric varieties with tori T_X and T_Y, respectively.

Definition 1.8. A *toric morphism* (resp. a *toric rational map*) is a regular (resp. rational) map

$$X \to Y \quad (\text{resp. } X \dashrightarrow Y)$$

that extends a group homomorphism $T_X \to T_Y$.

Remark 1.9. There is a correspondence between toric rational maps $X \dashrightarrow Y$ and maps of lattices $N_X \to N_Y$. Given a toric rational map, its maximum domain of definition can be determined from the map on lattices: it is the union of all U_σ where σ is a cone of X mapping into a cone of Y. So, under this correspondence, the regular maps (i.e. classes of toric morphisms) correspond to fan-preserving maps of lattices.

Definition 1.10. A toric rational map $\Phi : X \dashrightarrow Y$ is called a *toric fibration* if it is dominant.

Remark 1.11. If $\Phi : X \dashrightarrow Y$ is a toric fibration then the restriction to the tori $\Phi|_{T_X} : T_X \to T_Y$ is surjective.

Remark 1.12. One could equivalently define a toric fibration to be a toric rational map $X \dashrightarrow Y$ so that the the map on lattices $\overline{\Phi} : N_X \to N_Y$ satisfies $\operatorname{rank}(\overline{\Phi}) = \operatorname{rank}(N_Y)$.

1.3. **Functoriality of divisors**.

Let $\pi : X \to Y$ be a toric morphism. There is a pullback:

$$\pi^* \colon \operatorname{Div}_T(Y) \to \operatorname{Div}_T(X).$$

First we describe how to compute this pullback in terms of support functions. Let $D \subset Y$ be a torus-invariant Cartier divisor. The morphism $\pi : X \to Y$ induces a map on lattices

$$\overline{\pi} : (N_X)_{\mathbb{R}} \to (N_Y)_{\mathbb{R}}.$$

The support function

$$\psi_D : (N_Y)_{\mathbb{R}} \to \mathbb{R}.$$

on Y pulls back to a support function

$$\psi := \overline{\pi} \circ \psi_D : (N_X)_{\mathbb{R}} \to \mathbb{R}$$

on X. Then (by Remark 1.2) there exists a unique torus-invariant Cartier divisor D' on X so that $\psi = \psi_{D'}$ and we have $D' = \pi^* D$.

When $L = \mathcal{O}(D)$ is globally generated, we can describe this pullback in terms of polytopes. The map π induces a map on the dual lattices

$$\overline{\pi}^\vee \colon M_Y \to M_X$$

that sends the lattice points of $P(Y, L)$ to the lattice points of $P(X, \pi^* L)$. This corresponds to the pullback of global sections (as in Remark 1.5). This extends to a map on the polytopes:

$$\pi_P \colon P(Y, L) \to P(X, \pi^*(L))$$

In fact, as L is globally generated, the map π_P is surjective.

Proposition 1.13. *If $\pi : X \to Y$ is a toric morphism and L is a globally generated T-line bundle on Y then $\pi_P \colon P(Y, L) \to P(X, \pi^*(L))$ is surjective.*

Proof. The polytope $P(X,\pi^*(L))$ is the convex hull of its vertices, which are lattice points. As π_P is linear and $P(Y,L)$ is convex, it suffices to show that every vertex of $P(X,\pi^*(L))$ is in the image of π_P. For every vertex $m \in P(X,\pi^*(L))$ there is a point $x \in X^T$ such that the face map (Equation 1.3) satisfies

$$f(x) = m.$$

In fact, (by Equation 1.4) χ^m is the unique character of $H^0(X,\pi^*L)$ that doesn't vanish at x. The map

$$\pi\colon X \to Y$$

sends x to a distinguished point of Y corresponding to a T-invariant closed subvariety $V(\sigma)$ in Y. As L is globally generated, there is necessarily a point $m' \in P(Y,L)$ such that

$$\pi(x) \notin \mathrm{Supp}(\mathrm{div}(\chi^{m'}))$$

(any $m' \in f(V(\sigma))$ suffices). Therefore, $\pi^*(\chi^{m'})$ does not vanish on x and thus

$$\pi_P(m') = m. \qquad \square$$

Example 1.14. The map on lattice points

$$\pi_P\colon P(Y,L) \cap M_Y \to P(X,\pi^*(L)) \cap M_X$$

is not surjective in general. Consider the toric map

$$\begin{aligned} \pi\colon \mathbb{P}^1 &\to \mathbb{P}^2 \\ [x_0,x_1] &\mapsto [x_0^3 : x_0^2x_1 : x_1^3]. \end{aligned}$$

If $L = \mathcal{O}_{\mathbb{P}^2}(1)$ then $P(\mathbb{P}^2,L)$ has 3 lattice points but $P(\mathbb{P}^1,\pi^*L)$ has 4 lattice points.

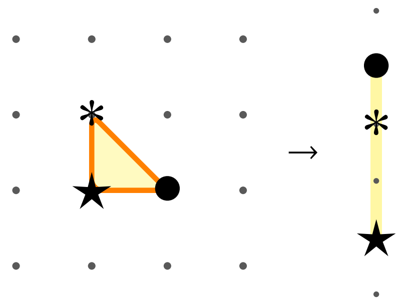


Figure: A depiction of the map on polytopes.

1.4. **Intersection theory of one-parameter curves and torus-invariant divisors.**

Definition 1.15. Let $0 \neq v \in N$. Then the *one-parameter subgroup associated to v*, is the map

$$\phi_v\colon \mathbb{G}_m \to T$$

induced by the map of lattices

$$\begin{aligned} \overline{\phi_v}\colon \mathbb{Z} &\to N \\ 1 &\mapsto v. \end{aligned}$$

Definition 1.16. The *one-parameter curve associated to v*, is the induced map $\phi_v\colon \mathbb{P}^1 \to X$ (which we also denote by ϕ_v by abuse of notation) and the cycle associated to v is

$$[C_v] := (\phi_v)_*([\mathbb{P}^1]) \in Z_1(X).$$

For $v \in N$ be nonzero and D a torus-invariant Cartier divisor, we recall how to compute the intersection number $[C_v] \cdot D$. Let σ_0 be the smallest cone of X containing v, and σ_∞ be the smallest cone of X containing $-v$. Then the images of 0 and ∞ are the distinguished points x_{σ_0} and x_{σ_∞}, respectively.

Proposition 1.17. *Let $v \in N$ be nonzero and $[C_v]$ the associated cycle such that the limits of ϕ_v at* 0 *(resp. ∞) are x_{σ_0} (resp. x_{σ_∞}). If $D \subset X$ is a torus-invariant Cartier divisor with support function ψ_D then the local intersection numbers at x_{σ_0} and x_{σ_∞} are:*

$$([C_v] \cdot D)_{x_{\sigma_0}} = -\psi_D(v) \quad \text{and} \quad ([C_v] \cdot D)_{x_{\sigma_\infty}} = -\psi_D(-v).$$

Consequently,

$$[C_v] \cdot D = -(\psi_D(v) + \psi_D(-v)).$$

Proof. We compute $\phi_v^* \mathcal{O}_X(D)$ by pulling back the support function ψ_D. The support function $\psi_D : N_{\mathbb{R}} \to \mathbb{R}$ pulls back along $\overline{\psi_v}$ to a support function

$$\begin{aligned} \psi_D \circ \overline{\psi_v} : \mathbb{R} &\to \mathbb{R} \\ 1 &\mapsto \psi_D(v). \end{aligned}$$

This support function corresponds to a torus-invariant Cartier divisor D' on $\mathbb{P}^1$ such that $\psi_v^* \mathcal{O}_X(D) \cong \mathcal{O}_{\mathbb{P}^1}(D')$. It follows that

$$([C_v] \cdot D)_{\sigma_0} = \operatorname{ord}_0(D') = -\psi_{D'}(1) = -\psi_D(v)$$

and

$$([C_v] \cdot D)_{\sigma_\infty} = \operatorname{ord}_\infty(D') = -\psi_{D'}(-1) = -\psi_D(-v)$$

as desired. □

Corollary 1.18. *If D is a torus-invariant prime divisor and v the primitive generator of the ray in the fan determined by D, then $[C_v] \cdot D = 1$.*

Proof. In this case, σ_0 is the ray determined by v. The cone σ_∞ contains $-v$, so it cannot contain the ray determined by D as σ_∞ is strongly convex. Then $\psi_D|_{\sigma_\infty} \equiv 0$ and thus (by Remark 1.2)

$$[C_v] \cdot D = -\psi_D(v) = 1. \qquad \square$$

2. Minimal fibering degree of toric varieties

This section contains the proof of the main theorem. To prove Theorem A we show that the minimal fibering degree can be computed by torus equivariant fibrations. We then relate the degree of these torus equivariant fibrations to the lattice width of the polytope of X.

Definition 2.1. Let X be an n-dimensional projective variety over k an algebraically closed field with line bundle L. Let

$$\Phi \colon X \dashrightarrow \mathbb{P}^{n-1}$$

be a dominant rational map. The L-degree of ψ is

$$\deg_L(\Phi) = \deg([C_b] \cdot L)$$

where C_b is the closure of a general fiber of ψ. We denote the rational equivalence class of any such $[C_b]$ by $[C_\Phi]$.

Definition 2.2. With (X, L) as above, if L is effective, then the *minimal fibering degree* of the pair (X, L) is

$$\operatorname{mfd}(X, L) := \min\{\deg_L(\Phi) \mid \Phi \colon X \dashrightarrow \mathbb{P}^{n-1}\}.$$

Lemma 2.3. *Let X be a smooth n-dimensional projective toric variety.*

(1) For every one-parameter curve $\phi_v \colon \mathbb{P}^1 \to X$ there is a torus equivariant fibration

$$\Phi_v \colon X \dashrightarrow \mathbb{P}^{n-1}$$

(where $\mathbb{P}^{n-1}$ has the standard torus action) such that $[C_{\Phi_v}] \equiv_{\mathrm{rat}} [C_v] \in \mathrm{CH}_1(X)$.

(2) Likewise, for any torus equivariant fibration:

$$\Phi \colon X \dashrightarrow \mathbb{P}^{n-1}$$

there exists a 1-parameter curve ϕ_v such that $[C_v] = [C_\Phi]$.

Proof. For (1), first consider the case v is primitive. Choose a basis $u_1, \dots, u_{n-1} \in M$ for $v^\perp$. These characters define a dominant map

$$\begin{aligned} T &\to \mathbb{G}_m^{n-1} \\ t &\mapsto (\chi^{u_1}(t), \dots, \chi^{u_{n-1}}(t)) \end{aligned}$$

that uniquely extends to a toric fibration $\Phi \colon X \dashrightarrow \mathbb{P}^{n-1}$. We claim that for every T-divisor $D \subset X$:

$$[C_\Phi] \cdot D = [C_v] \cdot D.$$

On the one hand, the closure of the fiber over $1 \in \mathbb{G}^{n-1}$ has class $[C_v]$ (this uses that v is primitive). Consider a general point $b \in \mathbb{G}^{n-1} \subset \mathbb{P}^{n-1}$. Let $t \in T$ be a point in $\Phi^{-1}(b)$. So

$$\overline{\Phi^{-1}(\Phi(t))} = C_\Phi.$$

Then

$$\begin{aligned} [C_v] \cdot D &= [t^{-1}(C_\Phi)] \cdot D \\ &= [t(t^{-1}(C_\Phi))] \cdot t(D) && \text{(as } t \text{ gives an automorphism of } X\text{)} \\ &= [C_\Phi] \cdot D. && \text{(as } D \text{ is } T\text{-invariant)} \end{aligned}$$

As numerical equivalence and rational equivalence coincide on smooth toric varieties this implies $[C_\Phi] \equiv_{\mathrm{rat}} [C_v]$.

The case that $v = mv_0$ is not primitive (with $m > 1$ and v_0 primitive) is resolved by postcomposing the above map

$$T \to \mathbb{G}_m^{n-1}$$

with a degree m homomorphism $\mathbb{G}_m^{n-1} \to \mathbb{G}_m^{n-1}$.

For (2), such a torus equivariant fibration gives rise to a 1-parameter subgroup (the reduced connected component of the identity in the kernel). Consider the induced vector $v_0 \in N$. By equivariance, there is a factorization

$$\begin{tikzcd} T \arrow[r, "\phi_{v_0}|_T"] \arrow[dr, "\Phi|_T"'] & \mathbb{G}_m^{n-1} \arrow[d, "\xi"] \\ & \mathbb{G}_m^{n-1}. \end{tikzcd}$$

By a computation

$$[C_\Phi] = \deg(\xi)[C_{v_0}] = [C_{\deg(\xi)\cdot v_0}],$$

so we may set $v = v_0$. □

Lemma 2.4. *Let X be smooth, n-dimensional, projective variety. If*

$$\Phi\colon X \dashrightarrow \mathbb{P}^{n-1}$$

is any dominant fibration and $D \subset X$ is any effective divisor then $[C_\Phi] \cdot D \geq 0$.

Proof. Let $b \in \mathbb{P}^{n-1}$ be a general point and consider the graph of Φ

$$\Gamma_\Phi \subset X \times \mathbb{P}^{n-1}.$$

Let π_i denote the projection of $X \times \mathbb{P}^{n-1}$ onto each factor. Then

$$[C_\Phi] = (\pi_1)_*(\Gamma_\Phi \cdot X \times b).$$

Thus

$$\begin{aligned}[C_\Phi] \cdot D &= (\pi_1)_*(\Gamma_\Phi \cdot X \times b) \cdot D \\ &= (\Gamma_\Phi \cdot (X \times b)) \cdot \pi_1^* D \\ &= (\Gamma_\Phi \cdot \pi_2^*(b)) \cdot \pi_1^* D \\ &= (\Gamma_\Phi \cdot \pi_1^* D) \cdot \pi_2^*(b) \\ &= (\pi_2)_*(\Gamma_\Phi \cdot \pi_1^* D) \cdot b.\end{aligned}$$

Lastly, $\Gamma_\Phi \cdot \pi_1^* D$ is an effective divisor on Γ_Φ (as the map $\Gamma_\Phi \to X$ is dominant and Γ_Φ is irreducible). Thus

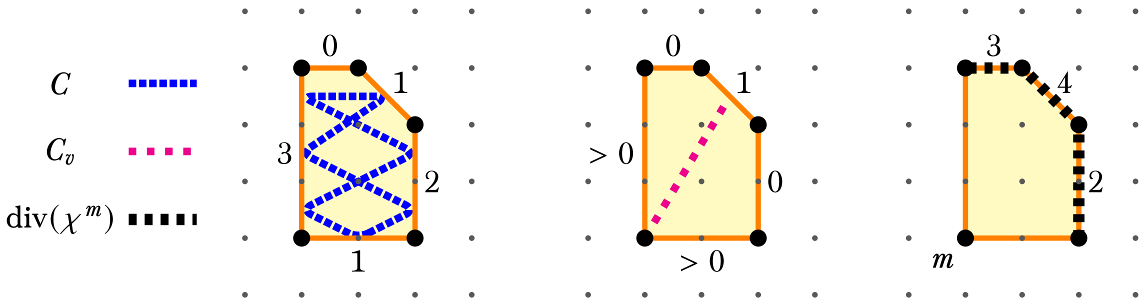
$$\begin{aligned}[C_\Phi] \cdot D &= (\pi_2)_*(\Gamma_\Phi \cdot \pi_1^* D) \cdot b \\ &= \deg((\Gamma_\Phi \cap \pi_1^* D) \to \mathbb{P}^{n-1}) \\ &\geq 0.\end{aligned}$$

□

Lemma 2.5. *Let X be a smooth projective toric variety and let $[C]$ be an effective curve class that meets all effective, T-invariant divisors non-negatively. There exists a 1-parameter curve ϕ_v such that for all globally generated line bundles L:*

$$[C] \cdot L \geq [C_v] \cdot L.$$

Example 2.6. Here we illustrate the proof of Lemma 2.5 by an example. To start we are given a curve C with prescribed intersection numbers with the torus-invariant divisors, as in the left figure. Consider any divisor D with positive intersection number (in this case the top right divisor). From this divisor we construct a 1-parameter curve C_v. This has intersection multiplicity 1 with D and meets one other stratum of the toric variety (in our example, the intersection of the bottom and left curves, as illustrated in the middle figure). The other intersection numbers are 0. Given any globally generated line bundle there is a section that does not vanish at the distinguished point of the stratum (in this example we take the section m in the right figure), the corresponding divisor is particularly convenient to intersect with $[C_v]$ as most of the terms vanish.



By direct comparison we complete the proof, in this example we have:

$$\begin{aligned}[C]\cdot L =&[C]\cdot \mathrm{div}(\chi^m)\\ =&1\cdot 0+2\cdot 2+1\cdot 4+0\cdot 3+3\cdot 0\\ \geq&(>0)\cdot 0+0\cdot 2+1\cdot 4+0\cdot 3+(>0)\cdot 0\\ =&[C_v]\cdot L.\end{aligned}$$

Proof of Lemma 2.5. Let $D_1 \dots D_r$ be the T-invariant divisors. As the curve class $[C]$ corresponds to an effective curve and X is projective, it cannot intersect all the effective T-divisors to multiplicity 0. Assume that

$$[C]\cdot D = a > 0$$

for one such T-invariant prime divisor D.

Let $v \in N$ be the integral generator of the ray corresponding to D. This gives rise to a 1-parameter curve

$$\phi_v : \mathbb{P}^1 \longrightarrow X.$$

Let σ be the smallest cone containing $-v \in N$. Then the image of $0 \in \mathbb{P}^1$ is the distinguished point of D and the image of $\infty \in \mathbb{P}^1$ is the distinguished point of σ. These are the only intersection points of C_v with the T-invariant divisors of X.

As L is globally generated there is a character $m \in M$ such that $\chi^m \in \Gamma(X, L)$ and χ^m does not vanish at the distinguished point of σ. Then

$$\mathrm{div}(\chi^m) = b_1 D_1 + \cdots bD + \cdots + b_r D_r \quad (b, b_i \geq 0).$$

As χ^m does not vanish at the distinguished point of σ, it follows that for every D_i such that $x_\sigma \in D_i$ the coefficient $b_i = 0$. So, as D is the unique prime divisor containing its distinguished point, by Corollary 1.18:

$$[C_v]\cdot L = [C_v]\cdot (bD) = b.$$

Therefore:

$$\begin{aligned}[C]\cdot L =&[C]\cdot \mathrm{div}(\chi^m)\\ =&[C]\cdot (b_1D_1+\cdots+bD+\cdots+b_rD_r)\\ =&[C]\cdot (b_1D_1)+\cdots+[C]\cdot(bD)+\cdots+[C]\cdot(b_rD_r)\\ \geq&[C]\cdot(bD) && \text{(by assumption on } [C])\\ =&b\cdot a \geq b\cdot 1 = [C_v]\cdot L. && \square\end{aligned}$$

Remark 2.7. Interestingly, for any curve class $0 \neq [C] \in \mathrm{CH}_1(X)$ such that $[C]\cdot D \geq 0$ for all T-invariant divisors D Payne constructs ([9, Prop. 2]) a family of curves with class $[C]$ that sweeps out X.

Definition 2.8. Let $P \subset M_{\mathbb{R}}$ be a lattice polytope (that is, the convex hull of finitely many point in M). For any nonzero $v \in N$, the *lattice width of P in the direction of v* is

$$\ell\mathrm{w}_v(P) := \mathrm{width}\left(\{\langle x, v\rangle \in \mathbb{R} | x \in P\}\right).$$

In other words, v induces a linear projection (mapping M to $\mathbb{Z}$) from P to $\mathbb{R}$, and $\ell\mathrm{w}_v(P)$ is the width of the image. The *lattice width of P* is simply:

$$\ell\mathrm{w}(P) := \min\{\ell\mathrm{w}_v(P) \mid 0 \neq v \in N\}.$$

Example 2.9. For the figure in the introduction, the lattice width is 6 (as this map shows). To argue this, we see that there are 8 points on a line in P that are identified under this projection. Any projection that does not identify these points, must have width at least 7.

Proposition 2.10. *Let D be a globally generated torus-invariant divisor and $v \in N$. Then*

$$C_v \cdot D = \ell\mathrm{w}_v(P(X, \mathcal{O}_X(D)).$$

Proof. By Remark 1.7, for each $u \in N$ we have

$$\psi_D(u) = \min_{x \in P(X, \mathcal{O}_X(D))} \langle x, u\rangle.$$

It follows that

$$\ell\mathrm{w}_v(P(X, \mathcal{O}_X(D)) = \max_{x \in P(X, \mathcal{O}_X(D))} \langle x, v\rangle - \min_{x \in P(X, \mathcal{O}_X(D))} \langle x, v\rangle = -(\psi_D(v) + \psi_D(-v)).$$

So, we are done by Proposition 1.17. □

Proof of Theorem A. Throughout we write $L = \mathcal{O}(D)$ for some effective torus invariant divisor $D \subset X$.

If X is not smooth, take a projective, toric resolution of singularities of X to obtain

$$\mu \colon X' \to X.$$

Now the pair (X', μ^*L) satisfies

$$P(X', \mu^*L) = P(X, L),$$

and μ^*L is globally generated. Finally for any rational fibration:

$$\Phi \colon X \dashrightarrow \mathbb{P}^{n-1}$$

there is an induced rational fibration

$$\Phi' \colon X' \dashrightarrow \mathbb{P}^{n-1}$$

and Φ' satisfies

$$\mu_*[C_{\Phi'}] = [C_\Phi].$$

So by the projection formula:

$$\mathrm{mfd}(X, L) = \mathrm{mfd}(X', \mu^*L).$$

Therefore, it suffices to prove Theorem A in the case X is smooth.

By Lemma 2.3 for any nonzero $v \in N$, there is a torus equivariant fibration:

$$\Phi_v \colon X \dashrightarrow \mathbb{P}^{n-1}$$

such that $[C_{\Phi_v}] \equiv_{\mathrm{rat}} [C_v]$. By Proposition 2.10,

$$\deg_L(\Phi_v) = [C_v] \cdot D = \ell\mathrm{w}_v(P(X, L)).$$

Thus, by the definitions:

$$\mathrm{mfd}(X,L) \leq \ell\mathrm{w}(P(X,L)).$$

On the other hand, by Lemma 2.4, if

$$\Phi\colon X \dashrightarrow \mathbb{P}^{n-1}$$

is any fibration, then $[C_\Phi] \cdot E \geq 0$ for any effective divisor E on X. Then by Lemma 2.5, there exists a $v \in N$ such that

$$\deg_L(\Phi_v) = [C_v] \cdot D \leq [C_\Phi] \cdot D = \deg_L(\Phi).$$

Therefore:

$$\ell\mathrm{w}(P(X,L)) \leq \ell\mathrm{w}_v(P(X,L)) = [C_v] \cdot D \leq \mathrm{mfd}(X,L)$$

which proves the Theorem. □

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