A computability-theoretic characterization of [0,1] up to homeomorphism among Polish spaces

Eissa Haydar
REU Mentor: Matthew Harrison-Trainor*

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1 Preliminaries

Dana Scott [Sco65] proved in 1965 that, by strengthening the language of logic with countably infinite conjunctions and disjunctions (infinitary logic, or, in particular, $\mathcal{L}_{\omega_1\omega}$), countable structures could be described up to isomorphism via what are now called Scott sentences. Good introductions to this work and contemporary results can be found in [HT22] and [Mon]. The goal of this paper is to work on extending these methods to continuous topological spaces up to homeomorphism, in particular using infinitary logic coupled with a notion of distance. Some work already done here can be found in [FKFN] and [HTMN20]. Our goal, in particular, will be characterizing the unit interval $[0,1]$ among Polish, or separable completely metrizable, topological spaces. For now, we stipulate only the relevant definitions and notations.

Notation 1.1. We denote a disjunction over a set of countable propositions P with $\bigvee\!\!\!\bigvee_{\varphi\in P}\varphi$ and a conjunction over the same set with $\bigwedge\!\!\!\bigwedge_{\varphi\in P}\varphi$. These are called *countable disjunctions* and *countable conjunctions*, respectively.

Definition 1.2. A *computable presentation* of a Polish space L is a countable metric space (X,d) with d computable on X and $\overline{X} = L$. This gives us that every point x in L has a sequence $(x_n)_{n\in\mathbb{N}}$ in X which converges to x. Points in X are *special points*, and points in $L - X$ are *non-special points*.

Definition 1.3. We will use the term *continuous Scott sentence* of a Polish space L to mean a sentence φ for which, given a separable presentation X of a Polish space, $\overline{X}$, $X \vDash \varphi \Leftrightarrow \overline{X} \cong L$.

Definition 1.4. It follows that the *continuous Scott complexity* of a Polish space L will be the least complexity of a continuous Scott sentence for L.

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Now we must define complexity, and we will use the same notion present in infinitary logic, where: Σ_0 and Π_0 are the complexity of quantifier free formulas with no countable disjunctions or conjunctions. For an ordinal α, Σ_α is the complexity of formulas which are themselves countable disjunctions of formulas of the form $\exists x\varphi$, where each φ is Π_β for $\beta < \alpha$. Likewise, Π_α is the complexity of formulas which are themselves countable conjunctions of formulas of the form $\forall x\varphi$, where each φ is Σ_β for $\beta < \alpha$

2 An upper-bounding continuous Scott sentence:

2.1 Axioms

By [Kur68] (see §47 V. Theorem 1), we have that any space L satisfying the following axioms is homeomorphic to a the unit interval:

A1. L is nonempty.

A2. L is compact.

A3. L is connected.

A4. L is a metric space.

A5. L has exactly two non-cut points.

For A1, will stipulate that our presentation, X, is infinite, via the following sentence, where $\neq(x_1,\dots,x_n)$ is shorthand for x_1 through x_n being pairwise unequal:

$$\mathrm{INF} := \bigvee\!\!\!\bigvee_{n\in\mathbb{N}} \exists x_1,\dots,x_n \neq(x_1,\dots,x_n)$$

Now we state the metric axioms in order to meet A4:

$$\begin{aligned}\mathrm{MET} := \forall x\forall y\forall z(x \neq y \longrightarrow d(x,y) > 0) \wedge (x = y \longrightarrow d(x,y) = 0)\\ \wedge\ d(x,y) = d(y,x) \wedge d(x,z) \leq d(x,y) + d(y,z))\end{aligned}$$

Since L is Polish, L satisfies MET for some d. Due to [FKFN], compactness, equivalent to totally-boundedness in metric spaces, is:

$$\mathrm{CPCT} := \bigwedge\!\!\!\bigwedge_{q\in\mathbb{Q}^+} \bigvee\!\!\!\bigvee_{n\in\mathbb{N}} \exists x_0\cdots x_{n-1}\forall y \bigvee_{i<n} d(x_i) < q$$

And we meet A2. Given compactness, A3 is equivalent to the following sentence:

$$\begin{aligned}\mathrm{CON} := \bigwedge\!\!\!\bigwedge_{\varepsilon\in\mathbb{Q}^+} \bigwedge\!\!\!\bigwedge_{m,n\in\mathbb{N}} \forall x_1,\dots,x_m,y_1,\dots,y_n\exists z\\ z \in B_\varepsilon(x_1)\cup\cdots\cup B_\varepsilon(x_m)\cup B_\varepsilon(y_1)\cup\cdots\cup B_\varepsilon(y_n) \longrightarrow \bigvee_{i<m,j<n} z \in B_{2\varepsilon}(x_i)\cap B_{2\varepsilon}(y_j)\end{aligned}$$

This states that L cannot be covered by a finite union of balls that are themselves disjoint; specifically, we cannot label the covering with as and bs such that no a intersects a b. We use

2ε in the consequent of the implication since, for $L = \overline{X}$, an ε-cover of X has the property that those same balls must cover $\overline{X}$ when enlarged sufficiently, and doubling their sizes guarantees this. Compactness is what allows us to only consider finite unions; any infinite disjoint cover $\mathcal{U}$ of L has at least one finite sub-cover, and each finite sub-cover of $\mathcal{U}$ is also disjoint. CON precludes the possibility of a disjoint finite cover, thus preventing any disjoint cover given CPCT. This leaves us with A5.

2.2 Local Connectedness

We begin our stipulation of A5 with an expression of weak local connectedness via the following sentence:

$$
\begin{aligned}
&\text{LOCC} := \bigwedge\!\!\!\bigwedge_{r\in\mathbb{Q}^+} \forall c \bigvee\!\!\!\bigvee_{n\in\mathbb{N}} \exists x_1,\dots,x_n \bigvee\!\!\!\bigvee_{\varepsilon\in\mathbb{Q}^+} \\
&(\forall y(d(y,c) < r \longrightarrow (\bigvee_i \bigvee\!\!\!\bigvee_{m\in\mathbb{N}} \exists a_1 = y, a_2, \dots, a_n = x_i(\bigwedge_j d(a_j,a_{j+1}) < \varepsilon \wedge \bigwedge_j d(a_j,c) < 2r))) \\
&\wedge \bigwedge_{i,j:i\neq j} (\bigwedge\!\!\!\bigwedge_{m\in\mathbb{N}} \forall a_1,\dots,a_m(a_1 \neq x_i \vee a_m \neq x_j \vee \bigvee_k d(a_k,a_{k+1} > \varepsilon) \vee \bigvee_k d(a_k,b) > 2r)) \\
&\wedge \bigwedge_i \bigwedge\!\!\!\bigwedge_{\varepsilon'\in\mathbb{Q}^+} \forall y(d(y,c) < r \longrightarrow (\bigvee\!\!\!\bigvee_{m\in\mathbb{N}} \exists a_1 = y, a_2,\dots,a_m = x_i(\bigwedge_j d(a_j,a_{j+1}) < \varepsilon \wedge \bigwedge_j d(a_j,c) < 2r)) \\
&\qquad\qquad \longrightarrow (\bigvee\!\!\!\bigvee_{l\in\mathbb{N}} \exists b_1 = y, b_2,\dots,b_l = x_i(\bigwedge_j d(a_j,a_{j+1}) < \varepsilon' \wedge \bigwedge_j d(b_j,c) < 2r))))
\end{aligned}
$$

Less formally, the sentence states the following property, quantifying over special points:

LOCC := For all basic open sets $A = B_r(c)$ and $B = B_{2r}(c)$ there are $x_1,\dots,x_n \in A$ and $\varepsilon > 0$ such that:

1. for all $y \in A$, there is some i such that there is an ε-path from y to x_i in B.
2. there is no ε-path from x_i to x_j for distinct i, j, in B.
3. for all i and for all $y \in A$ and $\varepsilon' > 0$ if there is an ε-path from y to x_i in B then there is an ε'-path from y to x_i in B.

Lemma 2.1. *If $L = \overline{X}$ satisfies LOCC, then L is weakly locally connected and hence locally connected (See [Wil70], Theorem 27.16).*

Proof. Let x be any point, and U an open set containing x. We can choose basic open balls $x \in B_r(c) \subseteq \overline{B_r(c)} \subseteq B_{2r}(c) \subseteq \overline{B_{2r}(c)} \subseteq U$. Let $A = B_r(c)$ and $B = B_{2r}(c)$. We will find a connected neighbourhood D of x contained in $\overline{B_{2r}(c)}$. To witness that D is a neighbourhood, we will have $D \cap A$ open.

By LOCC, there are $x_1,\dots,x_n \in A$ and ε satisfying (1), (2), and (3). Define

$$D_i = \{y \in \overline{B} : \text{for all } \varepsilon' > 0 \text{ there is an } \varepsilon'\text{-path from } y \text{ to } x_i \text{ in } B\}.$$

Clearly $D_i \subseteq B$. We prove the following five claims.

Claim 1. *Each D_i is closed.*

Proof. Given $y \in \overline{B}$, suppose that y is a limit point of D_i. We argue that $y \in D_i$. Given $\varepsilon' > 0$, there is some point y' of D_i with $d(y, y') < \varepsilon'/2$. Then there is an $\varepsilon'/2$-path from y' to x_i in B, which, by replacing y' by y, is an ε'-path from y to x_i. □

Claim 2. *Each D_i is connected.*

Proof. Suppose that D_i is not connected. Since D_i is closed and compact, there are disjoint relatively clopen $U, V \subseteq D_i$ which partition D_i and points $u \in U$ and $v \in V$. There is some distance ε' between U and V. Then there is no ε'-path from u to v in B. □

Claim 3. *The D_i are pairwise disjoint.*

Proof. Suppose that there is some $y \in D_i \cap D_j$. Then there is an $\varepsilon/2$-path from y to x_i and an $\varepsilon/2$-path from y to x_j. Putting them together, and deleting y, we get an ε-path from x_i to x_j. This contradicts (2). □

Claim 4. *Each $D_i \cap A$ is open. Indeed, given $y \in A \cap D_i$ and $z \in A$ with $d(y, z) < \varepsilon/2$, $z \in D_i$.*

Proof. Choose some ε'. We wish to find a ε'-path from z to x_i, which would place $z \in D_i$. Since $y \in D_i$, we have an ε-path from y to x_i. Since the special points are dense, we may choose a path among them, other than the y, which may be non-special. Call the path-elements a_i, where $a_1 = y$ and $a_n = x_i$. We have that y is in $B_{\varepsilon/2}(z)$ and $B_\varepsilon(a_2)$, and since both of these are open, there is some ball around y in both of these as well. Choose in particular a special y' within these two and also within ε' of y. Choose also a z' with $d(y, z') < \varepsilon/2$ and $d(z, z') < \varepsilon'$. Then we have an ε-path $z', y', a_2, \ldots, a_n = x_i$ among the special points, which by (3) gives us an ε'-path from z' to x_i, which by our choice of z' gives us an ε'-path from z to x_i. Thus, $z \in D_i$.

Together with the fact that A is open, this shows that $D_i \cap A$ is open. □

Claim 5. *The D_i cover A.*

Proof. For any $y \in A$, choose some special y' with $d(y, y') < \varepsilon/2$. By the "Indeed" statement in the above claim, we have that $y' \in D_i \Rightarrow y \in D_i$ for any i. Since y' is special, (1) and (3) give us that $y \in D_i$ for some i. □

Now since the D_i cover A, $x \in D_i$ for some i. Then D_i is a connected neighbourhood of x contained in U. This completes the proof of the lemma. □

Lemma 2.2. *If $\overline{X} \cong [0,1]$ then X satisfies LOCC.*

Proof. Firstly, take basic open sets with $A = B_r(c)$ and $B = B_{2r}(c)$ for some c and r. We want to show that there is a finite set of x_i together with an ε that satisfy (1)-(3) of LOCC.

Let $C_1, C_2, \ldots$ be the connected components of B, which are open sets because $\overline{X}$ is locally connected. Define an equivalence relation $\sim$ on the C_i saying that $C_i \sim C_j$ if for every $\varepsilon' > 0$ and $x \in C_i$ and $y \in C_j$, there is an ε'-path from x to y in B. For each equivalence class, define an open set D which is the union of the C_i in that equivalence class. Let $D_1, D_2, \ldots$ be these sets.

For each i, $\overline{D_i}$ is connected. This is because D_i is a union of intervals in $\overline{X}$, and by construction the closure is nowhere dense. So each $\overline{D_i}$ is a closed interval in $\overline{X}$.

Each pair of distinct $\overline{D_i}$ and $\overline{D_j}$ are disjoint. Otherwise, they would have shared a limit point and would then have been of a distance less than any ε, contradicting that $i \neq j$.

The D_i are an open cover of B, and hence of the compact set $\overline{A}$. So there are finitely many many of them, say $D_1, \ldots, D_n$, which cover $\overline{A}$.

For each i, j, choose $\varepsilon_{i,j} > 0$ such that there is no $\varepsilon_{i,j}$-path from any point $x \in D_i$ to any point $y \in D_j$. Let $\varepsilon = \min_{i,j} \varepsilon_{i,j}$, and choose a special point x_i in each D_i.

Claim 1. *ε and $x_1, \ldots, x_n$ satisfy (1).*

Proof. Since the D_i cover $\overline{A}$, they cover A, and so since $y \in A$, $y \in D_i$ for some i. By the construction of the D_i and since $x_i \in D_i$, this means that there is an ε'-path from y to x_i for all $\varepsilon' > 0$, including ε. □

Claim 2. *ε and $x_1, \ldots, x_n$ satisfy (2).*

Proof. Suppose that the converse of (2) was true. Then for some $i \neq j$, there would be an ε-path from x_i to x_j in B. But this contradicts the choice of ε in the construction above. □

Claim 3. *ε and $x_1, \ldots, x_n$ satisfy (3).*

Proof. This follows from the construction of the D_i as the union of connected components which have ε'-paths between any two elements for all ε'. By the choice of ε in the above construction, y having an ε-path to some x_i places y in D_i, which then, for any ε', has a ε'-path to any other point within D_i, including x_i. □

And the lemma is proven. □

Now, by [Eng89] 6.3.11, LOCC, together with INF, MET, CPCT, and CON, gives us path-connectedness and local path-connectedness for L.

2.3 Betweenness

Now we propose the following sentence as a notion of betweenness for special points:

$$\begin{aligned} B(x,y,z) := \bigwedge_{\delta \in \mathbb{Q}} \bigvee_{\delta' < \delta} \bigvee_{\varepsilon < \delta'} \bigwedge_{n \in \mathbb{N}} & \forall a_1 = x, a_2, \ldots, a_n = z \\ & \bigwedge_i d(a_i, a_{i+1}) < \varepsilon \longrightarrow \bigvee_i d(y, a_i) < \delta' \\ & \wedge x \neq y \wedge x \neq z \wedge y \neq z \end{aligned}$$

In words, this sentence states that for any three distinct special x, y, and z, and any arbitrary ball size δ, there is $\delta' < \delta$ and ε such that any ε-path from sufficiently close to x to sufficiently close to z must intersect with the δ'-ball around y.

Now we state what will be part of our continuous Scott sentence, which is that for any three distinct special points, one is between the other two:

$$\mathrm{BTW} := \forall x \forall y \forall z ((x \neq y \wedge y \neq z \wedge x \neq z) \longrightarrow (B(x,y,z) \vee B(y,z,x) \vee B(z,x,y))$$

We will assume from here on that our space satisfies INF, MET, CPCT, CON, LOCC, and BTW.

Lemma 2.3. *If $\overline{X} \cong [0,1]$ then X satisfies BTW.*

Proof. Take distinct $x, y, z \in X$. Since $\overline{X} \cong [0,1]$, we have that $\overline{X}$ is an arc, and any arc between two elements in $\overline{X}$ is a sub-arc and thus unique, up to its image. Then, relabel x, y, and z so that the sub-arc in $\overline{X}$ beginning at x and ending at z contains y. We wish to show that for all rational δ, there exists $\delta' < \delta$ and $\varepsilon < \delta'$ such that any ε-path from x to z goes within δ' of y. Suppose that this is not the case; i.e. for some δ, for all $\delta' < \delta$ and $\varepsilon < \delta'$, there is some ε-path in X which avoids y. This gives us that no matter how finely we approximate the arc from x to z, we need not go through y. Since the arc is unique and since X is dense in $\overline{X}$, we must not go through y if we need not go through it for arbitrarily small approximations, which is a contradiction. □

2.4 A5

Finally, we will show that any two points have a unique arc between them, and using this, along with INF, MET, CPCT, CON, LOCC, and BTW, we will demostrate A5.

Lemma 2.4. *Suppose that there is an arc from x to z that does not pass through y, where x, y, and z are special. Then $\neg B(x, y, z)$*

Proof. Let f be the arc from x to z and suppose that y is not on this arc. There is some distance d between y and f. For sufficiently small ε, a ε-path from x to z approximating f avoids $B_{d/2}(y)$. Thus $\neg B(x, y, z)$. □

Lemma 2.5. *For $x, y \in L$, there is a unique arc from x to y, thinking of an arc as its image, rather than by parametrization.*

Proof. The existence of such an arc follows from path-connectedness (which is equivalent to arc-connectedness in Hausdorff spaces). To prove uniqueness, suppose we have two distinct arcs, f and g, from x to y. Take a point z on f that is not on g. Take a path-connected set around z that is disjoint from g, say U, and take a special point $z' \in U$. Then there is an arc h from z to z' within U, and thus disjoint from g. We may assume by moving z' and shortening h that z is the only point on both h and f. Break up f at the point z into two segments f_1 from x to z and f_2 from z to y. Choose also special points x' and y' close to x and y, respectively, with arcs from x to x' near x and from y to y' near y.

Thus we have arcs containing x and y but not z, y and z but not x, and x and z but not y. By Lemma 2.4 we have that $\neg B(y', z', x')$, $\neg B(x', y', z')$, and $\neg B(z', x', y')$. This contradicts B6 and proves the lemma. □

Lemma 2.6. *There are at most two non-cut points.*

Proof. Suppose that there were three non-cut points a, b, c. Without loss of generality, we may assume that the arc f from a to b does not contain c. (If it did, then take the shorter arc from a to c and swap b and c.) We may also assume that the arc g from b to c does not contain a, as if it did, then we could take the arc from a to c and swap a and b.

Now we argue that b is in fact a cut point. Suppose not; then $X - \{b\}$ is connected, and since it is still locally path connected, it is path connected. Thus there is an arc from a to c avoiding b, a contradiction. □

It is a theorem of R. L. Moore (see [Kur68] §47 IV. Theorem 5) that there are at least two non-cut points. Thus, we have exactly two non-cut points, and we meet A5.

2.5 Sentence

Theorem 2.7. *The* Π_4 *sentence* $\varphi := INF \wedge MET \wedge CPCT \wedge CON \wedge LOCC \wedge BTW$ *is a continuous Scott sentence for [0,1].*

Proof. We already have that $\mathcal{M} \models \varphi \Rightarrow \mathcal{M} \cong [0,1]$ by meeting A1 through A5. All that remains to be shown is that $\mathcal{M} \cong [0,1] \Rightarrow \mathcal{M} \models \varphi$. This is immediate for INF, MET, CPCT, and CON, and we have already proven it for LOCC and BTW. □

This gives us that [0, 1] is Π_4 in complexity, making Π_4 an upper bound for the continuous Scott complexity of [0, 1].

3 A lower-bounding reduction:

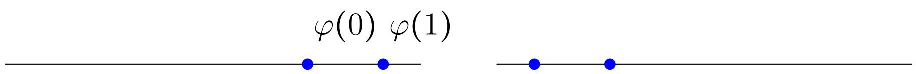
Note that this section uses a number of computability-theoretic notions, the most important of which being the following: there are only countably many partial computable programs, and the partial functions they compute may thus be numbered using $\mathbb{N}$. Thus, fixing some listing and numbering, $\varphi_{e,s}(x)$ refers to the eth function on input x computed to s stages.

We have that Coinf, the set of indices of partial computable functions whose domains are coinfinite (in particular, cocountable, since the set of possible values which may or may not be in the domain is ω), is Π_3-complete. Thus, a reduction from Coinf to [0, 1] would show its Π_3-hardness. In particular, we wish to describe a stage-by-stage construction which, given an index e, builds a structure homeomorphic to [0, 1] if, and only if, $e \in$ Coinf (equivalently, the structure is not homeomorphic to [0, 1] if, and only if, $e \in$ Cof, the index set of partial computable functions whose domains are cofinite). Throughout this construction, we will be placing only finitely-many points at each stage, but at each subsequent stage we will "fill in" the gaps between the points, thus letting us generalize our placing of points to placing compact sets, since that is what they will be at the limit.

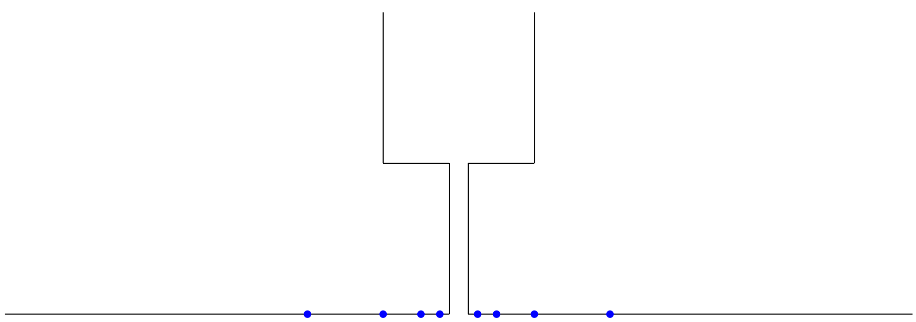
At each stage s, we will compute $\varphi_{e,s}(0), \ldots, \varphi_{e,s}(s)$. We will keep a "guess" going as to which computations will never halt. A computation enters the guess at some stage when it is not currently in the guess and is computed and does not halt. A computation leaves the guess at some stage if either it is computed and halts or if a computation which entered the guess prior to it and was still in the guess is computed to halt. Thus, when the guess realizes it made a mistake, it resets to what it was before that incorrectly-guessed computation entered. Elements which leave in the second way may re-enter if they have not halted by the next stage. In this manner, if $e \in$ Cof, we will have a correct guess, which is some finite number of elements which will never halt, which we will return to infinitely-often, and if $e \in$ Coinf, the guess will continue increasing in size arbitrarily as computations which will never halt enter for the last time, and this happens when all prior computations which will halt do so, and there is no finite value it will return to infinitely-often.

We begin by placing two inline line segments, as in the following:

For the sake of standardization, we make each one one unit, u, in length, with a distance of $1u$ between them. We will refer to these are the "main segments." We add to our construction at each stage depending on whether our guess reset at that stage or if computations are added to the guess at that stage. If computations enter the guess, we add line segments to the inside end of each main segment so as to decrease the distance between our main segments by a factor of one half for each computation that has entered at that stage. For example, suppose we guess that $\varphi_e(0)$ and $\varphi_e(1)$ will never enter, and that this takes place at the 1st stage. Then, our construction will look like the following (where the blue dots are not part of the construction, and only mark endpoints of the segments added due to our guess growing):



When our guess resets, we add line segments such that we move upward by half of the amount we moved upward last, beginning with $0.5u$, and we add line segments so as to move horizontally back to the end of the line segment added for the last element of the newly reset-to guess, and then move up the same amount again. These highest points are where will be our new main segments. For example, consider if we had guessed that $\varphi_e(0)$, $\varphi_e(1)$, $\varphi_e(2)$, and $\varphi_e(3)$, would not enter, and then $\varphi_e(1)$ enters, and the guess resets. The construction would look like the following:



In this manner, at the limit, the construction will either end up with a shape similar to the topologist's sine curve if $e \in \mathrm{Cof}$, or with a shape homeomorphic to [0, 1] if $e \in \mathrm{Coinf}$. This is because there will be some finite guess size to which we will return infinitely-many times, and this occuring to the upward limit will result in oscillations with length to the limit. If there is no finite guess we will return to infinitely-many times, because $e \in \mathrm{Cof}$, the upward limit will coincide with the two main segments intersecting at a limit point, and this will have no length (i.e. it will oscillate to a point similar to the tip of a V).

This gives us that [0, 1] is Π_3-hard, making Π_3 a lower bound for the continuous Scott complexity of [0, 1]. Thus:

Theorem 3.1. *The continuous Scott complexity of [0, 1] is* Π_3 *or* Π_4*.*

It is the belief of the authors that the continuous Scott complexity of [0, 1] is Π_4, and that a Π_4 reduction may be created which, given an index, produces a structure that is compact and connected in any case but is locally connected if, and only if, the index is in a chosen Π_4-complete set.

Conjecture 3.2. *The continuous Scott complexity of [0, 1] is* Π_4.

But this remains to be formally proven.

4 Future work

The scope of this paper is very narrow, and so the following are a few questions which would continue its endeavours:

Question 4.1. What is the continuous Scott complexity of the topologist's sine curve?

Question 4.2. What is the continuous Scott complexity of the real line?

Question 4.3. What is the continuous Scott complexity of $[0, 1] \times [0, 1]$?

Question 4.4. In general, what sorts of sentences are preserved up to homeomorphism? Would knowing this give us a method of generating continuous Scott sentences for Polish spaces?

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