Hardy-Littlewood Circle Method

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1 Introduction

Recent work of Browning & Heath-Brown explored the density of rational points on the biprojective hypersurface of bidegree (1, 2) in 8 variables cut out by the equation

$$x_1y_1^2 + x_2y_2^2 + x_3y_3^2 + x_4y_4^2 = 0$$

in $\mathbb{P}^3 \times \mathbb{P}^3$.

Specifically, they, in [1], proved a modified Manin conjecture for this Fano variety, where a removal of a "thin subset" of problematic points , which yields a greater density of rational points than predicted by the Manin conjecture, is allowed. Indeed, "points tend to accumulate on thin subsets which are images of non-trivial finite morphisms" (Peyre, [6]).

We wish to follow a similar line of reasoning for an infinite family of biprojective hypersurfaces of bidegree (1, k) in $\mathbb{P}^{s-1} \times \mathbb{P}^{s-1}$ cut out by equations of the form

$$x_1y_1^k + \cdots + x_sy_s^k = 0,$$

and prove the modified Manin conjecture for all $s \geq 2^k + 1$.

This would extend a result of Hu ([5]), which established the conjecture for biprojective hypersurfaces of bidegree (1, 2) for $s \geq 26$, to $s \geq 5$ for diagonal equations, of the type investigated for $s = 4$ by [1], and to biprojective hypersurfaces of bidegree (1, k) given by diagonal equations for all integers $k > 2$, holding for $s \geq 2^k + 1$.

Working towards a resolution of a modified Manin conjecture for these varieties, we apply the Hardy-Littlewood circle method to establish an asymptotic formula for integer solutions to these forms, and establish a result for $G(k)$.

We must note that the circle method produces answers for a high number of variables in comparison to the exponent, so that, in general, $s \gg k$. We establish specific functions of k, which we call $s_0(k)$ and $s_1(k)$, for the convergence and positivity of the singular series, but our asymptotic formula holds when $s > 2^k$.

To establish notation, in this paper we say that

$$F(x) \ll G(x), \text{ or } G(x) \gg F(x)$$

if there are constants $c \in \mathbb{R}_{>0}$ and $x_0 \in \mathbb{R}$ such that

$$|F(x)| \leq cG(x)$$

for all $x \geq x_0$.

We divide this paper into 3 sections. In section 1 we gave basic introduction and prove an asymptotic for our Waring's problem. In section 2, we analyzed the convergence criteria for the singular series. In Section 3 we prove a lower bound for the number of solutions under weaker conditions.

Part I

Define

$$T_j(\alpha) = \sum_{y=1}^{P} e(\alpha x_j y^k) = \sum_{y=1}^{P} e^{2\pi i \alpha x_j y^k}.$$

We wish to study the equation

$$x_1y_1^k + ... + x_sy_s^k = 0, \tag{1}$$

for $s > s_0(k)$, which is a certain function of k, and $y_j \leq Q$ for $j \in \mathbb{N}$, where $j \leq P$. Following the Hardy-Littlewood circle method, we notice that the number of solutions of the above is precisely:

$$\int_0^1 T_1(\alpha)T_2(\alpha)...T_s(\alpha)d\alpha. \tag{2}$$

Since the x_js are in fact variants instead of constants in the equations, we wish to evaluate major and minor arcs, as well as their errors, in terms of the x_js.

Lemma 1.1. *(Weyl's Inequality) Let $f_j(x)$ be polynomial of degree k with top coefficient αx_j. Suppose that α has rational approximation a/q:*

$$(a, q) = 1, q > 0, \left|\alpha - \frac{a}{q}\right| \leq \frac{1}{q^2},$$

then for any $\epsilon > 0$, *we have*

$$\Big|\sum_{x=1}^{P} e(f(x))\Big| \ll P^{1-\frac{1}{K}} + P^{1+\epsilon}\big(x_j^{\frac{1}{K}}(q^{-\frac{1}{K}} + P^{-\frac{1}{K}}) + (\frac{P^k}{q})^{-\frac{1}{K}}\big), \tag{3}$$

where $K = 2^{k-1}$.

Proof. We seek to prove the case when $f(y) = \alpha x_j y^k$ Define

$$S_k(\alpha x_j y_j^k) = \sum_{y=1}^{P} e(\alpha x_j y^k).$$

The k subscript standing for the degree of the function evaluated inside. We notice from complex conjugation that

$$\begin{aligned}\big|S_k(\alpha x_j y_j^k)\big|^2 &= \sum_{y_1=1}^{P}\sum_{y_2=1}^{P} e(\alpha x_j(y_1^k - y_2^k)) \\ &= P + 2\sum_{y_1>y_2}^{1\le y_1,y_2\le P} Re(e(\alpha x_j(y_1^k - y_2^k))) \\ &= P + 2\sum_{y=1}^{P}\sum_{y_j} Re(e(\alpha x_j(\Delta_y(y_j^k)))),\end{aligned}$$

where

$$\Delta_y(y_j^k) = (y_j + y)^k - y_j^k,$$

and the summation is taking place over y_j's such that $y_j + y, y_j \in N$ and $y_j + y, y_j \le P$. One can thus replace the inside sum by $S_{k-1}(\Delta_y(\alpha x_j y_j^k))$, and note that

$$\big|S_k(\alpha x_j y_j^k)\big|^2 \le P + 2\sum_{y=1}^{P}\big|S_{k-1}(\Delta_y(\alpha x_j y_j^k))\big|.$$

In particular, one sees that, replacing k with $k-1$ and summations with appropriate intervals, we have

$$\big|S_{k-1}(\Delta_y(\alpha x_j y_j^k))\big|^2 \le P + 2\sum_{z=1}^{P}\big|S_{k-2}(\Delta_{y,z}(\alpha x_j y_j^k))\big|,$$

where

$$\Delta_{y,z}(\alpha x_j y_j^k) = \Delta_y(\alpha x_j(y_j + z)^k) - \Delta_y(\alpha x_j y_j^k).$$

In particular, we see that this is polynomial of degree $k-2$ in y_j. Combined, we have the following:

$$\begin{aligned}\big|S_k(\alpha x_j y_j^k)\big|^4 &\le \big[P + 2\sum_{y=1}^{P}\big|S_{k-1}(\Delta_y(\alpha x_j y_j^k))\big|\big]^2 \\ &\le P^2 + 4P\sum_{y=1}^{P}\big|S_{k-1}(\Delta_y(\alpha x_j y_j^k))\big| + 4\big[\sum_{y=1}^{P}\big|S_{k-1}(\Delta_y(\alpha x_j y_j^k))\big|\big]^2 \\ &\le P^2 + 4P\sum_{y=1}^{P}\big|S_{k-1}(\Delta_y(\alpha x_j y_j^k))\big| + 4P\sum_{y=1}^{P}\big|S_{k-1}(\Delta_y(\alpha x_j y_j^k))\big|^2 \text{ (Cauchy-Schwartz Inequality)} \\ &\ll P^2 + P\sum_{y=1}^{P}\big|S_{k-1}(\Delta_y(\alpha x_j y_j^k))\big| \text{ (AM-GM Inequality)} \\ &\ll P^3 + P\sum_{y=1}^{P}\sum_{z=1}^{P}\big|S_{k-2}(\Delta_{y,z}(\alpha x_j y_j^k))\big|.\end{aligned}$$

One may thus follow through similar process and show that

$$\big|S_k(\alpha x_j y_j^k)\big|^{2^\upsilon} \ll P^{2^\upsilon - 1} + P^{2^\upsilon - \upsilon - 1}\sum_{y_1,y_2\ldots y_\upsilon=1}^{P}\big|S_{k-\upsilon}(\Delta_{y_1,y_2\ldots y_\upsilon}(\alpha x_j y_j^k))\big|.$$

Letting $\upsilon = k-1$, one sees that

$$S_{k-\upsilon}(\Delta_{y_1,y_2\ldots y_\upsilon}(\alpha x_j y_j^k)) = k!\alpha y_1\ldots y_\upsilon x_j y_j + \beta,$$

where β is a constant. Rearranging, we see that

$$\left|S_{k-v}(\Delta_{y_1,y_2...y_v}(\alpha x_j y_j^k))\right| = \sum_x e(k!\alpha y_1...y_v x_j y_j).$$

Here we note that

$$\sum_{x=x_1}^{x_2-1} e(\lambda x) \ll \frac{1}{||\lambda||},$$

which gives us that, letting $v = k-1$ and $K = 2^{k-1}$,

$$\left|S_k(\alpha x_j y_j^k)\right|^K \ll P^{K-1} + P^{K-k+\epsilon} \sum_{m=1}^{x_j k! P^{k-1}} \min\left(P, ||\alpha m||^{-1}\right),$$

where in the last sum we have rearranged into all possible values for $k!x_j y_1...y_{k-1}$. Reputting the summations into blocks of sizes q each, we see that the number of blocks $\ll \frac{x_j y_1...y_{k-1}}{q} + 1$. The rest follows through the text by Davenport, in which it was proved that:

$$\sum_{r=0}^{q-1} \min\left(P, ||\alpha m||^{-1}\right) \ll P + q\log q.$$

Therefore we have that

$$\left|S_k(\alpha x_j y_j^k)\right|^K \ll P^{K-1} + P^{K+\epsilon}(\frac{x_j P^{-1}}{q} + P^{-k})(P + q\log q).$$

Taking K^{th} root gives the result. □

Lemma 1.2. *(Hua's inequality) Given any j, we have*

$$\int_0^1 |T_j(\alpha)|^{2^k} d\alpha \ll P^{2^k-k+\epsilon}.$$

Proof. Denoting integral on the left as I_k, we prove the theorem through inducting on k. The base case is trivial: if $k = 1$, we have

$$\int_0^1 \left|T_j(\alpha)\right|^2 d\alpha = \int_0^1 \sum_{y_j=1}^{P} e(\alpha x_j y_j^k) \sum_{y_j=1}^{P} e(-\alpha x_j y_j^k) = P \ll P^{2^1-1+\epsilon}.$$

Suppose case holds for $k = v$, we show that it also holds for $k = v+1$. In particular, we have from proof of the previous theorem that

$$\left|T_j(\alpha)\right|^{2^v} \ll P^{2^v-1} + P^{2^v-v-1} \sum_{z_1,z_2...=1}^{P} \Re\left|S_{k-v}(\Delta_{z_1,z_2...z_v}(\alpha x_j y_j^k))\right|,$$

where

$$S_{k-v}(\Delta_{z_1,z_2...z_v}(\alpha x_j y_j^k) = \sum_{y_j} e(\Delta_{z_1,z_2...z_v}(\alpha x_j y_j^k).$$

Summing over ranges of y_j for which $y_j + \sum_{i=1}^{n<v} z_i$ are contained in $[1, P]$. After multiplying $\left|T_j(\alpha)\right|^{2^v}$ on both sides and integrating with respect to α from 0 to 1, we obtain:

$$I_{v+1} \ll P^{2^v-1} I_v + P^{2^v-v-1} \sum_{z_1...z_v} \Re \int_0^1 S_{k-v}\left|T_j(\alpha)\right|^{2^v} d\alpha, \tag{4}$$

the last integral being

$$\int_0^1 \sum_{y_j} e(\Delta_{z_1,z_2...z_v}(\alpha x_j y_j^k) \sum_{\substack{u_1,u_2...u_{2^v-1}\\ v_1,v_2...v_{2^v-1}}} e(\alpha x_j(u_1^k + u_2^k + ...))e(-\alpha x_j(v_1^k + v_2^k + ...))d\alpha.$$

Hence the integral counts the number of solutions to

$$\Delta_{z_1...z_v}(x_j y_j^k) + x_j u_1^k + ... - x_j v_1^k = 0. \tag{5}$$

In particular, we note that the first term is either a strictly increasing or a strictly decreasing function of y_j. Furthermore it is divisible by $z_1, z_2...z_v$. Thus only one possible y_j works for each choice of parameters. From previous theorem, we have that the number of solutions to (5), denoted N, has

$$N \ll P^{2^v + v2^v\epsilon} = P^{2^v+\epsilon}.$$

Substituting back into (4) along with inductive hypothesis, we have

$$I_{v+1} \ll P^{2^v-1} P^{2^v-v+\epsilon} + P^{2^v-v-1} P^{2^v+\epsilon} = P^{2^{v+1}-(v+1)+\epsilon},$$

giving the proof. □

With these tools it is possible to evaluate the integral along the minor arcs.

Definition 1.3. *Define*

$$\mathfrak{M}_{a,q} = \left\{ \alpha \in [0,1] : \left|\alpha - \frac{a}{q}\right| < P^{-k+\delta} \right\},$$

for $a \le q$, $(a,q) = 1$ *and* $1 \le q \le P^{\delta}$ *for some small* δ. *Define*

$$\mathfrak{M} = \bigcup_{q \le P^{\delta}} \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} \mathfrak{M}_{a,q},$$

$$m = [0,1] \backslash \mathfrak{M}.$$

Here $\mathfrak{M}$ *is our major arc and* m *is the minor arc.*

Lemma 1.4. *If* $s > 2^k$, *then we have*

$$\int_m |T_1(\alpha) T_2(\alpha) ... T_s(\alpha)| d\alpha \ll P^{s-k-\delta'} \left(x_1 ... x_s\right)^{(s-2^k)/Ks},$$

where $T_j(\alpha)$ *are defined as before.*

Proof. By Holder's inequality, one has

$$\int_m |T_1(\alpha) T_2(\alpha) ... T_s(\alpha)| d\alpha \ll \Big(\int_m |T_1(\alpha)|^s d\alpha \Big)^{\frac{1}{s}} ... \Big(\int_m |T_s(\alpha)|^s d\alpha \Big)^{\frac{1}{s}},$$

so we simply need to evaluate one of these integrals. We note that by Dirichlet's approximation theorem, one can find a, q such that

$$1 \le q \le P^{k-\delta}, \ \left|\alpha - \frac{a}{q}\right| \le q^{-1} P^{-k+\delta}.$$

Hence if $\alpha \in m$, then necessarily we have $q > P^{\delta}$. In particular, we see that, since $\left|\alpha - \frac{a}{q}\right| \le q^{-2}$ and $\frac{P^k}{q} > P^{\delta}$, we can use Weyl's Inequality to conclude that for $\alpha \in m$,

$$|T_j(\alpha)| \ll P^{1+\epsilon-\delta/K} x_j^{1/K}.$$

Thus using Hua's Inequality, we have

$$\begin{aligned} \left(\int_m |T_j(\alpha)|^s d\alpha \right)^{1/s} &= \left(\int_m |T_j(\alpha)|^{s-2^k} d\alpha \right)^{1/s} \left(\int_m |T_1(\alpha)|^{2^k} d\alpha \right)^{1/s} \\ &\ll \left(\left(P^{1+\epsilon-\delta/K} x_j^{1/K}\right)^{s-2^k} P^{2^k-k+\epsilon} \right)^{1/s} \\ &= (P^{s-k-\delta'} x_j^{(s-2^k)/K})^{1/s}. \end{aligned}$$

Combining using Holder's inequality, we have finally the desired Holder's Inequality:

$$\int_m |T_1(\alpha) T_2(\alpha) ... T_s(\alpha)| d\alpha \ll P^{s-k-\delta'} \left(x_1 ... x_s\right)^{(s-2^k)/Ks}.$$

□

In fact this concludes the proof of theorem for the minor arcs. We now seek to evaluate the expression for the major arcs. The following lemma transforms the summation into easier forms to handle.

Lemma 1.5. *For* α *in* $\mathfrak{M}_{a,q}$, *let* $\alpha = \beta + a/q$, *then we have*

$$T_j(\alpha) = q^{-1} S_{x_j a,q} I_j(\beta) + O(P^{2\delta} x_j),$$

where

$$S_{x_j a,q} = \sum_{r=0}^{q-1} e\left(\frac{a x_j}{q} r^k\right), \ I_j(\beta) = \int_0^P e(x_j \beta u^k) du.$$

Proof. We see that

$$\begin{aligned} T_j(\alpha) &= \sum_{y=1}^{P} e(\alpha x_j y^k) \\ &= \sum_{r=0}^{q-1} e(\frac{a x_j}{q} r^k) \sum_b e(x_j \beta (bq+r)^k) \\ &= S_{x_j a,q} \sum_b e(x_j \beta (bq+r)^k), \end{aligned}$$

where the summation for b takes place such that $bq+r$ runs over $1, 2...P$. In particular, we now seek to replace the second sum by an integral as indicated, and then reevaluate the error terms. Note that

$$\begin{aligned} &\int_0^{\frac{P}{q}} e(x_j \beta (yq+r)^k) dy - \sum_{0 \le b < \frac{P}{q}} e(x_j \beta (bq+r)^k) \\ &= \sum_{j=0}^{\lfloor \frac{P}{q} \rfloor} \int_j^{j+1} e(x_j \beta (yq+r)^k) - e(x_j \beta (jq+r)^k) dy. \end{aligned}$$

We note that for $|y-j| \le \frac{1}{2}$, if f is continuously differentiable, then we have:

$$\left| f(y) - f(j) \right| \le \frac{1}{2} \max |f'(j)|,$$

for j in that region. In particular, after substitution we see that

$$\begin{aligned} &\int_0^{\frac{P}{q}} e(x_j \beta (yq+r)^k) dy - \sum_{0 \le b < \frac{P}{q}} e(x_j \beta (bq+r)^k) \\ &\ll \sum_{j=0}^{\lfloor \frac{P}{q} \rfloor} \frac{1}{2} \max_{y \in [j, j+1]} |f'(y)| \\ &\ll \max_{y \in [0, \frac{P}{q}]} |f'(y)| \lfloor \frac{P}{q} \rfloor. \end{aligned}$$

where $f(y) = e^{2\pi i x_j \beta (qy+r)^k}$ and therefore $|f'(y)| \ll \beta x_j q P^{k-1}$, therefore combining all terms we have

$$\int_0^{\frac{P}{q}} e(x_j \beta (yq+r)^k) dy - \sum_{0 \le b < \frac{P}{q}} e(x_j \beta (bq+r)^k) \ll P^\delta x_j,$$

since by construction, $\beta \le P^{-k+\delta}$. Multiplying from outside by $S_{x_j a,q}$ we obtain the error term $O(P^{2\delta} x_j)$. Finally a change of variables in the integral gives the result. □

With this it is enough to determine the value along the major arcs

Lemma 1.6.

$$\int_{\mathfrak{M}} T_1(\alpha)...T_s(\alpha)\, d\alpha = P^{s-k} \frac{C(P)}{(x_1...x_s)^{\frac{1}{k}}} \sum_{q \le P^\delta} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} q^{-s} S_{ax_1,q} ... S_{ax_s,q} + O(P^{s-k-\delta'} max(x_1...x_s)),$$

where

$$C(P) = \int_{\gamma \le P^\delta} \left(\int_0^1 e(\gamma u^k)\, du \right)^s d\gamma.$$

Proof. Note that

$$\begin{aligned} \int_{\mathfrak{M}} T_1(\alpha) \dots T_s(\alpha) d\alpha &= \sum_{q \le p^\delta} \sum_{\substack{a=1 \\ (a,q)=1}} \int_{\mathfrak{M}_{a,q}} T_1(\alpha) \dots T_s(\alpha) d\alpha \\ &= \sum_{q \le p^\delta} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} q^{-s} S_{x_1 a,q} \dots S_{x_s a,q} \int_{|\beta| < p^{-k+\delta}} I_1(\beta) \dots I_s(\beta) d\beta \end{aligned}$$

by the definition of the major arcs, where

$$I_j(\beta) = \int_0^P e(x_j \beta u^k)\, du.$$

Define I to be the right-most integral

$$I := \int_{|\beta|<p^{-k+\delta}} I_1(\beta)\dots I_s(\beta)\,d\beta.$$

Then we have that

$$\begin{aligned}
I &= \int_{|\beta|<p^{-k+\delta}} \prod_{j=1}^{s} \int_0^P e(x_j\beta u_j^k)\,du_j\,d\beta \\
&= \int_{|\beta|<P^{-k+\delta}} \prod_{j=1}^{s} \int_0^{x_j^{1/k}} e(\beta v_j^k P^k)\,dv_j \frac{P}{x_j^{1/k}}\,d\beta \\
&= \frac{P^s}{x_1^{1/k}\dots x_s^{1/k}} \int_{|\beta|<P^{-k+\delta}} \prod_{j=1}^{s} \int_0^{x_j^{1/k}} e(\beta v_j^k P^k)\,dv_j\,d\beta,
\end{aligned}$$

when we set $u_j = Pv_j$.

Now, setting $\gamma = \beta P^k$, we have

$$I = P^{s-k} \int_{|\gamma|<P^\delta} \prod_{j=1}^{s} \int_0^1 e(x_j\gamma v_j^k)\,dv_j\,d\gamma.$$

Now, we introduce another change of variables, and set $\zeta = v_j^k$, and our I now becomes

$$I = P^{s-k} \int_{|\gamma|<P^\delta} \prod_{j=1}^{s} \int_0^1 e(x_j\gamma\zeta)\,\frac{d\zeta\,d\gamma}{k\zeta^{(k-1)/k}}.$$

Another change of variables, setting $\mu = \gamma\zeta$, yields

$$\begin{aligned}
I &= P^{s-k} \int_{|\gamma|<P^\delta} \prod_{j=1}^{s} \int_0^\gamma e(x_j\mu)\,\frac{d\mu\,d\gamma}{\gamma k\mu^{(k-1)/k}\gamma^{(k-1)/k}} \\
&= P^{s-k} \int_{|\gamma|<P^\delta} \prod_{j=1}^{s} \{\gamma^{-\frac{1}{k}} k^{-1} \int_0^\gamma e(x_j\mu)^{-1+\frac{1}{k}}\,d\mu\}\,d\gamma,
\end{aligned}$$

and

$$\left| \int_{|\gamma|\ge P^\delta} \gamma^{-\frac{s}{k}} k^{-s} \prod_{j=1}^{s} \int_0^\gamma e(x_j\mu)\mu^{-1+\frac{1}{k}} \right| \ll p^{(-\frac{s}{k}+1)\delta}.$$

□

Finally! We have the desired result:

Theorem 1.7.

$$\int_0^1 T_1(\alpha)...T_s(\alpha)\,d\alpha = P^{s-k}\frac{C(P)}{(x_1...x_s)^{\frac{1}{k}}} \sum_{q\le P^\delta} \sum_{\substack{a=1\\(a,q)=1}}^{q} q^{-s} S_{ax_1,q}...S_{ax_s,q} + O(P^{s-k-\delta'}T), \tag{6}$$

where

$$\begin{aligned}
C(P) &= \int_{\gamma\le P^\delta} \left(\int_0^1 e(\gamma u^k)\,du\right)^s d\gamma, \text{ and} \\
T &= \max((x_1...x_s)^{(s-2^k)/sK}, \max(x_1,...,x_s)).
\end{aligned}$$

2 Part II

In this section, we focus on the double series

$$\mathfrak{S}_s := \sum_{q=1}^{\infty} \sum_{\substack{a=1\\(a,q)=1}}^{q} q^{-s} S_{ax_1,q}...S_{ax_s,q},$$

which we call the *singular series* in the tradition of Hardy & Littlewood, and the integral

$$C(P) = \int_{\gamma\le P^\delta} \left(\int_0^1 e(\gamma u^k)\,du\right)^s d\gamma.$$

Let us consider $C(P)$ first.

Note that if we take the inner integral, we can apply a change of variables $\zeta = u^k$, to yield, as in Davenport,

$$\int_0^1 e(\gamma u^k)\, du = k^{-1} \int_0^1 \zeta^{-1+\frac{1}{k}} e(\gamma\zeta)\, d\zeta$$
$$= k^{-1}\gamma^{-\frac{1}{k}} \int_0^\gamma \zeta^{-1+\frac{1}{k}} e(\zeta)\, d\zeta,$$

where γ in the last integral is positive.

The above is absolutely convergent at 0. By Dirichlet's test, we know that $k^{-1}\gamma^{-\frac{1}{k}} \int_0^\gamma \zeta^{-1+\frac{1}{k}} e(\zeta)\, d\zeta$ is a bounded function of γ, so we know that

$$\left| \int_0^1 e(\gamma u^k)\, du \right| \ll |\gamma|^{-\frac{1}{k}},$$

and so we can extend the integration over γ to infinity to obtain

$$C(P) = C + O(P^{-(\frac{s}{k}-1)\delta}),$$

where

$$C = \int_{-\infty}^{\infty} \left(k^{-1} \int_0^1 e(\gamma\zeta)\, d\zeta \right)^s d\gamma,$$

which we call the *singular integral.*

This treatment is identical to the one in Davenport.

It suffices for our purpose to just show that $C > 0$.

We do this as in Davenport, using Fourier's integral theorem.

Setting $\xi = \zeta_1 + \cdots + \zeta_s$, define

$$\varphi(\xi) = \int_0^1 \cdots \int_0^1 \{\zeta_1 \ldots \zeta_{s-1}(\xi - \zeta_1 - \cdots - \zeta_{s-1})\}^{-1+\frac{1}{k}} d\zeta_1 \ldots d\zeta_{s-1},$$

taken over values of $\zeta_1, \ldots, \zeta_{s-1}$ such that $\xi - 1 < \zeta_1 + \cdots + \zeta_{s-1} < \xi$.

The application of Fourier's integral theorem requires certain conditions to be met, and it suffices for $\varphi(\xi)$ to be of bounded variation.

To show this, let $\zeta_j = \xi t_j$, so that

$$\varphi(\xi) = \xi^{\frac{s}{k}-1} \int_0^{\frac{1}{\xi}} \cdots \int_0^{\frac{1}{\xi}} \{t_1 \ldots t_{s-1}(1 - t_1 - \cdots - t_{s-1})\}^{-1+\frac{1}{k}} dt_1 \ldots dt_{s-1},$$

taken over values of $t_1, \ldots, t_{s-1}$ such that $1 - \frac{1}{\xi} < t_1 + \cdots + t_{s-1} < 1$.

As ξ increases, the region of integration becomes smaller, and since the integrand does not involve ξ, we see that $\varphi(\xi)$ is a function of bounded variation, being a product of the power of ξ and a positive monotonic decreasing function of ξ trivially.

Applying Fourier's integral theorem for a finite interval, which says that for $A < B < D$, and certain conditions that we have already satisfied,

$$\lim_{\lambda\to\infty} \int_A^B \varphi(\xi) \frac{\sin(2\pi\lambda(\xi - D)}{\pi(\xi - D)}\, dD = \varphi(D).$$

Hence, in our case, we have

$$k^s C = \varphi(1) = \int_0^1 \cdots \int_0^1 \{\zeta_1 \ldots \zeta_{s-1}(1 - \zeta_1 - \cdots - \zeta_{s-1})\}^{-1+\frac{1}{k}} d\zeta_1 \ldots d\zeta_{s-1},$$

with the integral taken over $\zeta_1, \ldots, \zeta_{s-1}$ for which $0 < \zeta_1 + \cdots + \zeta_{s-1} < 1$.

As Davenport states, this integral was explicitly evaluated by Dirichlet to yield

$$C = \left(\frac{1}{k}\right)^s \frac{\Gamma(\frac{1}{k})^s}{\Gamma(\frac{s}{k})} = \frac{\Gamma(1 + \frac{1}{k})^s}{\Gamma(\frac{s}{k})},$$

which indeed tells us that $C > 0$.

We now shift our focus to the singular series $\mathfrak{S}_s$.

Recall that our asymptotic formula is

$$N(P) = P^{s-k} \frac{C(P)}{(x_1 \ldots x_s)^{\frac{1}{k}}} \sum_{q \le P^\delta} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} q^{-s} S_{ax_1,q} \ldots S_{ax_s,q} + O(P^{s-k-\delta' T}),$$

for $s > 2^k$.

To show that the main term is significant, we wish to work with the double sum

$$\mathfrak{S}_s := \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} q^{-s} S_{ax_1,q} ... S_{ax_s,q},$$

the singular series, and prove that it is positive, is absolutely convergent for $s > 2k$.

This singular series is related to the number of solutions to the congruence

$$x_1 y_1^k + \cdots + x_s y_s^k \equiv 0 \pmod{p^\nu}, \qquad 0 \le x < p^\nu.$$

We will show that $\mathfrak{S}_s$ is always positive, in a similar way to how Davenport does in Chapter 8 of his *Analytic Methods for Diophantine Equations and Diophantine Inequalities.*

Define

$$\chi(p) = 1 + \sum_{\nu=1}^{\infty} \sum_{\substack{a=1 \\ (a,p^\nu)=1}}^{p^\nu} p^{-\nu s} S_{ax_1,p^\nu} \dots S_{ax_s,p^\nu},$$

and consider the fact that

$$\mathfrak{S}_s = \prod_p \chi(p),$$

for $s \ge 2^k + 1$ since

$$\begin{aligned} \mathfrak{S}_s &= \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} q^{-s} S_{ax_1,q} \dots S_{ax_s,q} \\ &= \prod_p \left\{ \sum_{\nu=0} \sum_{\substack{a=1 \\ (a,p^\nu)=1}}^{p^\nu} q^{-s} S_{ax_1,q} \dots S_{ax_s,q} \right\} \\ &= \prod_p \chi(p), \end{aligned}$$

which follows from the fact that

$$\sum_{\substack{a=1 \\ (a,q_1q_2)=1}}^{q_1q_2} \left(\frac{S_{ax_1,q_1q_2} \dots S_{ax_s,q_1q_2}}{q_1q_2} \right)^s = \left(\sum_{\substack{a=1 \\ (a,q_1)=1}}^{q_1} \left(\frac{S_{ax_1,q_1} \dots S_{ax_s,q_1}}{q_1} \right)^s \right) \left(\sum_{\substack{a=1 \\ (a,q_2)=1}}^{q_2} \left(\frac{S_{ax_1,q_2} \dots S_{ax_s,q_2}}{q_2} \right)^s \right),$$

when $(q_1, q_2) = 1$.

In fact, when $p > p_0$ for some p_0, then

$$\prod_{p > p_0} \chi(p) \ge \frac{1}{2}.$$

Because of the above, to show that $\mathfrak{S}_s$ is positive, it suffices to show that $\chi(p) > 0$ for all primes p.

Define $\tau(p,k)$ to be the highest exponent of p which divides k, and define

$$\gamma(p,k) = \begin{cases} \tau(p,k) + 1 \text{ if } p > 2 \\ \tau(p,k) + 2 \text{ if } p = 2. \end{cases}$$

Furthermore, let $C_p = p^{-\gamma(s-1)} > 0$.

To show that $\chi(p) > 0$ for all p, it suffices to show that the form

$$x_1 y_1^k + \cdots + x_s y_s^k$$

represents zero p-adically for all p. That is, for all p,

$$x_1 y_1^k + \cdots + x_s y_s^k \equiv 0 \pmod{p^\gamma}$$

has a solution with terms $x_i y_i^k$ not all divisible by p. This is also called the *congruence condition.*

Lemma 2.1. *If the congruence condition holds, so that*

$$x_1 y_1^k + \cdots + x_s y_s^k \equiv 0 \pmod{p^\gamma}$$

has a solution with terms $x_i y_i^k$ not all divisible by p, then

$$\chi(p) > 0.$$

Proof. It suffices to let

$$a_1b_1^k + \cdots + a_sb_s^k \equiv 0 \pmod{p^\gamma}$$

with $a_1b_1^k \not\equiv 0 \pmod{p}$ be a solution, from which we will construct more solutions for $\nu > \gamma$.

Choose $x_2y_2^k, \ldots, x_sy_s^k$ arbitrary, but subject to the condition

$$x_iy_i^k \equiv a_ib_i^k \pmod{p^\gamma}, \quad 0 < x_iy_i^k \le p^\nu.$$

There are $p^{(\nu-\gamma)(s-1)}$ such choices.

Then choose $x_1y_1^k$ to satisfy

$$x_1y_1^k \equiv -x_2y_2^k - \cdots - x_sy_s^k \pmod{p^\nu}.$$

This is possible since the right-hand side of the congruence is congruent to $a_1b_1^k \pmod{p^\nu}$ and $a_1b_1^k$ by the assumption, which means that the congruence

$$x_1y_1^k \equiv -x_2y_2^k - \cdots - x_sy_s^k$$

is soluble for every $\nu > \gamma$, as we will show at the end of this proof.

Thus the number of solutions of the congruence

$$x_1y_1^k + \cdots + x_sy_s^k \equiv 0 \pmod{p^\nu}, \qquad 0 \le x_i < p^\nu.$$

is at least $p^{(\nu-\gamma)(s-1)}$.

To finish the proof, we show that if the congruence

$$h_ig_i^k \equiv m \pmod{p^\gamma}$$

is soluble for $m \not\equiv 0 \pmod{p}$, then the congruence

$$r_js_j^k \equiv m \pmod{p^\nu}$$

is soluble for all $\nu > \gamma$. We tackle this exactly as in Davenport.

Let $p > 2$.

The relatively prime residue classes form a cyclic group of order $p^{\nu-1}(p-1)$, and they have as representatives the powers of a primitive root g modulo p^ν. In particular, if $\nu > \gamma$, then g is also a primitive root modulo p^γ.

Let

$$m \equiv g^\mu, \quad h_ig_i^k \equiv g^\eta, \quad r_js_j^k \equiv g^\xi \pmod{p^\nu}.$$

The hypothesis that $h_ig_i^k \equiv m \pmod{p^\gamma}$ is equivalent to

$$\eta \equiv \mu \pmod{p^{\gamma-1}(p-1)}.$$

FIX THIS LAST PART OF THE PROOF LATER. □

Showing $\chi(p) > 0$ is achieved in the literature by showing

$$M(p^\nu) \ge C_p p^{\nu(s-1)},$$

for sufficiently large ν, where $M(p^\nu)$, in our case, denotes the total number of solutions of the congruence

$$x_1y_1^k + \cdots + x_sy_s^k \equiv 0 \pmod{p^\nu}, \qquad 0 \le y_i < p^\nu.$$

We now find an explicit function $s_1(k)$ such that

$$M(p^\nu) \ge C_p p^{\nu(s-1)},$$

holds for each prime p when $s \ge s_0(k)$.

Theorem 2.2.

$$s_1(k) = k^2 + 1,$$

that is, when $s > k^2 + 1$, *we have that* $\mathfrak{S}_s > 0$.

Proof. This follows directly from the analogous result for the form

$$c_1x_1^k + \cdots + c_sx_s^k = 0,$$

where $c_1, \ldots, c_s$ are given integers, not all of the same sign if k is even.
This proof is given in "Homogeneous additive equations" (Davenport & Lewis, [4]). □

Finally, we show absolute convergence for the singular series for $s > s_0(k) = 2k$.
To show that the singular series is absolutely convergent for $s > 2k$, we show that

$$\left| \sum_{\substack{a=1 \\ (a,q)=1}}^{q} q^{-s} S_{ax_1,q} \ldots S_{ax_s,q} \right| \ll q^{1-\frac{s}{k}},$$

which is directly implied by each of the terms $S_{ax_i,q}$ being bounded:

$$|S_{ax_i,q}| \ll q^{1-\frac{1}{k}},$$

and this latter estimate is the one we show.

Theorem 2.3.

$$|S_{ax_i,q}| \ll q^{1-\frac{1}{k}}$$

for $(a,q) = 1$.

Proof. Let

$$T(ax_i, q) = q^{-1+\frac{1}{k}} S_{ax_i,q}.$$

We show that $T(ax_i, q)$ is bounded, independently of q, so that $|S_{ax_i,q}| \ll q^{1-\frac{1}{k}}$ will hold.
By the multiplicativity property of $S_{ax_1,q} \ldots S_{ax_s,q}$ we discuss above in the above discussion of the product form of the singular series,

$$T(ax_i, q) = T(a_1x_i, p_1^{\nu_1})T(a_2x_i, p_2^{\nu_2}) \ldots,$$

for $q = p_1^{\nu_1}p_2^{\nu_2} \ldots$, and for suitable $a_1, a_2, \ldots$, with each a_j subject to $(a_j, p_j^{\nu_j}) = 1$.
Since when $\nu_j > k$ and $a_j \not\equiv 0 \pmod{p_j}$ for some j it holds that

$$S_{a_jx_i,p_j^{\nu_j}} = p_j^{k-1} S_{a_jx_i,p_j^{\nu_j-k}},$$

we have that

$$\begin{aligned} T(a_jx_i, p_j^{\nu_j}) &= (p_j^{\nu_j})^{-1+\frac{1}{k}} S_{a_jx_i,p_j^{\nu_j}} \\ &= (p_j^{\nu})^{-1+\frac{1}{k}} p_j^{k-1} S_{a_jx_i,p_j^{\nu_j-k}} \\ &= p_j^{-\nu_j+k+\frac{\nu_j}{k}-1} S_{a_jx_i,p_j^{\nu_j-k}} \\ &= (p_j^{\nu_j-k})^{-1+\frac{1}{k}} S_{a_jx_i,p_j^{\nu_j-k}} \\ &= T(a_jx_i, p_j^{\nu_j-k}), \end{aligned}$$

which tells us that $T(a_jx_i, p_j^{\nu_j})$ is bounded independently of $p_j^{\nu_j}$ for $\nu_j > k$.
On the other hand, supposing that $\nu_j \leq k$ for all j, since when $a_j \not\equiv 0 \pmod{p}$, we have that

$$|S_{a_jx_i,p_j}| \leq ((k, p_j - 1) - 1)\sqrt{p_j}$$

it holds that

$$T(a_jx_i, p_j) \leq k\sqrt{p_j} p_j^{-1+\frac{1}{k}} \leq kp_j^{-\frac{1}{6}}.$$

Furthermore, since when $a_j \not\equiv 0 \pmod{p_j}$ and $p_j \nmid k$, for $\nu_j \leq k$, we have that

$$S_{a_jx_i,p_j^{\nu}} = p_j^{\nu_j-1},$$

it is the case that

$$T(a_jx_i, p_j^{\nu_j}) = p^{\nu_j-1}p^{\nu_j(-1+\frac{1}{k})} \leq 1.$$

Therefore

$$T(a_j x_i, p_j^{\nu_j}) \le 1,$$

except in the case that $p_j \le k^6$ and $\nu_j = 1$.

Combining the two results above for $\nu_j \le k$, we have that

$$T(ax_i, q) \le \prod_{p_j \le k^6} (kp_j^{-\frac{1}{6}}),$$

thus $T(ax_i, q)$ is bounded independently of q, as desired, exactly as in the case for the singular series for Waring's problem.

Lastly, for this proof, it remains to check that indeed if $a_j \not\equiv 0 \pmod{p_j}$, that we have

1. $|S_{a_j x_i, p_j}| \le ((k, p_j - 1) - 1)\sqrt{p_j}$,

2. if $p \nmid k$ and $\nu_j \le k$, then $S_{a_j x_i, p_j^{\nu_j}} = p_j^{\nu_j - 1}$, and

3. if $\nu_j > k$, then $S_{a_j x_i, p_j^{\nu_j}} = p_j^{k-1} S_{a_j x_i, p_j^{\nu_j - k}}$.

We start with 1. In the following proof of 1., sums are always over a complete set of residues mod p. We have that

$$S_{ax_i, p} = \sum_r e\left(\frac{ax_i}{p} r^{(k,p-1)}\right).$$

Letting χ be a primitive character of order $(k, p-1)$ modulo p, the number of solutions to $r^\delta \equiv t \pmod{p}$ is $1 + \chi(t) + \cdots + \chi^{(k,p-1)}(t)$, so that

$$S_{ax_i, p} = \sum_t \{1 + \chi(t) + \cdots + \chi^{(k,p-1)-1}(t)\} e\left(\frac{ax_i}{p} t\right).$$

Let ψ denote any one of $\chi, \ldots, \chi^{(k,p-1)-1}$. The Gauss sum

$$T(\psi) = \sum_t \psi(t) e\left(\frac{ax_i}{p} t\right)$$

is such that $|T(\psi)| = \sqrt{p}$, since

$$\begin{aligned} |T(\psi)|^2 &= \sum_t \sum_u \psi(t)\overline{\psi}(t) e\left(\frac{ax_i}{p}(t - u)\right) \\ &= \sum_v \sum_{u \ne 0} \psi(v) e\left(\frac{ax_i}{p} u(v-1)\right) \end{aligned}$$

with a change of variables $t \equiv vu \pmod{p}$. If $v = 1$, then

$$\sum_{u \ne 0} \psi(1) e\left(\frac{ax_i}{p} u(1-1)\right) = 1,$$

and if $v \ne 1$, then

$$\sum_{u \ne 0} \psi(v) e\left(\frac{ax_i}{p} u(v-1)\right) = -\psi(v),$$

so that

$$|T(\psi)|^2 = p\psi(1) - \sum_v \psi(v) = p.$$

Since in

$$\sum_t \{1 + \chi(t) + \cdots + \chi^{(k,p-1)-1}(t)\} e\left(\frac{ax_i}{p} t\right)$$

there are $(k, p-1) - 1$ non-zero terms in the bracket, we have that

$$|S_{ax_i, p}| \le ((k, p-1) - 1)\sqrt{p}.$$

For 2., note that

$$S_{ax_i,p^\nu} = \sum_{r=0}^{p^{\nu-1}-1}\sum_{z=0}^{p-1} e\left(\frac{ax_i}{p^\nu}r^k + \frac{ax_i}{p}kr^{k-1}z\right),$$

and with a change of variables $r = pw$, we have

$$S_{ax_i,p^\nu} = p\sum_{w=0}^{p^{\nu-2}-1} e\left(\frac{ax_i}{p^{\nu-k}}w^k\right),$$

where all of the terms are 1 if $\nu \le k$, so that

$$S_{ax_i,p^\nu} = p^{\nu-1},$$

as desired.

Finally, for 3., taking the sum above when $p \nmid k$, if $\nu > k$, we have a period function in w of period $p^{\nu-k}$, so that

$$S_{ax_i,p^\nu} = p^{k-1}S_{ax_i,p^{\nu-k}}.$$

If instead we have $p \mid k$, consider $k = p^{\tau(p,k)}k_0$, and note

$$\nu > p^{\tau(p,k)}k_0 \ge 2^{\tau(p,k)} \ge \tau(p,k) + 1,$$

so, in particular $\nu \ge \tau(p,k) + 2$.

Parallel to the above case when $p \nmid k$, we have

$$\begin{aligned}S_{ax_i,p^\nu} &= \sum_{r=0}^{p^{\nu-\tau(p,k)-1}-1}\sum_{z=0}^{p^{\tau(p,k)+1}-1} e\left(\frac{ax_i}{p^\nu}r^k - \frac{ax_i}{p}k_0r^{k-1}r\right)\\ &= p^{\tau(p,k)+1}\sum_{w=0}^{p^{\tau(p,k)+1}-1} e\left(\frac{ax_i}{p^{\nu-k}}w^k\right)\\ &= p^{\tau(p,k)+1}p^{k-\tau(p,k)-2}S_{ax_i,p^{\nu-k}}\\ &= p^{k-1}S_{ax_i,p^{\nu-k}}.\end{aligned}$$

□

Thus, we have proved the following.

Theorem 2.4.

$$s_0(k) = 2k + 1.$$

That is, the singular series $\mathfrak{S}_s$ is absolutely convergent for $s > 2k$.

Lastly, in this section, we make a note about $G(k)$.

We define $G(k)$ to be the smallest value for s such that infinite solutions exist for the equation with y_i's bounded by P.

From the above 2 sections, we know that $G(k) \le 2^k + 1$, but we wish to obtain tighter bounds.

It suffices to study solutions to the form

$$x_1y_1^k + \dots x_sy_s^k = M.$$

This is the topic of the following section.

3 Part III

In this section we give an upper bound for $G(k)$, the smallest number for s for which infinite solutions exists for the equation with y_i's bounded by P

$$x_1y_1^k + x_2y_2^k + ... + x_sy_s^k = 0, \tag{7}$$

as P approaches infinity. From Part II we note that to solve for the number of solutions to the form

$$x_1y_1^k + x_2y_2^k + ... + x_sy_s^k = M.$$

it suffices if

$$M \equiv 0 \pmod{p^{\gamma(p,k)}}$$

for all primes p less than some fixed p_0 not depending on M. For such choices of M, we note from Part II that the singular series is guaranteed to be positive.

Definition 3.1. *We define major arcs to be*

$$\mathfrak{M}_{a,q} = \left\{ \alpha : |q\alpha - a| \le \frac{1}{2kP^{k-1}\max|x_j|} \right\},$$
$$\mathfrak{M} = \bigcup \mathfrak{M}_{a,q} \text{ for } 1 \le a \le q, (a,q) = 1, q \le P^{\frac{1}{2}},$$

and we define minor arcs to be the complement of major arcs:

$$m = [0,1] \backslash \mathfrak{M}.$$

We will make use of the following lemma

Lemma 3.2. *(Van der Corput) Let f be twice differentiable function. Suppose we have:*

$$0 \le f'(x) \le \frac{1}{2} \text{ and } f''(x) \ge 0.$$

Then the following holds:

$$\sum_{A \le n \le B} e(f(n)) = \int_A^B e(f(x))dx + O(1).$$

Proof. We can start by assuming that A and B are both integers and that the difference between the summation and the integral is real by replacing $f(x)$ with $f(x)+c$ if necessary.
Define $\Psi(x) = x - [x] - \frac{1}{2}$, then we note that

$$\int_n^{n+1} \Psi(x)F'(x)dx = \frac{1}{2}\big(F(n+1) + F(n)\big) - \int_n^{n+1} F(x)dx.$$

Thus following such procedure, we have

$$\sum_{A \le n \le B} F(n) = \int_A^B \Psi(x)F'(x)dx + \int_A^B F(x)dx + O(1),$$

noting that $F(x) = e(f(x))$, which we can then replace, since the difference can be made real between the second integral and the sum, with $\cos 2\pi f(x)$. It remains to show that

$$I = \int_A^B \Psi(x)F'(x)dx$$

is bounded in absolute value.
Quoting results from Fourier analysis, we have

$$\Psi(x) = -\sum_{v=1}^{\infty} \frac{\sin 2\pi vx}{\pi v}.$$

We note that this series is absolutely convergent

$$\begin{aligned} I &= \int_A^B \Psi(x)F'(x)dx \\ &= -\sum_{v=1}^{\infty} \int_A^B \frac{\sin 2\pi vx}{\pi v} \{\cos 2\pi f(x)\}' dx \\ &= \sum_{v=1}^{\infty} \frac{2}{v} \int_A^B \sin(2\pi vx) \sin(2\pi f(x)) f'(x) dx \\ &= \sum_{v=1}^{\infty} \frac{1}{v} \int_A^B f'(x)\{\cos 2\pi(vx - f(x)) - \cos 2\pi(vx + f(x))\} dx. \end{aligned}$$

We will show that

$$\left| \int_A^B f'(x) \cos 2\pi(vx \pm f(x)) dx \right| \le \frac{1}{\pi(2v-1)},$$

from which the convergence of the series immediately follows. Rewriting $\phi(x) = \sin 2\pi(vx \pm f(x))$, we can reformulate the integral as

$$\frac{1}{2\pi} \int_A^B \frac{f'(x)}{v \pm f'(x)} \phi'(x) dx.$$

We note that integral for the second term is bounded above by 2, and the first term is monotone, since its derivative is

$$\frac{vf''(x)}{v \pm f'(x)} \ge 0,$$

and since the first term is bounded by $\frac{1}{2v-1}$, we conclude the proof. ☐

Remark: the theorem also holds in the case

$$\frac{1}{2} \le f'(y) \le 0 \ \ and\ f''(y) \le 0.$$

As in Part I, define

$$T_j(\alpha) = \sum_{y_j=1}^{P} e(\alpha x_j y_j^k).$$

Then we have the following approximation on major arcs:

Lemma 3.3. *For α in $\mathfrak{M}_{a,q}$, we have*

$$T_j(\alpha) = q^{-1} S_{x_j a,q} I_j(\beta) + O(q),$$

where (as before)

$$S_{x_j a,q} = \sum_{z=1}^{q} e(a x_j z^k / q),$$

and

$$I_j(\beta) = \int_0^P e(\beta x_j \eta^k) d\eta.$$

Proof. We see that after rewriting $\alpha = \frac{a}{q} + \beta$, we have

$$T_j(\alpha) = \sum_{z=1}^{q} e(a x_j z^k / q) \sum_{y} e(\beta x_j (qy+z)^k),$$

where the summation is over y for which $0 \le qy + z \le P$. In particular, let $f(y) = \beta x_j (qy+z)^k$, then we note that for x_j and β of the same sign we have

$$f'(y) = k\beta x_j q (qy+z)^{k-1} \le k x_j q \frac{1}{2kqP^{k-1} max|x_j|} P^{k-1} \le \frac{1}{2},$$

and the same holds for β and x_j of different sign. In particular, we see that we can replace this inner sum with

$$\int_{0 \le qy+z \le P} e(\beta x_j (q\eta + z)^k) d\eta + O(1),$$

by lemma 3.2. A simple change of variables lead to Lemma 3.3. □

We now seek to evaluate integral along the major arcs. Denote s_0 as the smallest possible value for s such that the associated singular series in Part I converges. We have the following lemma:

Lemma 3.4. *Suppose $s \ge s_0$, then for*

$$\frac{1}{5} P^k \le M \le P^k, \max |x_j|^{\frac{2s}{k} - 1} \ll P^{1-\epsilon},$$

we have

$$\int_{\mathfrak{M}} T_1(\alpha) T_2(\alpha) ... T_s(\alpha) e(-M\alpha) d\alpha \gg P^{s-k}.$$

Proof. We first try to find error terms associated: from Part II we have that

$$q^{-1} |S_{ax_j,q}| \ll |x_j|^{\frac{1}{k}} q^{-\frac{1}{k}}$$

and we also have:

$$I_j(\beta) \ll \min(P, \beta^{-\frac{1}{k}} |x_j|^{-\frac{1}{k}})$$

Where P comes from the trivial estimate, and the second estimate comes from change of coordinates $u = \beta x_j \eta^k$, giving:

$$I_j(\beta) = k^{-1} \beta^{-\frac{1}{k}} x_j^{-\frac{1}{k}} \int_0^{\beta x_j P^k} e(u) u^{-1+\frac{1}{k}} du$$

Where by the definition of the integral we can assume that $\beta x_j P^k$ is positive. In particular, we note that the integral is bounded: note that

$$\int_0^\infty e^{2\pi i u} u^{-1+\frac{1}{k}} du = \int_0^1 e^{2\pi i u} u^{-1+\frac{1}{k}} du + \int_1^\infty e^{2\pi i u} u^{-1+\frac{1}{k}} du.$$

In particular the first integral is bounded. For the second integral, we have

$$\begin{aligned} \Big|\int_1^\infty e^{2\pi i u} u^{-1+\frac{1}{k}} du\Big| &= \Big|\sum_{n=1}^\infty \int_n^{n+1} e^{2\pi i u} u^{-1+\frac{1}{k}} du\Big| \\ &\ll \Big|\sum_{n=1}^\infty \int_n^{n+1} u^{-1+\frac{1}{k}} de^{2\pi i u}\Big| \\ &\ll \Big|\sum_{n=1}^\infty (n+1)^{-1+\frac{1}{k}} - n^{-1+\frac{1}{k}} + (-1+\frac{1}{k}) \int_n^{n+1} e^{2\pi i u} u^{-2+\frac{1}{k}} du\Big| \\ &\ll \Big|\sum_{n=1}^\infty n^{-2+\frac{1}{k}}\Big| + \Big|\int_1^\infty e^{2\pi i u} u^{-2+\frac{1}{k}} du\Big|, \end{aligned}$$

which converges and hence the integral converges. This gives the estimate for the size of the main term:

$$q^{-1} S_{x_j a, q} I_j(\beta) \ll |x_j|^{\frac{1}{k}} q^{-\frac{1}{k}} \min(P, \beta^{-\frac{1}{k}} |x_j|^{-\frac{1}{k}}).$$

In particular, we note that error term q doesn't exceed either of these inside min, since

$$q^{1+\frac{1}{k}} \le P|x_j|^{\frac{1}{k}} \ and \ q^{1+\frac{1}{k}} \le \beta^{-\frac{1}{k}}$$

By our construction. Thus we have (assuming $|x_s|$ is smallest among all $x_i's$.)

$$\begin{aligned} T_1(\alpha)T_2(\alpha)...T_s(\alpha) = q^{-s} S_{x_1 a,q}...S_{x_s a,q} I_1(\beta)...I_s(\beta) + \\ O\Big(q|x_1|^{\frac{1}{k}}...|x_{s-1}|^{\frac{1}{k}} q^{-\frac{s-1}{k}} \min(P, |x_1|^{-\frac{1}{k}}\beta^{-\frac{1}{k}})... \min(P, |x_{s-1}|^{-\frac{1}{k}}\beta^{-\frac{1}{k}})\Big) \end{aligned}$$

Once we integrate over $(-\infty, \infty)$ with respect to β, we have the error term being bounded:

$$\ll q^{1-\frac{s-1}{k}} |x_1|^{\frac{1}{k}}...|x_{s-1}|^{\frac{1}{k}} \int_{-\infty}^\infty \min(P, |\beta|^{-\frac{1}{k}}|x_1|^{-\frac{1}{k}})... \min(P, |\beta|^{-\frac{1}{k}}|x_{s-1}|^{-\frac{1}{k}}) d\beta$$

By Holder's inequality, we have:

$$\Big|\int_{-\infty}^\infty \min(P, |\beta|^{-\frac{1}{k}}|x_1|^{-\frac{1}{k}})... \min(P, |\beta|^{-\frac{1}{k}}|x_{s-1}|^{-\frac{1}{k}}) d\beta\Big| \le \prod_{j=1}^{s-1} \Big(\int_{-\infty}^\infty |\min(P, |\beta|^{-\frac{1}{k}}|x_1|^{-\frac{1}{k}})|^{s-1} d\beta\Big)^{\frac{1}{s-1}}$$

In particular, for each term in the product, we have

$$\begin{aligned} &\int_{-\infty}^\infty |\min(P, |\beta|^{-\frac{1}{k}}|x_1|^{-\frac{1}{k}})|^{s-1} d\beta \\ &\le \int_{-(|x_j|P^k)^{-1}}^{(|x_j|P^k)^{-1}} P^{s-1} d\beta + \int_{(|x_j|P^k)^{-1}}^\infty |x_j|^{-\frac{s-1}{k}} \beta^{-\frac{s-1}{k}} d\beta + \int_{-\infty}^{-(|x_j|P^k)^{-1}} |x_j|^{-\frac{s-1}{k}} |\beta|^{-\frac{s-1}{k}} d\beta \\ &\ll P^{s-1-k}|x_j|^{-1} + |x_j|^{-\frac{s-1}{k}} \big((|x_j|P^k)^{-1}\big)^{1-\frac{s-1}{k}} \\ &= |x_j|^{-1} P^{s-1-k} \end{aligned}$$

Thus we have the entire error term bounded by, after integrating:

$$\Big|\int_{-\infty}^\infty \min(P, |\beta|^{-\frac{1}{k}}|x_1|^{-\frac{1}{k}})... \min(P, |\beta|^{-\frac{1}{k}}|x_{s-1}|^{-\frac{1}{k}}) d\beta\Big| \ll q^{1-\frac{s-1}{k}} |x_1 x_2 ... x_{s-1}|^{\frac{1}{k}-\frac{1}{s-1}} P^{s-k-1}$$

Summing over all possible values of a, which is at most q, and $q \le P^{\frac{1}{2}}$, we have

$$P^{s-k-1}|x_1...x_{s-1}|^{\frac{1}{k}-\frac{1}{s-1}} \sum_q q^{2-\frac{s-1}{k}} \ll P^{s-k-1}|x_1...x_{s-1}|^{\frac{1}{k}-\frac{1}{s-1}}$$

since the series converges.
Summing over all possible values of a and q and integrate along the major arcs we have

$$\sum_{q \le P^{\frac{1}{2}}} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} q^{-s} S_{x_1 a,q} S_{x_2 a,q} ... S_{x_s a,q} \int_{\beta \le (2kq \max|x_j|)^{-1} P^{1-k}} I_1(\beta)...I_s(\beta) e(-M\beta) d\beta$$

In particular, note that

$$\sum_q \sum_a q^{-s} S_{x_1 a,q} ... S_{x_s a,q} \int_{\beta \geq (2kq \max|x_j|)^{-1} P^{1-k}} I_1(\beta)...I_s(\beta) e(-M\beta) d\beta$$
$$\ll \sum_q q |x_1...x_s|^{\frac{1}{k}} q^{-\frac{s}{k}} \int_{\beta \geq (2kq \max|x_j|)^{-1} P^{1-k}} |x_1...x_s|^{-\frac{1}{k}} \beta^{-\frac{s}{k}} d\beta$$
$$\ll \sum_q q^{1-\frac{s}{k}} q^{\frac{s}{k}-1} \max|x_j|^{\frac{s}{k}-1} P^{(1-k)(1-\frac{s}{k})}$$
$$\ll \max|x_j|^{\frac{s}{k}-1} P^{s-k-1}$$

Consider

$$\int_{-\infty}^{\infty} I_1(\beta) \dots I_s(\beta) e(-M\beta)\, d\beta,$$

where

$$I_j(\beta) = \int_0^P e(\beta x_j \eta^k)\, d\eta,$$

where $\eta = Pu$.

Therefore,

$$I_j(\beta) = \int_0^1 e(\beta x_j P^k u^k) P\, du.$$

Thus,

$$\int_{-\infty}^{\infty} I_1(\beta) \dots I_s(\beta) e(-M\beta)\, d\beta = P^s \int_{-\infty}^{\infty} \prod_{j=1}^{s} \int_0^1 e(\beta x_j P^k u_j^k)\, du_j\, e(-M\beta)\, d\beta$$
$$= P^s \int_{-\infty}^{\infty} \prod_{j=1}^{s} \int_0^1 e(\gamma x_j u_j^k)\, du_j\, e\left(\frac{M}{P^k}\gamma\right) \frac{d\gamma}{P^k}$$
$$= P^{s-k} \int_{-\infty}^{\infty} \prod_{j=1}^{s} \int_0^1 e(\gamma x_j u_j^k)\, du_j\, e\left(\frac{M}{P^k}\gamma\right) d\gamma$$

where $\gamma = \beta P^k$, and thus $d\gamma = d\beta P^k$.

Setting $\zeta_j = u_j^k$, so that $d\zeta_j = k u_j^{k-1} du_j$.

We have

$$\int_{-\infty}^{\infty} I_1(\beta)...I_s(\beta) e(-M\beta)\, d\beta$$
$$= P^{s-k} \int_{-\infty}^{\infty} \{\prod_{j=1}^{s} \int_0^1 e(\gamma x_j u_j^k) du_j\} e(-\gamma \frac{M}{P^k}) d\gamma$$
$$= P^{s-k} \prod_{j=1}^{s} \int_{-\infty}^{\infty} \{ \int_0^1 e(\gamma x_j \zeta_j) k^{-1} \zeta_j^{\frac{1}{k}-1} d\zeta_j \} e(-\gamma \frac{M}{P^k})\, d\gamma \qquad (\zeta_j := u_j^k)$$
$$= \frac{P^{s-k}}{k^s} \int_{-\infty}^{\infty} \int_0^1 ... \int_0^1 (\zeta_1^{\frac{1}{k}-1} ... \zeta^{\frac{1}{k}-1}) e(\gamma(x_1\zeta_1 + ... + x_s\zeta_s - \frac{M}{P^k}))\, d\gamma\, d\zeta$$
$$= \frac{P^{s-k}}{k^s} \lim_{\lambda\to\infty} \int_0^1 ... \int_0^1 (\zeta_1^{\frac{1}{k}-1} ... \zeta_s^{\frac{1}{k}-1}) \int_{-\lambda}^{\lambda} e(\gamma(x_1\zeta_1 + ... + x_s\zeta_s - \frac{M}{P^k}))\, d\gamma\, d\zeta \qquad \text{(Dominated Convergence Theorem)}$$
$$= \frac{P^{s-k}}{k^s} \lim_{\lambda\to\infty} \int_0^1 ... \int_0^1 (\zeta_1 ... \zeta_s)^{\frac{1}{k}-1} \frac{\sin(2\pi\lambda(x_1\zeta_1 + ... + x_s\zeta_s - \frac{M}{P^k}))}{\pi(x_1\zeta_1 + ... + x_s\zeta_s - \frac{M}{P^k})}\, d\zeta.$$

Define $z = x_1\zeta_1 + ... x_s\zeta_s$, we have

$$= \frac{P^{s-k}}{k^s} \lim_{\lambda\to\infty} \int_0^1 ... \int_0^1 \int_{x_1\zeta_1+...+x_{s-1}\zeta_{s-1}}^{x_1\zeta_1+...+x_{s-1}\zeta_{s-1}+x_s} (\zeta_1...\zeta_{s-1})^{\frac{1}{k}-1} (\frac{z - x_1\zeta_1 - ... - x_{s-1}\zeta_{s-1}}{x_s})^{\frac{1}{k}-1} \frac{\sin(2\pi\lambda(z - \frac{M}{P^k}))}{\pi(z - \frac{M}{/}' P^k)} x_s^{-1}\, dz\, d\zeta$$
$$= \frac{P^{s-k}}{k^s} \lim_{\lambda\to\infty} \int_0^{x_1+...x_s} \frac{\sin(2\pi\lambda(z - \frac{M}{P^k}))}{\pi(z - \frac{M}{P^k})} \psi(z)\, dz,$$

where

$$\psi(z) = \int_0^1 ... \int_0^1 (\zeta_1...\zeta_{s-1})^{\frac{1}{k}-1} (z - x_1\zeta_1 ... x_{s-1}\zeta_{s-1})^{\frac{1}{k}-1} x_s^{-\frac{1}{k}} d\zeta_1 ... d\zeta_{s-1},$$

with the integration taking place for $z - x_s \leq x_1\zeta_1 + ... + x_{s-1}\zeta_{s-1} \leq z$. In particular we see that with the Fourier Integral Theorem, we have that

$$\frac{P^{s-k}}{k^s} \lim_{\lambda\to\infty} \int_0^{x_1+...x_s} \frac{\sin(2\pi\lambda(z - \frac{M}{P^k}))}{\pi(z - \frac{M}{P^k})} \psi(z) dz$$
$$= \frac{P^{s-k}}{k^s} \psi(\frac{M}{P^k})$$
$$= \frac{P^{s-k}}{k^s} \int_0^1 ... \int_0^1 (\zeta_1...\zeta_{s-1})^{\frac{1}{k}-1} (\frac{M}{P^k} - x_1\zeta_1 ... - x_{s-1}\zeta_{s-1})^{\frac{1}{k}-1} x_s^{-\frac{1}{k}} d\zeta_1 ... d\zeta_{s-1}.$$

In particular, we have that since $\frac{M}{P^k} - x_1\zeta_1 - ... - x_{s-1}\zeta_{s-1} \le x_1$ and $\frac{1}{k} - 1 \le 0$ where the integral is taking place, we have that

$$\int_{-\infty}^{\infty} I_1(\beta)\dots I_s(\beta)e(-M\beta)\,d\beta >> \frac{P^{s-k}}{\max|x_j|}$$

□

Lemma 3.5. *Define $U_l(X)$ the number of natural numbers M up to X that can be written in the form*

$$M = x_1y_1^k + ...x_ly_l^k.$$

Then we have:

$$U_l(X) \gg (x_1...x_s)^{-\frac{1}{k-1}}(1-\lambda^l), \lambda = 1 - \frac{1}{k}.$$

Proof. We prove the lemma through inducting on l. For $l = 1$, we note that

$$U_1(X) = x_1^{-\frac{1}{k}}X^{\frac{1}{k}} \gg (x_1)^{-\frac{1}{k-1}}X^{1-\lambda}.$$

For the inductive step, suppose the lemma holds for $l-1$. We show that it holds for l. Consider integers of the form $z + x_ly_l^k$ where $z = x_1y_1^k + ...x_{l-1}y_{l-1}^k$ for a certain choice of x_i's and y_i's, and also subject to the conditions

$$x_l^{-\frac{1}{k-1}}(\frac{1}{4}X)^{\frac{1}{k}} < y_l < x_l^{-\frac{1}{k-1}}(\frac{1}{2}X)^{\frac{1}{k}},$$

and

$$0 < z < \frac{1}{2}X^{1-\frac{1}{k}}.$$

In particular, we show that such representations are unique. Suppose that we have satisfying $z_1 > z_2, y_1, y_2$ such that

$$z_1 + x_ly_1^k = z_2 + x_ly_2^k.$$

Then we have the following inequalities:

$$x_ly_2^k - x_ly_1^k \ge x_lky_1^{k-1} > k(\frac{1}{4}X)^{\frac{k-1}{k}} > \frac{1}{2}X^{1-\frac{1}{k}},$$

meanwhile

$$z_1 - z_2 < z_1 < \frac{1}{2}X^{1-\frac{1}{k}}.$$

This gives us a contradiction. Hence such representations are unique. In particular, we also have for such choices of z, y_l

$$z + x_ly_l^k < \frac{1}{2}X^{1-\frac{1}{k}} + x_l^{1-\frac{k}{k-1}}(\frac{1}{2}X) = \frac{1}{2}X^{1-\frac{1}{k}} + x_l^{-\frac{1}{k-1}}(\frac{1}{2}X) < X.$$

We thus have, using inductive hypothesis:

$$\begin{aligned} U_l(X) &\gg U_{l-1}(\frac{1}{2}X^{1-\frac{1}{k}})x_l^{-\frac{1}{k-1}}X^{\frac{1}{k}} \\ &\gg X^{(1-\frac{1}{l})(1-\lambda^{l-1})}(x_1...x_{l-1})^{-\frac{1}{k-1}}x_l^{-\frac{1}{k-1}}X^{\frac{1}{k}} \\ &\gg (x_1...x_s)^{-\frac{1}{k-1}}X^{1-\lambda^l}, \end{aligned}$$

concluding the proof. □

We thus have the following corollary that will prove to be helpful:

Corollary 3.5.1. *Define*

$$R_1(\alpha) = \sum_{u<\frac{1}{4}P^k} e(\alpha u), \tag{8}$$

where u ranges over integers less than $\frac{1}{4}P^k$ that can be written in the form:

$$u = x_1y_1^k + ... + x_ly_l^k.$$

Then we have the following asymptotic bound:

$$\int_0^1 |R(\alpha)|^2 d\alpha = R(0) \ll (x_1...x_l)^{\frac{1}{k-1}}P^{-k(1-\lambda^l)}R^2(0).$$

Proof. The first equality is obvious. For the second one, note that $R(0) = U_l(\frac{1}{4}P^k) \gg (x_1...x_l)^{-\frac{1}{k-1}} P^{k(1-\lambda^l)}$. The second inequality immediately follows. □

The next lemma, due to Vinogradov, also plays an important role.

Lemma 3.6. *Let X_0, Y_0 be the size of 2 sets of different integers running over interval of length X, Y, respectively. Let $\alpha = \frac{a}{q} + O(q^{-2})$, then we have*

$$\Big|\sum_x \sum_y e(\alpha xy)\Big|^2 \ll X_0 Y_0 \frac{\log q}{q}(q + X)(q + Y),$$

where the summation is over x, y of the distinct 2 sets.

Proof. By Cauchy-Schwartz, we have

$$\begin{aligned}
\Big|\sum_x \sum_y e(\alpha xy)\Big|^2 &\leq \Big(\sum_x 1\Big)\Big(\sum_x \Big|\sum_y e(\alpha xy)\Big|^2\Big) \\
&\leq X_0 \sum_{x=x_1}^{x_1+X} \sum_{y_1} \sum_{y_2} e(\alpha x(y_1 - y_2)) \\
&\leq X_0 \sum_{y_1} \sum_{y_2} \min(X, \|\alpha(y_1 - y_2)\|^{-1}) && \text{(Part I, Lemma 1.1)} \\
&\ll X_0 Y_0 \sum_{|t|\leq Y} \min(X, \|\alpha t\|^{-1}) \\
&\ll X_0 Y_0 (\frac{Y}{q} + 1) \sum_{t=t_1+1}^{t_1+q} \min(X, \|\frac{at}{q} + O(q^{-1}) + \tau\|^{-1}) && \text{(Part I, Lemma 1.1)} \\
&\ll X_0 Y_0 (\frac{Y}{q} + 1)(\sum_{1\leq u\leq \frac{1}{2}q} \frac{q}{u} + X) \\
&\ll X_0 Y_0 \frac{\log q}{q}(q + X)(q + Y).
\end{aligned}$$

□

Corollary 3.6.1. *Denote*

$$S(\alpha) = \sum_y \sum_v e(\alpha y^k v),$$

where it is summed over $1 \leq y \leq P^{\frac{1}{2k}}$ and $1 \leq z \leq \frac{1}{4}P^{k-\frac{1}{2}}$, where z has the form $x_1 y_1^k + ... + x_l y_l^k$. In addition, if we have $\alpha = \frac{a}{q} + O(q^{-2})$ and $P^{\frac{1}{2}} \leq q \leq 2\max|x_j|kP^{k-1}$, then we have

$$|S(\alpha)| \ll S(0)(x_1...x_{s_0})^{\frac{1}{2(k-1)}} P^{-\frac{1}{2}(k-\frac{1}{2})(1-\lambda^l)-\frac{1}{4k}+\epsilon} A,$$

where

$$A = \max(2\max|x_j|kP^{k-1}, \frac{1}{4}P^{k-\frac{1}{2}}).$$

Proof. Following Vinogradov's theorem, we have

$$\begin{aligned}
X &= \frac{1}{4}P^{k-\frac{1}{2}},\ X_0 = U_l(\frac{1}{4}P^{k-\frac{1}{2}}), \\
Y &= P^{\frac{1}{2}}, Y_0 = P^{\frac{1}{2k}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|S(\alpha)|^2 &\ll X_0 Y_0 \frac{\log q}{q}(q + X)(q + Y) \\
&= X_0 P^{\frac{1}{2k}} (P^{\frac{1}{2}} + q)(\frac{1}{4}P^{k-\frac{1}{2}} + q)\frac{\log q}{q} \\
&\ll X_0 P^{\frac{1}{2k}} A \log P,
\end{aligned}$$

where we define

$$A = \max(2\max|x_j|kP^{k-1}, \frac{1}{4}P^{k-\frac{1}{2}}).$$

In particular, since

$$S(0) \gg P^{\frac{1}{2k}} X_0,$$

we have

$$|\frac{S(\alpha)}{S(0)}|^2 \ll X_0^{-1} P^{-\frac{1}{2k}+\epsilon} A,$$

and from lemma 3.5 we have

$$X_0 \gg P^{(k-\frac{1}{2})(1-\lambda)^l} (x_1...x_l)^{-\frac{1}{k-1}}.$$

We thus conclude that

$$|S(\alpha)| \ll S(0)(x_1...x_{s_0})^{\frac{1}{2(k-1)}} P^{-\frac{1}{2}(k-\frac{1}{2})(1-\lambda^l)-\frac{1}{4k}+\epsilon} A.$$

□

To come to the main idea of the proof, we consider the form

$$0 = x_1 y_1^k + ... x_{s_0} y_{s_0}^k + u_1 + u_2 + y^k v,$$

subject to the conditions that $1 \leq y_j \leq P$, u_1, u_2 runs through numbers less than $\frac{1}{4}P^k$ that's also of the form $x_{s_0+1} y_{s_0+1}^k + ... + x_{s_0+l} y_{s_0+l}^k$ and respectively for u_2, $1 \leq y \leq P^{\frac{1}{2k}}$ and $1 \leq z \leq \frac{1}{4}P^{k-\frac{1}{2}}$ such that z is of the form $\sum x_{s_0+2l+j} y_{s_0+2l+j}^k$. In particular we see that 0 is represented in the desired form with $s = s_0 + 3l$. We have all the tools we need to analyze the function along the minor arcs. stead of positive ones to ensure that the top integral actually converges.

Lemma 3.7. *(Minor Arc)*

$$\int_m T_1(\alpha)...T_{s_0}(\alpha)R_1(\alpha)R_2(\alpha)S(\alpha)d\alpha \ll (x_{s_0+1}...x_{s_0+3l})^{\frac{1}{2(k-1)}} P^{s_0-\frac{1}{2}(k-\frac{1}{2})(1-\lambda^l)-\frac{1}{4k}+\epsilon-k(1-\lambda^l)} R_1(0)R_2(0)S(0)A$$

Where the expressions are clearly defined as in Part III.

Proof. The proof follows directly once we apply corollary 3.6.1,3.5.1 and using the trivial bound that $|T_j(\alpha)| \leq P$. □

Theorem 3.8. *If we have*

$$\max |x_j|^{1+\frac{3l}{2(k-1)}} \leq P^{\frac{3}{2}k\lambda^l - \frac{1}{4k} - \epsilon}$$
$$\max |x_j| \leq P^{\frac{k}{s_0-k}-\epsilon}$$

for natural number l, provided that $l \geq 2k \log 3k$, then the expression

$$x_1 y_1^k + ... x_{s_0+3l} y_{s_0+3l}^k = 0$$

has infinitely many solutions.

Proof. We note that all the expressions previously suggested before Lemma 3.7 satisfy such result. Denote such number r, then we have

$$r = \int_0^1 T_1(\alpha)...T_{s_0}(\alpha)R_1(\alpha)R_2(\alpha)S(\alpha)d\alpha$$

In particular we see that for the major arc we have

$$\sum_{u_1}\sum_{u_2}\sum_{y}\sum_{v} \int_{\mathfrak{M}} T_1(\alpha)...T_{s_0}(\alpha)e(\alpha(-u_1 - u_2 - y^k v))d\alpha$$

Using the constraints suggested before Lemma 3.6, we see that they satisfy the criteria for Lemma 3.4, and thus we have major arc contribution bounded below:

$$\int_{\mathfrak{M}} T_1(\alpha)...T_{s_0}(\alpha)R_1(\alpha)R_2(\alpha)S(\alpha)d\alpha \gg R_1(0)R_2(0)S(0)\frac{P^{s_0-k}}{\max |x_j|}$$

One could also check that the minor arc is bounded above by

$$\int_m T_1(\alpha)...T_{s_0}(\alpha)R_1(\alpha)R_2(\alpha)S(\alpha)d\alpha \ll R_1(0)R_2(0)S(0)\frac{P^{s_0-k-\epsilon}}{\max |x_j|}$$

In particular, we see that infinite solutions exist provided that the constraints are satisfied. □

References

[1] Browning, Timothy D., & Heath-Brown, D.R. "Density of rational points on a quadric bundle in $\mathbb{P}^3 \times \mathbb{P}^3$." Duke Mathematical Journal, Volume 169, Issue 16, p. 3099 - 3165. https://doi.org/10.1215/00127094-2020-0031

[2] Browning, Timothy D. *Quantitative Arithmetic of Projective Varieties.* Progress in Mathematics 277, Birkhäuser Verlag, 2009.

[3] Davenport, Harold. *Analytic Methods for Diophantine Equations and Diophantine Inequalities.* Cambridge Mathematical Library, Cambridge University Press, 1963.

[4] Davenport, Harold, & Lewis, Donald J. "Homogeneous additive equations." Proceedings of the Royal Society A, Volume 274, Issue 1359, p. 443-460. https://doi.org/10.1098/rspa.1963.0143.

[5] Hu, Liqun. "Counting rational points on biprojective hypersurfaces of bidegree (1, 2)." Journal of Number Theory, Volume 214, p. 312-325. https://doi.org/10.1016/j.jnt.2020.04.002.

[6] Peyre, Emmanuel. "Chapter V: Beyond Heights: Slopes and Distribution of Rational Points." *Arakelov Geometry and Diophantine Applications*, p. 215-279. Lecture Notes in Mathematics, Volume 2276, Springer Verlag, 2021.