OPTIMAL CONNECTIVITY RESULTS FOR SPHERES IN THE CURVE GRAPH OF LOW AND MEDIUM COMPLEXITY SURFACES

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Abstract. Answering a question of Wright, we show that spheres of any radius are always connected on the curve graph of surfaces $\Sigma_{2,0}, \Sigma_{1,3}$, and $\Sigma_{0,6}$, and the union of two consecutive spheres is always connected for $\Sigma_{0,5}$ and $\Sigma_{1,2}$.

1. Introduction

1.1. **Main results.** Let $\Sigma = \Sigma_{g,n}$ be a connected surface with genus g and n punctures. We define the complexity of Σ to be $\xi(\Sigma) = 3g-3+n$. We say Σ is

- *exceptional* if $\xi(\Sigma) = 1$, i.e. $(g, n) \in \{(1, 1), (0, 4)\}$,
- *low complexity* if $\xi(\Sigma) = 2$, i.e. $(g, n) \in \{(1, 2), (0, 5)\}$,
- *medium complexity* if $\xi(\Sigma) = 3$, i.e. $(g, n) \in \{(2, 0), (1, 3), (0, 6)\}$,
- *high complexity* if $\xi(\Sigma) \geq 4$.

Let $\mathcal{C}\Sigma$ be the curve graph of Σ. For any vertex $c \in \mathcal{C}\Sigma$ and radius r, let

$$S_r = S_r(c) = \{a \in \mathcal{C}\Sigma : d(a, c) = r\}$$

be the sphere of radius r about c in $\mathcal{C}\Sigma$. We will say that a sphere is connected if the induced subgraph is connected.

The main results to be proved in this paper are as follows:

Theorem 1.1. Let $\Sigma_{g,n}$ be low complexity. Fix center $c \in \mathcal{C}\Sigma$. Then for all $r > 0$ we have that $S_r(c) \cup S_{r+1}(c)$ is connected.

Theorem 1.2. Let $\Sigma_{g,n}$ be medium complexity. Fix center $c \in \mathcal{C}\Sigma$. Then for all $r > 0$ we have that $S_r(c)$ is connected.

1.2. **Previous results.** The main contribution of this paper is to strengthen the results of the following theorem from [Wri23].

Theorem 1.3 ([Wri23], Theorem 1.1)**.** For all $r > 0$ and connected surface Σ,

(1) If Σ has high complexity, then S_r is connected.
(2) If Σ has medium complexity, then $S_r \cup S_{r+1}$ is connected.
(3) If Σ has low complexity, then $S_r \cup S_{r+1} \cup S_{r+2}$ is connected.

Our results answer [Wri23, Question 1.7] and prove Theorem 1.1 and Theorem 1.2. These results are now sharp, as S_r is never connected in low complexity [Wri23, Corollary 6.12].

1.3. **Organization of the proof.** In both the low and medium complexity cases, we utilize the same proof strategy and preliminary results from [Wri23] about the connectivity of spheres. Then we modify the paths in order to stay closer to S_r by using the Bounded Geodesic Image Theorem from [MM00] as our primary tool.

Our main contribution in the low complexity case is to construct improved "preliminary paths" (discussed in Section 3.4), and show this adjustment allows the argument to ultimately yield paths contained in two spheres instead of three.

In the medium complexity case, Wright's argument included an induction on radius, for which it was crucial to use essentially non-separating curves. We assume Wright's result and no longer need to argue by induction, so we are able to use curves which fail to be essentially non-separating to produce a variant of Wright's paths which stay in a single sphere.

1.4. **Motivation.** This paper continues the tradition of examining the relationship between fine and coarse geometry on the curve graph. As an example, the Bounded Geodesic Image Theorem uses coarse information to deduce a precise result about the vertices on this geodesic.

In particular, we can also gain a better understanding of the coarse geometry of the curve graph as a whole by understanding the fine results. This idea is exemplified in [Wri23] where the linear connectivity of the Gromov boundary (coarse) follows from an analysis on the connectivity of S_r (fine).

Our paper also develops techniques to perform constructions directly in the curve graph rather than spaces of lamination or Teichmüller space.

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2. Subsurface projections and Bounded Geodesic Image Theorem

In this section we introduce one of our key tools, the Bounded Geodesic Image Theorem, and recall some basic facts about subsurface projections.

Let U be a subsurface of Σ and $\alpha \in \mathcal{C}\Sigma$. We say curve α *cuts* U if it is not possible to isotope α out of U. We define $\mathcal{C}(\Sigma, U)$ to be the subgraph of $\mathcal{C}\Sigma$ whose vertices are all essential non-peripheral curves that cut U, and keeping all possible edges. Note that $\mathcal{C}U$ is contained in $\mathcal{C}(\Sigma, U)$.

Given a subsurface U of Σ, there exists a subsurface projection map, denoted ρ_U, from the set of curves cutting U to finite subsets of curves on U. We will want to recall some key facts about ρ_U:

(1) The values of ρ_U are uniformly bounded in diameter.
(2) The map ρ_U is 6-Lipschitz i.e.
$$d(\rho_U(\alpha), \rho_U(\beta)) \leq 6d(\alpha, \beta).$$
(3) Define
$$d_U(\alpha, \beta) = diam(\rho_U(\alpha) \cup \rho_U(\beta)).$$
It can easily be verified that d_U satisfies the triangle inequality.

The following theorem is known as the Bounded Geodesic Image Theorem:

Theorem 2.1. [MM00, Theorem 3.1] Let U be a subsurface of Σ. There exists $M > 0$ such that if $d_U(\alpha, \beta) \geq M$ then every geodesic from α to β in $\mathcal{C}\Sigma$ contains a curve not cutting U.

From here on, M will refer to the constant required for Theorem 2.1, which can be taken independent of Σ and U [Web15].

3. Low complexity

Throughout this section, we deal with $\Sigma = \Sigma_{0,5}$. Assume that a center vertex $c \in \mathcal{C}\Sigma_{0,5}$ is fixed and let $S_r = S_r(c)$.

3.1. **Organization.** The outcome of this section is to prove Theorem 1.1. We do so by first taking arbitrary $a \in S_r$ and $b, b' \in S_{r+1} \cap S_1(a)$ and constructing a preliminary path, described in Proposition 3.7, connecting b to b'. We then offer Lemma 3.20 to serve a similar function as [Wri23, Lemma 6.16] to push this path up to $S_r \cup S_{r+1}$ using Dehn twists, by observing that vertices on this preliminary path only enter $S_3(a)$ when they are close to $S_{r-1} \cup S_r$. This adjustment is sufficient in proving the path stays within two consecutive spheres rather than three.

3.2. Definitions.

Definition 3.1. A vertex $x \in S_r$ has *unique backtracking* if it has a unique neighbour in S_{r-1}.

Definition 3.2. A vertex $x \in S_r$ has *no sidestepping* if it does not have any neighbour in S_r.

Definition 3.3. A vertex $x \in S_r$ is *forward facing* if it has unique backtracking and no sidestepping.

3.3. Pentagons in $\mathcal{C}\Sigma_{0,5}$.

It is important to note that $\mathcal{C}\Sigma$ contains no cycles of length 3 or 4 [Wri23, Lemma 6.1]. Thus we often study paths on $\mathcal{C}\Sigma$ by using pentagons.

Definition 3.4. Label the 5 punctures of Σ with the elements of $\mathbb{Z}/5\mathbb{Z}$. The 5 tuple of curves $(a_1, a_2, a_3, a_4, a_5)$ is a *pentagon* if for $i \in \mathbb{Z}/5\mathbb{Z}$:

(1) a_i goes around punctures i and $i+1$,
(2) the intersection number between a_i, a_{i+1} and a_i, a_{i-1} is 2, and
(3) the intersection number between a_i, a_{i+2} and a_i, a_{i-2} is 2.

To obtain a 5-cycle from a pentagon with vertices $(a_1, a_2, a_3, a_4, a_5)$, we can traverse the curves in the following order: $(a_1, a_3, a_5, a_2, a_4)$. We use the following lemmas to find pentagons in $\mathcal{C}\Sigma_{0,5}$.

Lemma 3.5. [Wri23, Lemma 6.5] Suppose $a_1, a_3 \in S_{r-1}$ are adjacent. Then there are curves $a_2, a_3, a_5 \in S_r \cup S_{r+1}$ such that $(a_1, a_2, a_3, a_4, a_5)$ is a pentagon.

Lemma 3.6. [Wri23, Lemma 6.6] Suppose $a_1 \in S_{r-1}$ and $a_3, a_4 \in S_r \cap S_1(a_1)$ have $i(a_3, a_4) = 2$. Then there exist $a_2, a_5 \in S_r \cup S_{r+1}$ such that $(a_1, a_2, a_3, a_4, a_5)$ is a pentagon.

3.4. Preliminary path construction.

Proposition 3.7. Suppose $a \in S_r$ and $b, b' \in S_{r+1} \cap S_1(a)$. Then there exists a path γ from b to b' contained in $S_1(a) \cup S_2(a) \cup S_3(a)$ such that the following hold for all vertices v on the path γ:

(1) If $v \in S_3(a)$, then $d(v, (S_{r-1} \cup S_r) \cap S_1(a)) \leq 2$.
(2) If $v \in S_1(a)$, then $v \in S_{r+1}$.

First we recall the following lemmas:

Lemma 3.8. [Wri23, Lemma 6.10] For any $a \in \mathcal{C}\Sigma_{0,5}$ and $x \in S_1(a)$, x is forward facing with respect to a.

Lemma 3.9. [Wri23, Lemma 6.13] For any $a \in \mathcal{C}\Sigma_{0,5}$, $S_1(a) \cup S_2(a)$ is connected.

Lemma 3.10. [Wri23, Lemma 6.14] Suppose $x \in S_r$ is forward facing and $y, y' \in S_1(x) \cap S_{r+1}$. Then there exists a path from y to y' in $(S_{r+1} \cup S_{r+2}) \cap B_2(x)$.

Lemma 3.10 gives us the following corollary.

Corollary 3.11. Suppose x_{j-1}, x_j, x_{j+1} is a path in $S_1(a) \cup S_2(a)$ with $x_j \in S_1(a)$. Then there exists a path from x_{j-1} to x_{j+1} contained in $(S_2(a) \cup S_3(a)) \cap B_2(x_j)$.

Proof. This statement is exactly the conclusion of Lemma 3.10 with a as the center, $x = x_j$, $y = x_{j-1}$, and $y' = x_{j+1}$, so we only need to check the conditions are satisfied.

First we see x_j is forward facing with respect to a because $x_j \in S_1(a)$ by assumption and by Lemma 3.8, every vertex in $S_1(a)$ is forward facing with respect to a.

Second we have $x_{j-1}, x_{j+1} \in S_1(x_j)$ because x_{j-1}, x_j, x_{j+1} is a path by assumption.

Third we observe $x_{j-1}, x_{j+1} \in S_1(a) \cup S_2(a)$ and $S_1(a)$ is totally disconnected because $\mathcal{C}\Sigma_{0,5}$ has no triangles. Now x_{j-1}, x_{j+1} are adjacent to $x_j \in S_1(a)$. Thus, $x_{j-1}, x_{j+1} \notin S_1(a)$ so $x_{j-1}, x_{j+1} \in S_2(a)$. This verifies the conditions of Lemma 3.10. $\square$

Now we have the tools to construct the preliminary path as stated in Proposition 3.7.

Proof of Proposition 3.7. By Lemma 3.9, $S_1(a) \cup S_2(a)$ is connected. Since $b, b' \in S_1(a)$ this implies there exists a path $b = x_0, ..., x_l = b'$ contained in $S_1(a) \cup S_2(a)$. Now for each $x_j \in (S_{r-1} \cup S_r) \cap S_1(a)$ replace the path segment x_{j-1}, x_j, x_{j+1} with the path $x_{j-1} = x_j^0, x_j^1, ..., x_j^k = x_{j+1}$ for some $k \geq 0$ given by Corollary 3.11. Call this path γ. First we observe by construction that γ has no vertex in $(S_{r-1} \cup S_r) \cap S_1(a)$.

Now we check that γ satisfies the conclusions of Proposition 3.7 with each following sublemma:

Sublemma 3.12. The path γ is contained in $S_1(a) \cup S_2(a) \cup S_3(a)$.

Proof. By construction the vertices in γ are either in $S_1(a) \cup S_2(a)$ or in $(S_2(a) \cup S_3(a)) \cap B_2(x_j)$ for some $j \leq l$. $\square$

Sublemma 3.13. If v is a vertex in γ and $v \in S_3(a)$, then $d(v, (S_{r-1} \cup S_r) \cap S_1(a)) \leq 2$.

This establishes part (1) of Proposition 3.7.

Proof. The original path $b = x_0, ..., x_l = b'$ is contained in $S_1(a) \cup S_2(a)$ so if $v \in S_3(a)$, then v must have been obtained from replacing the segment x_{j-1}, x_j, x_{j+1} with the path $x_{j-1} = x_j^0, x_j^1, ..., x_j^k = x_{j+1}$ for some $k \geq 0$. In particular, $v = x_j^i$ for some $i \leq k$. By Corollary 3.11, $v = x_j^i \in (S_2(a) \cup S_3(a)) \cap B_2(x_j)$ so $d(v, x_j) \leq 2$. Additionally, $x_j \in (S_{r-1} \cup S_r) \cap S_1(a)$. Thus $d(v, (S_{r-1} \cup S_r) \cap S_1(a)) \leq 2$. □

Sublemma 3.14. The only vertices in γ which are in $S_1(a)$ are also in S_{r+1}.

This establishes part (2) of Proposition 3.7.

Proof. Let v be a vertex on γ such that $v \in S_1(a)$. Now $a \in S_r$ so $v \in S_{r-1} \cup S_r \cup S_{r+1}$. But by construction γ has no vertices in $(S_{r-1} \cup S_r) \cap S_1(a)$ because any such vertices in the original path were replaced by a path in $S_2(a) \cup S_3(a)$. Thus, $v \in S_{r+1}$. □

Since we have verified the conclusions of Proposition 3.7 for arbitrary $a \in S_r$ and $b, b' \in S_1(a) \cap S_{r+1}$, this finishes the proof. □

3.5. **Pushing the path up.** Now we will apply Dehn twists to the path obtained in Proposition 3.7 to make sure it lies in $S_r \cup S_{r+1}$.

We first fix some important notations.

Remark 3.15. Suppose $a, b \in \mathcal{C}\Sigma_{0,5}$. Let $T_a(b)$ denote the left Dehn twist of b around a. Henceforth, we will refer to left Dehn twists as just Dehn twists.

In addition, we use d_a to denote the distance between the projections to the curve graph of the annular subsurface associated to an element a of $\mathcal{C}\Sigma_{0,5}$.

We will make use of the following basic fact.

Proposition 3.16. Suppose a, b are vertices in $\mathcal{C}\Sigma_{0,5}$ such that $d(a, b) \geq 2$. Then

$$\lim_{N \to \infty} d_a(b, T_a^N(b)) = \infty. \tag{1}$$

Lemma 3.17. Suppose $a \in S_r$ and $d(b, a) \geq 2$. Then there exists a positive integer $N(a, b)$, such that for all $N' \geq N(a, b)$, we have $d_a(T_a^{N'}(b), c) \gg M$.

Proof. For all integers m, we have

$$d_a(T_a^m(b), c) \geq d_a(T_a^m(b), b) - d_a(b, c), \tag{2}$$

where $d_a(b, c)$ is a constant. Thus, the lemma follows from Proposition 3.16. □

Remark 3.18. For the rest of Section 3.5, we will continue to use $N(a, b)$ to denote the constant in Lemma 3.17. Note that $N(a, b)$ depends on a, b.

Corollary 3.19. Suppose $a \in S_r$ and $d(b,a) \geq 2$. If $N' \geq N(a,b)$, then

$$d(c, T_a^{N'}(b)) \geq r. \tag{3}$$

Proof. By Lemma 3.17 and Theorem 2.1, any geodesic from $T_a^{N'}(b)$ to c must contain a vertex that lies in $B_1(a)$. This implies that $d(T_a^{N'}(b), c) \geq r$. □

3.6. **Main lemma.**

Lemma 3.20. Suppose $a \in S_r$ and $b, b' \in S_{r+1} \cap S_1(a)$. Then there exists a path $b, x_1, \cdots, x_l, b'$ with four properties:

(1) $1 \leq d(x_i, a) \leq 3$.
(2) $r \leq d(x_i, c) \leq r+2$.
(3) If $d(x_i, c) = r$, then $d(x_i, a) = 2$, there exists a unique vertex z adjacent to both x_i and a, $z \in S_{r-1}$, and z is the unique backtrack of x_i.
(4) If $d(x_i, c) = r$ and if a has unique backtracking, then x_i has no sidestepping.

Remark 3.21. This lemma improves [Wri23, Lemma 6.16] in that our lemma also shows that $d(x_i, c) \leq r+2$.

Proof. We first construct a path and then prove that it satisfies the four listed properties.

We begin by considering the path α that Proposition 3.7 gives us. Let $b, y_1, \cdots, y_l, b'$ be the vertices of the path. By Lemma 3.17, for all i such that $d(y_i, a) \geq 2$, there exists a positive integer $N(a, y_i)$ such that if $N' \geq N(a, y_i)$, then $d_a(T_a^{N'}(y_i), c) \gg M$. Take $N = \max_i(N(y_i, a))$. Let γ be the path obtained by applying T_a^N to α. The vertices of γ are then

$$b, T_a^N(y_1), \cdots, T_a^N(y_l), b'. \tag{4}$$

Let $x_i = T_a^N(y_i)$ for all $1 \leq i \leq l$.

Proposition 3.7, as well as the fact that Dehn twists preserve distance (Remark 3.15), verifies property (1) above.

Now we verify property (2). We first claim that for all i, $d(x_i, c) \geq r$. Let us fix some i. If $d(y_i, a) \geq 2$, then Corollary 3.19 implies that $d(x_i, c) = d(T_a^N(y_i), c) \geq r$. On the other hand, if $d(y_i, a) \leq 1$, then the assumptions on the path α imply that $y_i \in S_{r+1}$. So $d(x_i, c) = d(T_a^N(y_i), c) = d(y_i, c) \geq r$.

Next, we claim that for all i, $d(x_i, c) \leq r+2$. This follows from the observation that if $y_i \in S_3(a)$, then by assumptions on the path α, there exists $z_i \in S_1(a) \cap (S_r \cup S_{r-1})$ such that $d(y_i, z_i) \leq 2$. But since Dehn twists preserve distances,

$$d(x_i, z_i) = d(T_a^N(y_i), T_a^N(z_i)) = d(y_i, z_i). \tag{5}$$

And so $d(x_i, z_i) \leq 2$. So

$$d(x_i, c) \leq d(x_i, z_i) + d(c, z_i) \leq r + 2. \tag{6}$$

This finishes the verification of property (2).

To verify property (3), we suppose $d(x_i, c) = r$. Recall that by definition, $x_i = T_a^N(y_i)$. If $d(y_i, a) = 1$, then by construction of α, we have $y_i \in S_{r+1}$. Since T_a^N fixes y_i, we conclude that $x_i = T_a^N(y_i)$ belongs to S_{r+1}. This contradicts the assumption that $d(x_i, c) = r$. So we must have $d(y_i, a) \geq 2$.

And so by Lemma 3.17 and Theorem 2.1, every geodesic from $x_i = T_a^N(y_i)$ to c must pass through $B_1(a)$. Let ζ be one such geodesic and z be one vertex in $\zeta \cap B_1(a)$. Since $d(x_i, a) \geq 2$, z must belong to S_{r-1}, implying that $d(x_i, a) = 2$. By construction, z is a vertex adjacent to both x_i and z. It is the unique such vertex because $\mathcal{C}\Sigma_{0,5}$ has no quadrilaterals.

To finish verifying property (3), it remains to show that z is the unique backtrack of x_i. Let z' be a backtrack of x_i. There is a geodesic $\tilde{\zeta}$ connecting z to c that passes through z'. By the Bounded Geodesic Image Theorem, $\tilde{\zeta}$ must intersect $B_1(a)$. Since $z' \in S_{r-1}(c)$, z' must in fact belong to $B_1(a)$. Because $\Sigma_{0,5}$ has no quadrilaterals, z and z' must coincide. This verifies property (3).

To verify property (4), assume a has unique backtracking and $x_i \in S_r(c)$. Suppose for the sake of contradiction that s is a sidestep of x_i. We note that s is not adjacent to a because otherwise x_i, s, a, z would form a quadrilateral, a contradiction. s is also not equal to a, since otherwise x_i, a, z form a triangle, a contradiction.

Let z be the unique neighbor of x_i and a constructed during the verification of property (3). During the verification of property (3), we proved that $d(y_i, a) \geq 2$. So by Lemma 3.17, $d_a(x_i, c) \gg M$. Additionally, since $d(x_i, s) = 1$, by the coarse-Lipschitz property of d_a, we have $d_a(x_i, s)$ is bounded. So by the triangle inequality, $d_a(s, c) \gg M$. By Theorem 2.1, we know that every geodesic from s to c passes through $B_1(a)$.

Let η be one such geodesic. Since $s \in S_r$ and s is not adjacent or equal to a, we have $\eta \cap B_1(a) \subset S_{r-1}$. But since a has unique backtracking, the only vertex in $B_1(a) \cap S_{r-1}$ is z. This shows that η must pass through z. But then s, z, x_i form a triangle, a contradiction. This proves property (4). $\square$

3.7. **Proving Theorem 1.1.** Before we begin the proof of Theorem 1.1, we will need to make use of the following lemmas.

Lemma 3.22. Suppose $a \in S_r$ has unique backtracking and $b, b' \in S_{r+1}$ are both adjacent to a. Then there exists a path from b to b' entirely in $(S_{r+1} \cup S_{r+2}) \cap B_4(a)$.

Proof. Consider the path from b to b' given by Lemma 3.20. Each vertex on this path that lies in S_r is forward facing and also in $B_2(a)$. Forward facing vertices have no side stepping, so this path has no adjacent vertices in S_r. Thus we can apply Lemma 3.10 to each vertex in S_r to obtain the appropriate path in $(S_{r+1} \cup S_{r+2}) \cap B_4(a)$. □

Lemma 3.23. Suppose $a \in S_r$ and $b, b' \in S_{r+1}$ are both adjacent to a. Then there exists a path from b to b' entirely in $(S_{r+1} \cup S_{r+2}) \cap B_6(a)$.

Proof. Lemma 3.20 gives a path from b to b' in $(S_r \cup S_{r+1} \cup S_{r+2}) \cap B_3(a)$ such that each vertex on this path that lies in S_r has unique backtracking and is in $B_2(a)$. By Lemma 3.5 we can modify the path at each pair of adjacent vertices that lie in S_r to obtain a new path in $(S_r \cup S_{r+1} \cup S_{r+2}) \cap B_4(a)$ with the additional assumption that no two adjacent vertices are in S_r. Now we can apply Lemma 3.22 to each vertex in S_r to obtain the appropriate path in $(S_{r+1} \cup S_{r+2}) \cap B_6(a)$. □

Next, we want to recall [Wri23, Lemma 2.1] for the sufficient conditions for connectivity of spheres:

Lemma 3.24. [Wri23, Lemma 2.1] Let Γ be an arbitrary graph and fix $c \in \Gamma$. Fix $w > 0$, and let $r > 0$ be arbitrary. Suppose the following conditions hold:

(1) For every $z \in S_r(c)$ and $x, y \in S_{r+1}(c) \cap B_1(z)$ there exists a path
$$x = x_0, x_1, \ldots, x_l = y$$
with
$$x_i \in S_{r+1}(c) \cup \cdots \cup S_{r+w}(c)$$
for $0 \leq i \leq l$.

(2) For every adjacent pair $x, y \in S_r(c)$ there exists a path
$$x = x_0, x_1, \ldots, x_l = y$$
with
$$x_i \in S_{r+1}(c) \cup \cdots \cup S_{r+w}(c)$$
for $0 < i < l$.

Then $S_r(c) \cup S_{r+1}(c) \cup \cdots \cup S_{r+w-1}(c)$ is connected.

Lemma 3.25. In $\mathcal{C}\Sigma_{0,5}$, for all $r \geq 0$ the following hold:

(1) For every $z \in S_r$ and $x, y \in S_{r+1} \cap B_1(z)$ there exists a path
$$x = x_0, \ldots, x_l = y$$
with
$$x_i \in (S_{r+1} \cup S_{r+2}) \cap B_6(z)$$
for $0 \leq i \leq l$.

(2) For every adjacent pair $x, y \in S_r$ there exists a path
$$x = x_0, x_1, x_2, x_3, x_4 = y$$
with
$$x_i \in S_{r+1} \cup S_{r+2}$$
for $0 < i < 4$.

Proof. The first claim is Lemma 3.23 and the second claim is Lemma 3.5. □

Proof of Theorem 1.1. Since the curve graphs $\mathcal{C}\Sigma_{0,5}$ and $\mathcal{C}\Sigma_{1,2}$ for the low complexity surfaces are isomorphic, it suffices to prove Theorem 1.1 for $\mathcal{C}\Sigma_{0,5}$. The result follows immediately from combining Lemma 3.24 and Lemma 3.25. □

4. Medium complexity

Throughout this section we assume Σ is medium complexity. Again we fix a center vertex c and let $S_r = S_r(c)$. In this section we upgrade the results from [Wri23, Theorem 1.1] to prove Theorem 1.2: S_r is connected for medium complexity surfaces.

4.1. **Organization.**

We use [Wri23, Theorem 1.1 (2)] that $S_r \cup S_{r+1}$ is connected and begin with a path in $S_r \cup S_{r+1}$. Then we use the definition $\mathcal{O}(z)$, introduced by Wright, as a tool to push the path into S_{r+1} by allowing the path to contain vertices which need not be essentially non-separating.

4.2. **Essentially non-separating curves.**

Definition 4.1. A curve on Σ is called a *pants curve* if it bounds a genus 0 subsurface with 2 punctures.

Definition 4.2. A curve on $\mathcal{C}\Sigma$ is *essentially non-separating* if it is non-separating or a pants curve. A two-component multi-curve $\alpha \cup \beta$ is *essentially non-separating* if α and β themselves are essentially non-separating, and either

(1) $\alpha \cup \beta$ is non-separating,
(2) at least one of α or β is a pants curve, or
(3) $\alpha \cup \beta$ bounds a genus 0 subsurface with 1 puncture.

For $c \in \mathcal{C}\Sigma$, we can define $\mathcal{C}_c\Sigma$ as the subgraph of $\mathcal{C}\Sigma$ whose vertex set is $\{c\}$ union all essentially non-separating curves on $\mathcal{C}_c\Sigma$. Disjoint curves α and β are joined by an edge if either $\alpha \cup \beta$ is essentially non-separating or they have different distances to c.

To fix notation, let $S_r^c = S_r \cap \mathcal{C}_c\Sigma$.

Remark 4.3. [Wri23, Lemma 5.2] Wright showed that S_r^c coincides with the sphere of radius r in $\mathcal{C}_c\Sigma$.

We now recall the following results:

Lemma 4.4. $S_r^c \cup S_{r+1}^c$ is connected.

Proof. [Wri23, Proposition 5.4] verifies that the sufficient conditions for the connectivity of spheres in Lemma 3.24 hold in $\mathcal{C}_c\Sigma$ with $w = 2$. $\square$

Lemma 4.5. [Wri23, Lemma 5.3] Suppose Σ has medium complexity. For all $x \in S_r$, then either $x \in S_r^c$ or there exists $x' \in S_r^c \cap S_1(x)$.

4.3. **Definition and properties of $\mathcal{O}(z)$.** In order to prove Theorem 1.2, we make use of the following definition and prove several of its properties.

Definition 4.6. For any $z \in S_r^c$, define

$$\mathcal{O}(z) = \{a \in S_1(z) \cap \mathcal{C}_c\Sigma : d_U(a, c) > M\}$$

where U is the unique component of $\Sigma - z$ that is not a pants. Observe that $\mathcal{O}(z) \subseteq S_{r+1}^c$.

Recalling [Wri23, Lemma 7.2], we know we can connect any essentially non-separating curve to $\mathcal{O}(z)$:

Lemma 4.7. [Wri23, Lemma 7.2] Let $z \in S_r^c$ and $U = \Sigma - z$. Then for all $N > 0$, any $x \in S_1(z) \cap S_{r+1}^c$ can be connected to some $e \in \mathcal{O}(z)$ by a path in $S_1(z) \cap S_{r+1}^c$. Moreover, e can be taken such that $d_U(e, c) > N$.

Additionally, we will make use of the following lemma:

Lemma 4.8. Let $z \in S_r^c$ and $a, b \in \mathcal{O}(z)$. Then a, b can be connected by a path contained entirely in S_{r+1}.

Proof. Let $U = \Sigma - z$ be the unique connected component that is not pants. Observe that the subsurface projection $\rho_U(c)$ is a finite set with diameter bounded by some constant k (see section 2). Thus there exists $c' \in \rho_U(c)$ such that $d_{\mathcal{C}\Sigma}(c, c') \leq k$. Since $a, b \in \mathcal{O}(z)$, both $d_U(a, c), d_U(b, c) \geq M + 1$, so by the triangle inequality,

$$a, b \in \bigcup_{r'=M+1+k}^{\infty} S_{r'}(c'), \tag{7}$$

where each $S_{r'}(c')$ is a sphere in $\mathcal{C}U$. This union is a subgraph of $\mathcal{C}U$. It is connected because U is low complexity, and so Theorem 1.1 gives that $S_{M+1+k}(c') \cup S_{M+2+k}(c')$ is connected. Thus we can find a path in $\mathcal{C}U$

$$a = p_0, \ldots, p_l = b$$

such that each $d_U(p_i, c') \geq M + 1 + k$. Then by the triangle inequality, $d_U(p_i, c) > M$.

Applying Theorem 2.1 for all $0 < i < l$, every geodesic from p_i to c must go through z, as this is the only vertex not cutting U because z is essentially non-separating. For all i, p_i lies on U, so $d(z, p_i) = 1$. Since $d(z, c) = r$ by assumption, $d(p_i, c) = r + 1$ for all i, as desired. □

4.4. **Proving Theorem 1.2.**

Proof of Theorem 1.2. Suppose $x, y \in S_{r+1}$ are arbitrary. By Lemma 4.5 we can connect x, y to $x', y' \in S^c_{r+1}$ respectively, so it suffices to find a path connecting x', y' inside S_{r+1}. By Lemma 4.4, $S^c_r \cup S^c_{r+1}$ is connected, so there exists a path $x' = x_0, x_1, ..., x_k = y'$ contained in $S^c_r \cup S^c_{r+1}$.

We now make use of a sublemma:

Sublemma 4.9. The path from x' to y' above can be taken to have no two consecutive vertices in S^c_r.

Proof. This follows from [Wri23, Lemma 5.4, part (2)] that for each $x_i, x_{i+1} \in S^c_r$, there exists a path $x_i = x_i^0, x_i^1, x_i^2 = x_{i+1}$ such that $x_i^1 \in S^c_{r+1}$. □

By Sublemma 4.9, for each vertex x_i in the path from x' to y', if $x_i \in S^c_r$, then both x_{i-1} and x_{i+1} must be in S^c_{r+1}. In particular, since x_{i-1}, x_i, x_{i+1} is a path, we have $x_{i-1}, x_{i+1} \in S_1(x_i) \cap S^c_{r+1}$.

Now applying Lemma 4.7, to x_i, there exists x'_{i-1} and x'_{i+1} in $\mathcal{O}(x_i)$ which can be connected to x_{i-1} and x_{i+1} respectively with paths contained in $S_1(x_i) \cap S^c_{r+1}$ such that $d_U(x'_{i-1}, c) \gg M$ and $d_U(x'_{i+1}, c) \gg M$.

Applying Lemma 4.8, we can connect x'_{i-1} and x'_{i+1} by a path entirely in S_{r+1}. Thus, for consecutive vertices x_{i-1}, x_i, x_{i+1} in the path from x' to y' where $x_{i-1}, x_{i+1} \in S^c_{r+1}$ and $x_i \in S^c_r$, we can remove x_i and connect x_{i-1} to x_{i+1} by a path contained in S_{r+1}. By Sublemma 4.9 no two consecutive vertices in the path were in S^c_r, so this construction eliminates all vertices in S_r and results in a path from x' to y' contained in S_{r+1} as desired. □

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