MODULAR PRINCIPAL SERIES REPRESENTATION OF GL_2 OVER FINITE RINGS

GAHL SHEMY

ABSTRACT. We construct a Jordan-Hölder series for the modulo p reduction of the principal series representation of $GL_2(\mathbb{F}_p[t]/(t^r))$, given any prime $p \geq 3$, $r \in \mathbb{N}$, and character χ on the Borel subgroup of $GL_2(\mathbb{F}_p[t]/(t^r))$. As a corollary we provide the semisimplifications of all characteristic p principal series representations of $GL_2(\mathbb{F}_p[t]/(t^r))$, and explain a process to compute such semisimplifications in small cases by the means of Brauer characters, apart from utilizing the known Jordan-Hölder series.

1. INTRODUCTION

A common quest in representation theory involves determining how the irreducible representations of a group "fit together" to make up some other representation of concern. For instance, given a complex representation $\rho : G \to GL(V)$ of a finite group G, Maschke's theorem guarantees that the representation is *completely reducible*, such that it can be uniquely expressed as a direct sum of irreducible representations of the group G, up to isomorphism. Maschke's theorem also holds when the representation V is over any field of characteristic 0 or over a field of characteristic p, so long as p does not divide the order of the group. In the case where V is over a field of characteristic p and p *does* divide the order of the group, Maschke's theorem no longer holds, forcing us to consider a different method of determining exactly how the irreducible modular representations of a finite group G "fit together" to make up the representation with which we are concerned. This is done through investigating Jordan-Hölder series of the representation, which are filtrations

$$0 \subset V_1 \subset \cdots \subset V_d = V$$

of subrepresentations with inclusions being proper and maximal, in the sense that each composition factor V_{i+1}/V_i is isomorphic to an irreducible representation of G. The Jordan-Hölder Theorem states that such composition series need not be unique, but that the *set* of composition factors (known as the irreducible constituents) of a representation is unique. We can then define

$$V^{ss} := \bigoplus_{i=1}^{d-1} V_{i+1}/V_i \tag{1}$$

to be the *semisimplification* of V, so that while V is not semisimple, the semisimplification of V does indeed have a direct sum decomposition of irreducible representations by construction. Since each quotient V_{i+1}/V_i is isomorphic to an irreducible representation of G, we have

$$V^{ss} = \bigoplus_{j} \rho_j^{d_j} \tag{2}$$

where ρ_j is an irreducible representation of G and d_j is its multiplicity in the semisimplification of V. A consequence of the Jordan-Hölder theorem is that V^{ss} is unique up to rearrangement of factors in the direct sum, so V^{ss} is unique up to isomorphism.

In this paper we fix a prime p and consider the non-archimedean local field $L = \mathbb{F}_p((t))$, the field of formal Laurent series in t with coefficients in $\mathbb{F}_p$. The ring of integers of L, denoted $\mathcal{O}_L$, is given by $\mathbb{F}_p[[t]]$ and consists of all formal power series in t with coefficients in $\mathbb{F}_p$. This ring has a unique maximal ideal generated by an element ϖ_L called a *uniformizer*, in this case taken to be t. For any $r \in \mathbb{N}$, we may consider the general linear group of the finite ring $(\mathbb{F}_p[t]/(t^r))^2$, that is, the group consisting of invertible 2×2 matrices with entries in $\mathbb{F}_p[t]/(t^r)$. We denote $G_r := GL_2(\mathbb{F}_p[t]/(t^r))$.

Date: August 11, 2022.
Key words and phrases. rep theory.

The choice of $L = \mathbb{F}_p((t))$ puts us in the equal characteristic setting, where the field L has the same characteristic as its residue field $\mathbb{F}_p$. For work done in the mixed characteristic setting, see the appendix in [**?**].

Given the finite group G_r, we let $B_r \leq G_r$ denote the *Borel subgroup* of G_r consisting of 2×2 upper triangular invertible matrices with entries in $\mathbb{F}_p[t]/(t^r)$. Fixing a field E of characteristic 0 whose residue field $k_E = \mathcal{O}_E/(\varpi_E)$ is of characteristic p, we let $\chi_1, \chi_2 : (\mathbb{F}_p[t]/(t^r))^\times \to E^\times$ be group homomorphisms, and define

$$\chi : B_r \to E^\times$$
$$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \mapsto \chi_1(a)\chi_2(d).$$

The *principal series representation* of G_r is the induced representation $\mathrm{Ind}_{B_r}^{G_r}(\chi)$, which is a vector space

$$\mathrm{Ind}_{B_r}^{G_r}(\chi) := \{f : G_r \to E \mid f(bg) = \chi(b)f(g) \ \forall g \in G_r, b \in B_r\} \tag{3}$$

with a G_r-action given by

$$\begin{aligned} \vartheta_\chi : G_r &\to GL(\mathrm{Ind}_{B_r}^{G_r}(\chi)) \\ \vartheta_\chi(x)(f(g)) &= f(gx) \end{aligned} \tag{4}$$

for all $x, g \in G_r, f \in \mathrm{Ind}_{B_r}^{G_r}(\chi)$. In this paper we explore the modulo p reduction of the principal series representation, where now χ maps to $k_E = \mathcal{O}_E/(\varpi_E) \cong \overline{\mathbb{F}_p}$ and all maps $f \in \mathrm{Ind}_{B_r}^{G_r}(\chi)$ have codomain k_E. From hereon we abuse notation and write $\mathrm{Ind}_{B_r}^{G_r}(\chi)$ to mean the principal series representation after reducing modulo p. Hence $\mathrm{Ind}_{B_r}^{G_r}(\chi)$ is a characteristic p vector space of dimension $[G_r : B_r] \cdot \dim(\chi) = (p+1)p^{r-1}$, with a G_r-action still given by (4).

The main result of the paper is an inductive construction of a Jordan-Hölder series for $\mathrm{Ind}_{B_r}^{G_r}(\chi)$:

Theorem 1.1. *Let $p \geq 3$ be a prime, let $r \in \mathbb{N}_{\geq 2}$, and let $\chi : B_r \to \overline{\mathbb{F}_p}^\times$ be a character. There exists a filtration for $\mathrm{Ind}_{B_r}^{G_r}(\chi)$ given by*

$$0 \subset \mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(1)}) \subset \cdots \subset \mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(p-1)}) \subset \mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma) = \mathrm{Ind}_{B_r}^{G_r}(\chi), \tag{5}$$

where $I_r^{r-1} := \{\begin{bmatrix} a & b \\ ct^{r-1} & d \end{bmatrix} \in GL_2(\mathbb{F}_p[t]/(t^r)) : c \in \mathbb{F}_p\}$, $\sigma := \mathrm{Ind}_{B_r}^{I_r^{r-1}}(\chi)$, and $\sigma^{(k)}$ is an I_r^{r-1}-invariant k-dimensional subspace of σ. Furthermore, we have that

$$\mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k+1)}) / \mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k)}) \cong \mathrm{Inf}_{G_{r-1}}^{G_r} \mathrm{Ind}_{B_{r-1}}^{G_{r-1}}(\chi \cdot (\frac{a}{d})^k) \tag{6}$$

for $0 \leq k \leq p-1$, where $\chi \cdot (\frac{a}{d})^k$ is the character $\chi \cdot (\frac{a}{d})^k : B_r \to \overline{\mathbb{F}_p}^\times$ mapping $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \mapsto \chi(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}) \cdot (\frac{a}{d})^k$.

We prove Theorem 1.1 in §3 after providing some preliminaries in §2. In §4 we give a corollary of the main theorem regarding semisimplification numbers. Finally, since determining the semisimplification of a given representation can be done without a Jordan-Hölder series via a computational process of computing Brauer characters, we compute a small example using this method in §5, and show that the semisimplification matches with what is deduced from our main theorem.

2. Preliminaries

2.1. **Basic Representation Theory.** We begin by providing key definitions from representation theory.

Definition 2.1. (Modular representation of a finite group) A *characteristic p representation* of a finite group G is a group homomorphism

$$\rho : G \to GL(V)$$

where V is a finite-dimensional vector space over a field of characteristic p and $GL(V)$ is the general linear group of V. Equivalently we may define a representation of a finite group as a group action of G on a vector space V, such that $g \cdot v = \rho(g)(v)$.

Remark 2.2. Although a representation of a group G is specified by both a vector space V and a group homomorphism ρ, we will often refer to the vector space V as the representation of G, keeping in mind that V is equipped with a G-action.

Definition 2.3. (Subrepresentations) Let $\rho : G \to GL(V)$ be a representation, and consider a subspace $W \leq V$. We say W is a *subrepresentation* of V if

$$\rho(g)(w) \in W$$

for all $g \in G, w \in W$.

Definition 2.4. (Irreducible representation) A representation $\rho : G \to GL(V)$ is *irreducible* if its only subrepresentations are the zero subspace and the whole vector space V. Otherwise we say V is *reducible*.

2.2. **Maschke's Theorem and its Converse.**

Proposition 2.5. *(Maschke's Theorem) Let G be a finite group and let $\mathbb{F}$ be a field whose characteristic does not divide $|G|$. If V is a representation of G over $\mathbb{F}$ and U is any subrepresentation of V, then V has a subrepresentation W such that $V = U \oplus W$.*

Maschke's theorem implies that every representation V of a finite group G over a field whose characteristic does not divide the order of the group can be expressed uniquely as a direct sum of irreducible representations. The converse of Maschke's theorem holds as well: if G is a finite group and V is a representation over a field $\mathbb{F}$ whose order *does* divide $|G|$, then V is not completely reducible, that is, there exists some subrepresentation U of V which has no complement subrepresentation W in V.

For a common example of Maschke's Theorem failing when the characteristic of $\mathbb{F}$ divides $|G|$, we consider the following:

Example 2.6. Let $G = \mathbb{Z}/p\mathbb{Z} = \langle g \rangle$ and let $V = \overline{\mathbb{F}_p}^2$ over $\overline{\mathbb{F}_p}$. Define an action of G on V via $g \cdot e_1 = e_1$ and $g \cdot e_2 = e_1 + e_2$, giving

$$\rho : \langle g \rangle \to GL(V)$$
$$\rho(g) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

We note that this is indeed a representation, as $\rho(0) = \rho(p \cdot g) = \rho(g)^p = \begin{bmatrix} 1 & p \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ since the characteristic of the underlying field is p. Notice that $\langle e_1 \rangle$ is stable under the action of G and that $\langle e_1 \rangle$ is isomorphic to the trivial representation. We claim that there does not exist V' a subrepresentation of V such that $V = \langle e_1 \rangle \oplus V'$. For, if there was, then $V/\langle e_1 \rangle \cong V'$. But $V/\langle e_1 \rangle$ is isomorphic to $\langle \overline{e_2} \rangle$, which, according to the action of G on V, is isomorphic to the trivial representation, as

$$g \cdot \overline{e_2} = \overline{e_1 + e_2} = \overline{e_2}.$$

This implies that V is isomorphic to the direct sum of two copies of the trivial representation, and hence that the fixed subspace of V, denoted V^G, is two-dimensional. But V^G is one-dimensional: if $\alpha_1 e_1 + \alpha_2 e_2 \in V^G$, then $g \cdot (\alpha_1 e_1 + \alpha_2 e_2) = \alpha_1 e_1 + \alpha_2(e_1 + e_2) = \alpha_1 e_1 + \alpha_2 e_2$ implies that $\alpha_2 = 0$ and hence that $V^G = \langle e_1 \rangle$.

3. Proof of Main Theorem

3.1. **Characters of B_r.**

It is known (see [1]) that every character $\chi : B_1 \to \overline{\mathbb{F}_p}^\times$ is of the form

$$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \mapsto a^\ell (ad)^s$$

for some $0 \leq \ell, s \leq p-2$. We claim that an analogue holds in the general B_r case, in the sense that every character $\chi : B_r \to \overline{\mathbb{F}_p}^\times$ is of the form

$$\begin{bmatrix} a_0 + \cdots + a_{r-1}t^{r-1} & b_0 + \cdots + b_{r-1}t^{r-1} \\ 0 & d_0 + \cdots + d_{r-1}t^{r-1} \end{bmatrix} \mapsto a_0^\ell (a_0 d_0)^s$$

for some $0 \leq \ell, s \leq p-2$, and hence only depends on the constant terms a_0, d_0 belonging to $\mathbb{F}_p^\times$.

Lemma 3.1. *Every character* $\chi_i : (\mathbb{F}_p[t]/(t^r))^\times \to \overline{\mathbb{F}_p}^\times$ *is completely determined by where it maps the constant terms belonging to* $\mathbb{F}_p^\times$. *That is,* $\chi_i(a_0 + a_1 t + \cdots + a_{r-1}t^{r-1}) = \chi_i(a_0)$.

Proof. We first show that $\chi_i : (\mathbb{F}_p[t]/(t^r))^\times \to \overline{\mathbb{F}_p}^\times$ must always map an element of the form $1 + a_1 t + \cdots + a_{r-1}t^{r-1}$ to 1. By raising such an element to the p^{th} power we see that such an element has order dividing p inside $(\mathbb{F}_p[t]/(t^r))^\times$, and thus any nonidentity element of such form has order p. Since χ_i is a group homomorphism, the image $\chi_i(1 + a_1 t + \cdots + a_{r-1}t^{r-1}) \in \overline{\mathbb{F}_p}^\times$ must have order dividing p. But no elements in $\overline{\mathbb{F}_p}^\times$ have order p, and thus $\chi_i(1 + a_1 t + \cdots + a_{r-1}t^{r-1}) = 1 = \chi_i(1)$.

Now $\chi_i(a_0 + \cdots + a_{r-1}t^{r-1}) = \chi_i(a_0 \cdot (1 + \frac{a_1}{a_0} t + \cdots + \frac{a_{r-1}}{a_0})) = \chi_i(a_0)\chi_i(1 + \frac{a_1}{a_0} t + \cdots + \frac{a_{r-1}}{a_0}) = \chi_i(a_0)$, completing the proof. □

Lemma 3.2. *Every multiplicative map* $\chi : B_r \to \overline{\mathbb{F}_p}^\times$ *is of the form*

$$\chi : B_r \to (\mathbb{F}_p[t]/(t^r))^\times$$

$$\begin{bmatrix} a_0 + \cdots + a_{r-1}t^{r-1} & b \\ 0 & d_0 + \cdots + d_{r-1}t^{r-1} \end{bmatrix} \mapsto a_0^\ell (a_0 d_0)^s$$

for some $0 \leq \ell, s \leq p-2$.

Proof. We first show that any matrix $\begin{bmatrix} 1 + \cdots + a_{r-1}t^{r-1} & b \\ 0 & 1 + \cdots + d_{r-1}t^{r-1} \end{bmatrix}$ must get mapped to 1 in $\mathbb{F}_p^\times$ under any multiplicative map χ. Notice that

$$\begin{bmatrix} 1 + \cdots + a_{r-1}t^{r-1} & b \\ 0 & 1 + \cdots + d_{r-1}t^{r-1} \end{bmatrix}^p = \begin{bmatrix} 1 + \cdots & pb(1 + \cdots) \\ 0 & 1 + \cdots \end{bmatrix}$$

and since $pb \equiv 0$ in $\mathbb{F}_p$, we must have that

$$\chi(\begin{bmatrix} 1 + \cdots & b \\ 0 & 1 + \cdots \end{bmatrix})^p = \chi(\begin{bmatrix} 1 + \cdots & b \\ 0 & 1 + \cdots \end{bmatrix}^p) = \chi(\begin{bmatrix} 1 + \cdots & 0 \\ 0 & 1 + \cdots \end{bmatrix}).$$

Because any multiplicative map on a diagonal matrix in G_r must be the product of two multiplicative maps on each entry in the diagonal, and since such diagonal elements belong to $(\mathbb{F}_p[t]/(t^r))^\times$, each of the two multiplicative maps must be of the form in Lemma 3.1. In particular this shows that $\chi(\begin{bmatrix} 1 + \cdots & b \\ 0 & 1 + \cdots \end{bmatrix}) = 1$.

Now any matrix $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \in B_r$ can be expressed as

$$\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \cdot \begin{bmatrix} 1 & a^{-1}b \\ 0 & 1 \end{bmatrix}$$

so $\chi(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}) = \chi(\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix})$. But a multiplicative map on a diagonal matrix is again just the product of multiplicative maps on its diagonal entries, implying that $\chi = \chi_1 \times \chi_2$ where each χ_i is a map as in Lemma 3.1. In particular, since Lemma 3.1 shows that $\chi_i(a_0 + a_1 t + \cdots + a_{r-1}t^{r-1}) = \chi_i(a_0)$ for an element $a_0 + \cdots + a_{r-1}t^{r-1} \in (\mathbb{F}_p[t]/(t^r))^\times$, then we conclude

$$\chi(\begin{bmatrix} a_0 + \cdots + a_{r-1}t^{r-1} & b \\ 0 & d_0 + \cdots + d_{r-1}t^{r-1} \end{bmatrix}) = \chi_1(a_0) \cdot \chi_2(d_0).$$

But both a_0 and d_0 belong to $\mathbb{F}_p^\times$, a cyclic group of order $p-1$, and hence $\chi_1(a_0)$ and $\chi_2(d_0)$ must be $(p-1)^{st}$ roots of unity in $\overline{\mathbb{F}_p}^\times$. Since all $p-1$ such roots of unity lie in $\mathbb{F}_p^\times \subset \overline{\mathbb{F}_p}^\times$, then both χ_1 and χ_2 map into $\mathbb{F}_p^\times$, which is cyclic of order $p-1$. This implies that $\chi_1(a_0) = a_0^m$ for some $0 \leq m \leq p-2$ and $\chi_2(d_0) = d_0^s$ for some $0 \leq s \leq p-2$. Alternatively, we can express $a_0^m \cdot d_0^s$ as $a_0^\ell (a_0 d_0)^s$ where $\ell = m - s \mod p$. ☐

Remark 3.3. In this paper we abuse notation and write, for instance, $\frac{a}{d} : B_r \to \overline{\mathbb{F}_p}^\times$ to mean the map sending $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \mapsto a_0 d_0^{-1} = a_0 d_0^{p-2}$, since the lemmas above guarantee that any character $\chi : \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \to \overline{\mathbb{F}_p}^\times$ is of the form $a_0^\ell (a_0 d_0)^s$.

3.2. **Induction from Borel subgroup.** Let $\chi : B_r \to \overline{\mathbb{F}_p}^\times$ be a character. For $r \geq 2$, we define the Iwahori subgroup

$$I_r^{r-1} := \{ \begin{bmatrix} a & b \\ ct^{r-1} & d \end{bmatrix} \in G_r \} \tag{7}$$

to be the invertible matrices in G_r whose $(2,1)$-entry have no terms of the form $c_k t^k$ for $0 \leq k \leq r-2$. Equivalently, we may define I_r^{r-1} to be the preimage of B_{r-1} under the surjective homomorphism

$$\begin{aligned} \pi : G_r &\twoheadrightarrow G_{r-1} \\ t^{r-1} &\mapsto 0. \end{aligned} \tag{8}$$

Let $\sigma := \mathrm{Ind}_{B_r}^{I_r^{r-1}}(\chi)$. Because $\dim(\sigma) = [I_r^{r-1} : B_r] = p$, we fix a basis $\{\delta_0, \ldots, \delta_{p-1}\}$ of σ by setting

$$\begin{aligned} \delta_j &: I_r^{r-1} \to \overline{\mathbb{F}_p}^\times \\ \delta_j(i) &= \mathbb{1}_{B_r x_j} \cdot \chi(i x_j^{-1}) \end{aligned} \tag{9}$$

where $B_r x_j := B_r \begin{bmatrix} 1 & 0 \\ jt^{r-1} & 1 \end{bmatrix}$ and $\mathbb{1}$ refers to the indicator function. It is clear that these p functions are linearly independent as they each have support on a distinct right coset of B_r in I_r^{r-1}, and that these functions truly belong to σ, as if $bi \in B_r x_j$, we have

$$\delta_j(bi) = \chi(bix_j^{-1}) = \chi(b)\delta_j(i)$$

and if $bi \notin B_r x_j$, then $i \notin B_r x_j$, and

$$\delta_j(bi) = 0 = \chi(b)\delta_j(i).$$

We note that by composition of induction, constructing a Jordan-Hölder series for $\mathrm{Ind}_{B_r}^{G_r}(\chi)$ is equivalent to constructing a Jordan-Hölder series for $\mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma)$. Therefore one may initially construct a Jordan-Hölder series for σ and then "induce up" to get a filtration for $\mathrm{Ind}_{B_r}^{G_r}(\chi)$, which can then be further refined to a full composition series for $\mathrm{Ind}_{B_r}^{G_r}(\chi)$. Since this is the approach we take in Theorem 1.1, we must first construct a Jordan-Hölder series for σ:

Proposition 3.4. *For every $0 \leq k \leq p$ there exists a k-dimensional I_r^{r-1}-invariant subspace $\sigma^{(k)}$ of σ, such that*

$$0 \subset \sigma^{(1)} \subset \cdots \sigma^{(p-1)} \subset \sigma$$

is a Jordan-Hölder series for σ.

Proof. The cases of $k = 0$ and $k = p$ are trivial. For each $1 \leq k \leq p-1$, we construct a k-dimensional subspace of σ, denoted $\sigma^{(k)}$, as follows:

$$\sigma^{(k)} = \langle \sum_{j=0}^{p-1} \binom{j}{j} \delta_j, \sum_{j=0}^{p-2} \binom{j+1}{j} \delta_j, \ldots, \sum_{j=0}^{p-k} \binom{j+k-1}{j} \delta_j \rangle. \tag{10}$$

To see that the vectors $\{\sum_{j=0}^{p-\ell}\binom{j+\ell-1}{j}\delta_j : 1 \leq \ell \leq k\}$ are linearly independent (and hence form a basis for $\sigma^{(k)}$), we notice that if we express each sum as a tuple in the basis $\{\delta_0, \ldots, \delta_{p-1}\}$, then putting the k p-tuples into a $p \times k$ matrix gives

$$A = \begin{bmatrix} \binom{0}{0} & \binom{1}{0} & \binom{2}{0} & \cdots & \binom{k-1}{0} \\ \binom{1}{1} & \binom{2}{1} & \binom{3}{1} & \cdots & \binom{k}{1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{p-2}{p-2} & \binom{p-1}{p-2} & 0 & \cdots & 0 \\ \binom{p-1}{p-1} & 0 & 0 & \cdots & 0 \end{bmatrix}_{p\times k}. \tag{11}$$

We verify that the columns $\{\vec{v_1}, \ldots, \vec{v_k}\}$ are linearly independent by noting that if

$$a_1\vec{v_1} + \cdots + a_k\vec{v_k} = 0$$

then in particular $a_1\binom{p-1}{p-1} = 0$, implying that $a_1 = 0$. Then since $a_1\binom{p-2}{p-2} + a_2\binom{p-1}{p-2} = 0$, we deduce that $a_2 = 0$. The fact that $A_{ij} = 0$ for $j \geq p - i + 2$ allows us to inductively deduce that $a_i = 0$ for $1 \leq i \leq k$.

To see that $\sigma^{(k)}$ is I_r^{r-1}-invariant and therefore a subrepresentation of σ, we check that it is invariant under every generator of I_r^{r-1}. By the Iwahori factorization of I_r^{r-1}, we have that any matrix $\begin{bmatrix} a & b \\ ct^{r-1} & d \end{bmatrix} \in I_r^{r-1}$ can be expressed as

$$\begin{bmatrix} a & b \\ ct^{r-1} & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ ca^{-1}t^{r-1} & 1 \end{bmatrix} \cdot \begin{bmatrix} a & 0 \\ 0 & -ca^{-1}bt^{r-1} + d \end{bmatrix} \cdot \begin{bmatrix} 1 & ba^{-1} \\ 0 & 1 \end{bmatrix}$$

which allows us to conclude that

$$I_r^{r-1} = \langle \begin{bmatrix} 1 & t^k \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ t^{r-1} & 1 \end{bmatrix}, \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \rangle \tag{12}$$

where k ranges from 0 to $r-1$ and a, d belong to $(\mathbb{F}_p[t]/(t^r))^\times$. In order to determine how I_r^{r-1} acts on each subspace $\sigma^{(k)}$, we first determine how each generator of I_r^{r-1} given in (12) acts on each basis vector δ_j of σ.

Lemma 3.5. *Let $\chi : B_r \to \overline{\mathbb{F}_p}^\times$ be a character of B_r and let $\sigma = \mathrm{Ind}_{B_r}^{I_r^{r-1}}(\chi)$. Let $\{\delta_0, \ldots, \delta_{p-1}\}$ be the ordered basis of σ given in* (9)*. Then the generators of I_r^{r-1} act on each δ_j via*

$$\begin{bmatrix} 1 & t^k \\ 0 & 1 \end{bmatrix} \cdot \delta_j = \delta_j \tag{13}$$

$$\begin{bmatrix} 1 & 0 \\ t^{r-1} & 1 \end{bmatrix} \cdot \delta_j = \delta_{j-1} \tag{14}$$

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \cdot \delta_j = \chi(\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}) \cdot \delta_{\frac{d}{a}j} \tag{15}$$

where all indices j are taken modulo p.

Proof. We have that

$$(\begin{bmatrix} 1 & t^k \\ 0 & 1 \end{bmatrix} \cdot \delta_j)(i) \neq 0 \iff \delta_j(i\begin{bmatrix} 1 & t^k \\ 0 & 1 \end{bmatrix}) \neq 0$$

by definition of the G_r action on σ. But

$$\delta_j(i\begin{bmatrix} 1 & t^k \\ 0 & 1 \end{bmatrix}) \neq 0 \iff i\begin{bmatrix} 1 & t^k \\ 0 & 1 \end{bmatrix} \in B_r\begin{bmatrix} 1 & 0 \\ jt^{r-1} & 1 \end{bmatrix} \iff i \in B_r\begin{bmatrix} 1 & 0 \\ jt^{r-1} & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -t^k \\ 0 & 1 \end{bmatrix} \iff i \in B_r\begin{bmatrix} 1 & 0 \\ jt^{r-1} & 1 \end{bmatrix}$$

such that $\begin{bmatrix}1 & t^k\\0 & 1\end{bmatrix}\cdot\delta_j$ only has support on B_rx_j. Now suppose $i\in B_rx_j$, so $i = b\cdot\begin{bmatrix}1 & 0\\jt^{r-1} & 1\end{bmatrix}$ for some $b\in B_r$. Then

$$\begin{aligned}(\begin{bmatrix}1 & t^k\\0 & 1\end{bmatrix}\cdot\delta_j)(i) = \delta_j(b\begin{bmatrix}1 & 0\\jt^{r-1} & 1\end{bmatrix}\cdot\begin{bmatrix}1 & t^k\\0 & 1\end{bmatrix}) = \delta_j(b\begin{bmatrix}1 & t^k\\jt^{r-1} & jt^{r-1+k}+1\end{bmatrix}) &= \chi(b\begin{bmatrix}1 & t^k\\jt^{r-1} & jt^{r-1+k}+1\end{bmatrix}\cdot\begin{bmatrix}1 & 0\\-jt^{r-1} & 1\end{bmatrix})\\ &= \chi(b\begin{bmatrix}1-jt^{r+k-1} & t^k\\-j^2t^{2r-2+k} & jt^{r-1+k}+1\end{bmatrix})\\ &= \chi(b)\chi(\begin{bmatrix}1-jt^{r+k-1} & t^k\\0 & 1+jt^{r-1+k}\end{bmatrix})\\ &= \delta_j(i)\end{aligned}$$

since $\chi(\begin{bmatrix}1+\cdots & b\\0 & 1+\cdots\end{bmatrix}) = 1$ by the proof of Lemma 3.2. Hence $\begin{bmatrix}1 & t^k\\0 & 1\end{bmatrix}\cdot\delta_j = \delta_j$. A similar argument shows that $\begin{bmatrix}1 & 0\\t^{r-1} & 1\end{bmatrix}\cdot\delta_j$ only has support on B_rx_{j-1}, and if $i = bx_{j-1}$ for some $b\in B_rx_{j-1}$, then

$$(\begin{bmatrix}1 & 0\\t^{r-1} & 1\end{bmatrix}\cdot\delta_j)(b\begin{bmatrix}1 & 0\\(j-1)t^{r-1} & 1\end{bmatrix}) = \delta_j(b\begin{bmatrix}1 & 0\\(j-1)t^{r-1} & 1\end{bmatrix}\begin{bmatrix}1 & 0\\t^{r-1} & 1\end{bmatrix}) = \delta_j(b\begin{bmatrix}1 & 0\\jt^{r-1} & 1\end{bmatrix}) = \chi(b) = \delta_{j-1}(i),$$

allowing us to conclude $\begin{bmatrix}1 & 0\\t^{r-1} & 1\end{bmatrix}\cdot\delta_j = \delta_{j-1}$. Finally, an analogous computation shows that $\begin{bmatrix}a & 0\\0 & d\end{bmatrix}\cdot\delta_j$ only has support on $B_rx_{\frac{d}{a}j}$, so we suppose $i = b\begin{bmatrix}1 & 0\\\frac{d}{a}jt^{r-1} & 1\end{bmatrix}$ for some $b\in B_r$, and find that

$$(\begin{bmatrix}a & 0\\0 & d\end{bmatrix}\cdot\delta_j)(i) = \delta_j(b\begin{bmatrix}1 & 0\\\frac{d}{a}jt^{r-1} & 1\end{bmatrix}\begin{bmatrix}a & 0\\0 & d\end{bmatrix}) = \delta_j(b\begin{bmatrix}a & 0\\djt^{r-1} & d\end{bmatrix}) = \chi(b\begin{bmatrix}a & 0\\djt^{r-1} & d\end{bmatrix}\begin{bmatrix}1 & 0\\-jt^{r-1} & 1\end{bmatrix}) = \chi(b)\chi(\begin{bmatrix}a & 0\\0 & d\end{bmatrix})$$

whereas

$$\delta_{\frac{d}{a}j}(b\begin{bmatrix}1 & 0\\\frac{d}{a}jt^{r-1} & 1\end{bmatrix}) = \chi(b)$$

by definition, which shows that $\begin{bmatrix}a & 0\\0 & d\end{bmatrix}\cdot\delta_j = \chi(\begin{bmatrix}a & 0\\0 & d\end{bmatrix})\delta_{\frac{d}{a}j}$ as desired.

□

Recall that we wish to show $\sigma^{(k)}$ is I_r^{r-1}-invariant. Consider the sum $\sum_{j=0}^{p-\ell}\binom{j+\ell-1}{j}\delta_j\in\sigma^{(k)}$ for $1\le\ell\le k$. Since $\begin{bmatrix}1 & t^k\\0 & 1\end{bmatrix}$ acts trivially on each δ_j, then certainly $\begin{bmatrix}1 & t^k\\0 & 1\end{bmatrix}\cdot\sum_{j=0}^{p-\ell}\binom{j+\ell-1}{j}\delta_j = \sum_{j=0}^{p-\ell}\binom{j+\ell-1}{j}\delta_j\in\sigma^{(k)}$ for each ℓ. The actions by the other generators are more involved, so we provide them as lemmas.

Lemma 3.6.

$$\begin{bmatrix}1 & 0\\t^{r-1} & 1\end{bmatrix}\cdot\sum_{j=0}^{p-\ell}\binom{j+\ell-1}{j}\delta_j = \sum_{m=1}^{\ell}\sum_{j=0}^{p-m}\binom{j+m-1}{j}\delta_j \tag{16}$$

so that if the basis vectors of $\sigma^{(k)}$ are ordered, then acting on each basis vector by $\begin{bmatrix}1 & 0\\t^{r-1} & 1\end{bmatrix}$ yields a sum of the vector being acted on and the preceding basis vectors, thus remaining in $\sigma^{(k)}$.

Proof. We prove (16) by induction on ℓ: when $\ell = 1$, we have

$$\begin{aligned}\begin{bmatrix}1 & 0\\ t^{r-1} & 1\end{bmatrix}\cdot\sum_{j=0}^{p-1}\binom{j}{j}\delta_j &= \sum_{j=0}^{p-1}\binom{j}{j}\begin{bmatrix}1 & 0\\ t^{r-1} & 1\end{bmatrix}\cdot\delta_j\\ &= \sum_{j=0}^{p-1}\binom{j}{j}\delta_{j-1}\\ &= \sum_{j=0}^{p-1}\binom{j}{j}\delta_j\end{aligned}$$

so that the base case holds. Now suppose (16) holds for some $\ell \in \mathbb{N}, \ell < k$. We wish to show the claim holds for $\ell+1$. By the binomial coefficient recurrence relation $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ (where $\binom{n-1}{k-1} = 0$ whenever $k < 1$), and by the fact that we can express $\sum_{j=0}^{p-(\ell+1)}\binom{j+\ell}{j}\delta_j = \sum_{j=0}^{p-\ell}\binom{j+\ell}{j}\delta_j$ since the coefficient $\binom{p}{p-\ell}$ in front of $\delta_{p-\ell}$ is zero mod p, we get

$$\begin{aligned}\begin{bmatrix}1 & 0\\ t^{r-1} & 1\end{bmatrix}\cdot\sum_{j=0}^{p-(\ell+1)}\binom{j+\ell}{j}\delta_j &= \begin{bmatrix}1 & 0\\ t^{r-1} & 1\end{bmatrix}\cdot\sum_{j=0}^{p-\ell}\binom{j+\ell}{j}\delta_j\\ &= \begin{bmatrix}1 & 0\\ t^{r-1} & 1\end{bmatrix}\cdot\left(\sum_{j=0}^{p-\ell}\binom{j+\ell-1}{j}\delta_j + \sum_{j=0}^{p-\ell}\binom{j+\ell-1}{j-1}\delta_j\right). \qquad (17)\end{aligned}$$

Our inductive hypothesis guarantees that

$$\begin{bmatrix}1 & 0\\ t^{r-1} & 1\end{bmatrix}\cdot\sum_{j=0}^{p-\ell}\binom{j+\ell-1}{j}\delta_j = \sum_{m=0}^{\ell}\sum_{j=0}^{p-m}\binom{j+m-1}{j}\delta_j, \qquad (18)$$

while

$$\begin{aligned}\begin{bmatrix}1 & 0\\ t^{r-1} & 1\end{bmatrix}\cdot\sum_{j=0}^{p-\ell}\binom{j+\ell-1}{j-1}\delta_j &= \sum_{j=0}^{p-\ell}\binom{j+\ell-1}{j-1}\delta_{j-1}\\ &= \sum_{j=1}^{p-\ell}\binom{j+\ell-1}{j-1}\delta_{j-1}\\ &= \sum_{j=0}^{p-(\ell+1)}\binom{j+\ell}{j}\delta_j \qquad (19)\end{aligned}$$

since the coefficient $\binom{j+\ell-1}{j-1} = 0$ for $j = 0$, by convention. Hence from (17), (18) and (19), we conclude that

$$\begin{bmatrix}1 & 0\\ t^{r-1} & 1\end{bmatrix}\cdot\sum_{j=0}^{p-(\ell+1)}\binom{j+\ell}{j}\delta_j = \sum_{m=1}^{\ell+1}\sum_{j=0}^{p-m}\binom{j+m-1}{j}\delta_j \qquad (20)$$

as desired, confirming $\sigma^{(k)}$ is indeed invariant under $\begin{bmatrix}1 & 0\\ t^{r-1} & 1\end{bmatrix}$. $\square$

It now suffices to show that $\sigma^{(k)}$ is invariant under $\begin{bmatrix}a & 0\\ 0 & d\end{bmatrix}$. As in the $\begin{bmatrix}1 & 0\\ t^{r-1} & 1\end{bmatrix}$ case, we show that acting by $\begin{bmatrix}a & 0\\ 0 & d\end{bmatrix}$ on $\sum_{j=0}^{p-\ell}\binom{j+\ell-1}{j}\delta_j \in \sigma^{(k)}$ yields an $\overline{\mathbb{F}_p}$-linear combination of $\sum_{j=0}^{p-m}\binom{j+m-1}{j}\delta_j \in \sigma^{(k)}$ for $m \leq \ell$, and hence belongs to $\sigma^{(k)}$. Explicitly, we claim:

Lemma 3.7. *Let* $\alpha_i = \binom{(p-i)\frac{a}{d}+\ell-1}{(p-i)\frac{a}{d}}$. *Then*

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \cdot \sum_{j=0}^{p-\ell} \binom{j+\ell-1}{j} \delta_j = \chi(\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}) \sum_{m=1}^{\ell} c_m \sum_{j=0}^{p-m} \binom{j+m-1}{j} \delta_j \tag{21}$$

where each c_m *is given by* $\sum_{i=1}^{m} (-1)^{i+1} \binom{m-1}{i-1} \alpha_i$.

Proof. By the action of $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ on each δ_j, we have

$$\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \cdot \sum_{j=0}^{p-\ell} \binom{j+\ell-1}{j} \delta_j = \chi(\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}) \sum_{j=0}^{p-\ell} \binom{j+\ell-1}{j} \delta_{\frac{d}{a}j}. \tag{22}$$

For $0 \le n \le p-1$, we see that δ_n appears in the right hand sum of (22) with a coefficient of $\chi(\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix})\binom{n\frac{a}{d}+\ell-1}{n\frac{a}{d}}$, and since δ_n appears in each vector $\sum_{j=0}^{p-m} \binom{j+m-1}{j}\delta_j$ with a coefficient of $\binom{n+m-1}{n}$ for the respective $1 \le m \le \ell$, it suffices to verify

$$c_1\binom{n}{n} + c_2\binom{n+1}{n} + \cdots + c_\ell\binom{n+\ell-1}{n} = \binom{n\frac{a}{d}+\ell-1}{n\frac{a}{d}}$$

for the proposed coefficients $c_1, \ldots, c_\ell$. That is, we wish to show

$$\sum_{r=1}^{\ell} \binom{n+r-1}{n} \sum_{i=1}^{r} (-1)^{i+1} \binom{r-1}{i-1} \alpha_i = \alpha_{p-n}. \tag{23}$$

Noticing how often each α_r appears in the left hand side of (23) allows us to express

$$\sum_{r=1}^{\ell} \binom{n+r-1}{n} c_r = \sum_{r=1}^{\ell} (-1)^{r+1} \left(\sum_{j=r-1}^{\ell-1} \binom{j+n}{n} \binom{j}{r-1} \right) \alpha_r \tag{24}$$

such that the new goal is to show

$$\sum_{r=1}^{\ell} (-1)^{r+1} \left(\sum_{j=r-1}^{\ell-1} \binom{j+n}{n} \binom{j}{r-1} \right) \alpha_r = \alpha_{p-n}. \tag{25}$$

When $n = 0$, we need to show that $\sum_{r=1}^{\ell} \binom{r-1}{0} c_r = \alpha_p = \binom{\ell-1}{0} = 1$. To see this, notice that by (24) we know that $\sum_{r=1}^{\ell} c_r = \sum_{r=1}^{\ell} (-1)^{r+1} \sum_{j=r-1}^{\ell-1} \binom{j}{0}\binom{j}{r-1}\alpha_r = \sum_{r=1}^{\ell}(-1)^{r+1}\binom{\ell}{r}\alpha_r$. Writing $\alpha_1 = \binom{(p-1)\frac{a}{d}+\ell-1}{(p-1)\frac{a}{d}} = \frac{1}{(\ell-1)!}(\ell-1-\frac{a}{d})\cdots(1-\frac{a}{d})$ and letting the variable x stand in for $\frac{a}{d}$, we have that

$$\alpha_1 = \frac{1}{(\ell-1)!}(a_{\ell-1}x^{\ell-1} + a_{\ell-2}x^{\ell-2} + \cdots + a_1 x + (\ell-1)!)$$

for some coefficients $a_{\ell-1}, \ldots, a_1$. Notice then that $\alpha_r = \frac{1}{(\ell-1)!}((-1)^{\ell-1}r^{\ell-1}x^{\ell-1} + \cdots + a_1 r x + (\ell-1)!)$, so that the constant term of $\sum_{r=1}^{\ell} c_r$, when viewed as a polynomial in $x = \frac{a}{d}$, is given by

$$\sum_{r=1}^{\ell}(-1)^{r+1}\binom{\ell}{r}\frac{(\ell-1)!}{(\ell-1)!} = (-1)\sum_{r=1}^{\ell}(-1)^r\binom{\ell}{r} = (-1)\sum_{r=0}^{\ell}(-1)^r\binom{\ell}{r} - (-1) = 1$$

since $\sum_{r=0}^{\ell}(-1)^r\binom{\ell}{r} = 0$. On the other hand, the coefficient of x^m in the polynomial $\sum_{r=1}^{\ell} c_r$ for $1 \le m \le \ell-1$ is given by

$$\sum_{r=1}^{\ell}(-1)^{r+1}r^m\binom{\ell}{r}\frac{a_m}{(\ell-1)!} = \frac{-a_m}{(\ell-1)!}\sum_{r=0}^{\ell}(-1)^r r^m\binom{\ell}{r} = 0$$

due to the combinatorial sum identity $\sum_{r=0}^{\ell}(-1)^r r^m\binom{\ell}{r} = 0$ given in [3]. We conclude that $\sum_{r=1}^{\ell} c_r = 1 = \alpha_p$ as desired.

To prove $\sum_{r=1}^{\ell}\binom{n+r-1}{n}c_r = \alpha_{p-n}$ for $1 \leq n \leq p-1$, we compare the coefficient of x^m in both expressions. Since the coefficient of x^m in α_r is given by $\frac{a_m}{(\ell-1)!}r^m$, then from (24) we deduce that the coefficient of x^m in $\sum_{r=1}^{\ell}\binom{n+r-1}{n}c_r$ must be $\sum_{r=1}^{\ell}(-1)^{r+1}\frac{a_m}{(\ell-1)!}r^m\sum_{j=r-1}^{\ell-1}\binom{j+n}{n}\binom{j}{r-1}$. On the other hand, the coefficient of x^m in α_{p-n} is given by $(-n)^m\frac{a_m}{(\ell-1)!}$, so it suffices to prove

$$\sum_{r=1}^{\ell}(-1)^{r+1}r^m\sum_{j=r-1}^{\ell-1}\binom{j+n}{n}\binom{j}{r-1} = (-n)^m. \tag{26}$$

Because $\binom{j}{r-1} = 0$ whenever $j < r-1$, we can express the left hand side of (26) as

$$\sum_{r=1}^{\ell}(-1)^{r+1}r^m\sum_{j=0}^{\ell-1}\binom{j+n}{n}\binom{j}{r-1}. \tag{27}$$

Identity 3.155 in [2] tells us that $\sum_{k=0}^{s-1}\binom{k}{n}\binom{k+m}{m} = \binom{s}{n}\binom{s+m}{m}\frac{s-n}{m+n+1}$, which allows us to express (27) as

$$\begin{aligned}
\sum_{r=1}^{\ell}(-1)^{r+1}r^m\sum_{j=0}^{\ell-1}\binom{j+n}{n}\binom{j}{r-1} &= \sum_{r=1}^{\ell}(-1)^{r+1}r^m\binom{\ell}{r-1}\binom{\ell+n}{n}\frac{\ell-r+1}{r+n}\\
&= \binom{\ell+n}{n}\sum_{r=1}^{\ell}(-1)^{r+1}r^m\binom{\ell}{r-1}\frac{\ell-r+1}{r+n}\\
&= \binom{\ell+n}{n}\sum_{r=1}^{\ell}(-1)^{r+1}r^m\cdot r\binom{\ell}{r}\frac{1}{r+n}\\
&= \binom{\ell+n}{n}\sum_{r=1}^{\ell}(-1)^{r+1}\binom{\ell}{r}\frac{r^{m+1}}{r+n}. \qquad (28)
\end{aligned}$$

Finally, identity 1.47 in [2] shows that $\sum_{k=0}^{\ell}(-1)^k\binom{\ell}{k}\frac{k^j}{x+k} = (-1)^j\frac{x^{j-1}}{\binom{x+\ell}{\ell}}$, and therefore (28) becomes

$$\begin{aligned}
\binom{\ell+n}{n}\sum_{r=1}^{\ell}(-1)^{r+1}\binom{\ell}{r}\frac{r^{m+1}}{r+n} &= \binom{\ell+n}{n}(-1)\sum_{r=0}^{\ell}(-1)^{r}\binom{\ell}{r}\frac{r^{m+1}}{r+n}\\
&= \binom{\ell+n}{n}(-1)(-1)^{m+1}\frac{n^m}{\binom{n+\ell}{\ell}}\\
&= (-1)^m n^m\\
&= (-n)^m \qquad (29)
\end{aligned}$$

as desired. This proves that there exist $c_1,\dots,c_\ell \in \overline{\mathbb{F}_p}^{\times}$ such that

$$\sum_{j=0}^{p-\ell}\binom{j+\ell-1}{j}\delta_{\frac{d}{a}j} = \sum_{m=1}^{\ell}c_m\sum_{j=0}^{p-m}\binom{j-m+1}{j}\delta_j \tag{30}$$

which means that there exist $c_1,\dots,c_\ell \in \overline{\mathbb{F}_p}^{\times}$ such that

$$\begin{bmatrix} a & 0\\ 0 & d\end{bmatrix}\cdot\sum_{j=0}^{p-\ell}\binom{j+\ell-1}{j}\delta_j = \sum_{m=1}^{\ell}\chi(\begin{bmatrix} a & 0\\ 0 & d\end{bmatrix})c_m\sum_{j=0}^{p-m}\binom{j-m+1}{j}\delta_j. \tag{31}$$

Because this holds for all $1 \leq \ell \leq k$, we have that $\sigma^{(k)}$ is invariant under action by $\begin{bmatrix} a & 0\\ 0 & d\end{bmatrix}$. □

Proposition 3.4 gives us a p-dimensional Jordan-Hölder series

$$0 \subset \sigma^{(1)} \subset \cdots \subset \sigma^{(p-1)} \subset \sigma.$$

Since each $\sigma^{(k)}$ is a subrepresentation of σ which is itself a representation of I_r^{r-1}, then inducing each $\sigma^{(k)}$ to G_r gives a filtration

$$0 \subset \operatorname{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(1)}) \subset \cdots \subset \operatorname{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(p-1)}) \subset \operatorname{Ind}_{I_r^{r-1}}^{G_r}(\sigma).$$

In order to refine this filtration to a composition series for $\operatorname{Ind}_{I_r^{r-1}}^{G_r}(\sigma) = \operatorname{Ind}_{B_r}^{G_r}(\chi)$, we note that it suffices to find a composition series for $\operatorname{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k+1)})$ which begins with $\operatorname{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k)})$ for each $0 \leq k \leq p-1$. But this is equivalent to finding a composition series for $\operatorname{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k+1)})/\operatorname{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k)})$ and then lifting the subrepresentations under the projection map $p : \operatorname{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k+1)}) \to \operatorname{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k+1)})/\operatorname{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k)})$. Furthermore, since

$$\operatorname{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k+1)})/\operatorname{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k)}) \cong \operatorname{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k+1)}/\sigma^{(k)})$$

then we only need to consider composition series of $\operatorname{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k+1)}/\sigma^{(k)})$ in order to answer our original question.

We claim that $\sigma^{(k+1)}/\sigma^{(k)}$ is equivalent to $\operatorname{Inf}_{B_{r-1}}^{I_r^{r-1}}(\chi\cdot(\frac{a}{d})^k)$ as one-dimensional I_r^{r-1} representations, where $\operatorname{Inf}_{B_{r-1}}^{I_r^{r-1}}(\chi\cdot(\frac{a}{d})^k)$ refers to the inflation to I_r^{r-1} of the character sending $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \mapsto \chi(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix})\cdot(\frac{a}{d})^k \in \overline{\mathbb{F}_p}^\times$. To prove this equivalence it suffices to show that I_r^{r-1} acts on $\sigma^{(k+1)}/\sigma^{(k)}$ via multiplication by $\chi\cdot(\frac{a}{d})^k$. Again we show this claim only for the three types of generators of I_r^{r-1}. □

Lemma 3.8. *The generators* $\begin{bmatrix} 1 & t^\ell \\ 0 & 1 \end{bmatrix}$ *and* $\begin{bmatrix} 1 & 0 \\ t^{r-1} & 1 \end{bmatrix}$ *act trivially on* $\sigma^{(k+1)}/\sigma^{(k)}$ *for* $0 \leq \ell \leq r-1$ *and* $0 \leq k \leq p-1$.

Proof. Notice that $\sigma^{(k+1)}/\sigma^{(k)} = \langle \sum_{j=0}^{p-(k+1)} \binom{j+k}{j}\overline{\delta_j}\rangle$, where $\overline{\delta_{p-1}}, \ldots, \overline{\delta_{p-k}}$ are defined by the equations $\sum_{j=0}^{p-m}\binom{j+m-1}{j}\overline{\delta_j} = 0$ for $1 \leq m \leq k$. Since $\begin{bmatrix} 1 & t^\ell \\ 0 & 1 \end{bmatrix}$ acts trivially on each δ_j, then clearly $\begin{bmatrix} 1 & t^\ell \\ 0 & 1 \end{bmatrix}$ acts trivially on $\sum_{j=0}^{p-(k+1)}\binom{j+k}{j}\overline{\delta_j}$, which generates $\sigma^{(k+1)}/\sigma^{(k)}$. On the other hand, by the proof of Lemma 3.6, we know that

$$\begin{aligned}
\begin{bmatrix} 1 & 0 \\ t^{r-1} & 1 \end{bmatrix} \cdot \sum_{j=0}^{p-(k+1)} \binom{j+k}{j}\overline{\delta_j} &= \sum_{m=1}^{k+1}\sum_{j=0}^{p-m}\binom{j+m-1}{j}\overline{\delta_j} \\
&= \sum_{m=1}^{k}\sum_{j=0}^{p-m}\binom{j+m-1}{j}\overline{\delta_j} + \sum_{j=0}^{p-(k+1)}\binom{j+k}{j}\overline{\delta_j} \\
&= \begin{bmatrix} 1 & 0 \\ t^{r-1} & 1 \end{bmatrix} \cdot \sum_{j=0}^{p-k}\binom{j+k-1}{j}\overline{\delta_j} + \sum_{j=0}^{p-(k+1)}\binom{j+k}{j}\overline{\delta_j} \\
&= \sum_{j=0}^{p-(k+1)}\binom{j+k}{j}\overline{\delta_j} \qquad (32)
\end{aligned}$$

where (32) follows from the fact that $\sum_{j=0}^{p-k}\binom{j+k-1}{j}\overline{\delta_j} = 0 \in \sigma^{(k+1)}/\sigma^{(k)}$. This proves that $\begin{bmatrix} 1 & 0 \\ t^{r-1} & 1 \end{bmatrix}$ acts trivially on $\sigma^{(k+1)}/\sigma^{(k)}$, completing the proof of our lemma. □

Lemma 3.9. *The generator* $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ *acts on* $\sigma^{(k+1)}/\sigma^{(k)}$ *via scaling by* $\chi(\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix})\cdot(\frac{a}{d})^k$.

Proof. Since $\sigma^{(k+1)}/\sigma^{(k)}$ is generated by $\sum_{j=0}^{p-(k+1)}\binom{j+k}{j}\overline{\delta_j}$, we wish to show that $\begin{bmatrix} a & 0 \\ 0 & d\end{bmatrix}\cdot\sum_{j=0}^{p-(k+1)}\binom{j+k}{j}\overline{\delta_j} = \chi(\begin{bmatrix} a & 0 \\ 0 & d\end{bmatrix})(\frac{a}{d})^k\sum_{j=0}^{p-(k+1)}\binom{j+k}{j}\overline{\delta_j}$. By Lemma 3.7 we have that

$$\begin{bmatrix} a & 0 \\ 0 & d\end{bmatrix}\cdot\sum_{j=0}^{p-(k+1)}\binom{j+k}{j}\overline{\delta_j} = \chi(\begin{bmatrix} a & 0 \\ 0 & d\end{bmatrix})\sum_{m=1}^{k+1}c_m\sum_{j=0}^{p-m}\binom{j+m-1}{j}\overline{\delta_j}$$

and since $\sum_{j=0}^{p-m}\binom{j+m-1}{j}\overline{\delta_j} = 0 \in \sigma^{(k+1)}/\sigma^{(k)}$ for $1 \leq m \leq k$, then in $\sigma^{(k+1)}/\sigma^{(k)}$ we have

$$\begin{bmatrix} a & 0 \\ 0 & d\end{bmatrix}\cdot\sum_{j=0}^{p-(k+1)}\binom{j+k}{j}\overline{\delta_j} = \chi(\begin{bmatrix} a & 0 \\ 0 & d\end{bmatrix})c_{k+1}\sum_{j=0}^{p-(k+1)}\binom{j+k}{j}\overline{\delta_j}.$$

Thus to prove our claim it suffices to show that $c_{k+1} = (\frac{a}{d})^k$. Recall that by Lemma 3.7, we have

$$c_{k+1} = \sum_{i=1}^{k+1}(-1)^{i+1}\binom{k}{i-1}\alpha_i$$

where here $\alpha_i = \binom{(p-i)\frac{a}{d}+k}{(p-i)\frac{a}{d}} = \frac{(k-i\frac{a}{d})\cdots(1-i\frac{a}{d})}{k!}$. In particular, since we may write out $\alpha_1 = \frac{(k-x)\cdots(1-x)}{k!} = \frac{1}{k!}((-1)^kx^k + a_{k-1}x^{k-1} + \cdots + a_1x + k!)$ where $x = \frac{a}{d}$, then we have that $\alpha_i = \frac{1}{k!}((-1)^ki^kx^k + a_{k-1}i^{k-1}x^{k-1} + \cdots + a_1ix + k!)$ for $1 \leq i \leq k+1$. Since the coefficient of x^m in α_i is given by $\frac{a_m}{k!}\cdot i^m$, then the coefficient of x^m in the expression of c_{k+1} is given by

$$\sum_{i=1}^{k+1}(-1)^{i+1}\binom{k}{i-1}\frac{a_m}{k!}i^m = \frac{a_m}{k!}\sum_{i=1}^{k+1}(-1)^{i+1}\binom{k}{i-1}i^m. \tag{33}$$

Since we wish to show that $c_{k+1} = x^k = (\frac{a}{d})^k$, it suffices to show that (33) is zero whenever $0 \leq m \leq k-1$ and is 1 whenever $m = k$. When $m = 0$, we have that $a_0 = k!$, so $\frac{a_0}{k!}\sum_{i=1}^{k+1}(-1)^{i+1}\binom{k}{i-1}i^0 = \sum_{i=1}^{k+1}(-1)^{i+1}\binom{k}{i-1} = \sum_{i=0}^{k}(-1)^i\binom{k}{i} = 0$, as desired. On the other hand, the identity

$$\sum_{i=0}^{k+1}(-1)^i\binom{k+1}{i}i^m = 0 \tag{34}$$

holds for $1 \leq m \leq k$ (see [3], #3 in 0.154), and since $\binom{k+1}{i} = \binom{k}{i} + \binom{k}{i-1}$, we deduce from (34) that

$$\sum_{i=0}^{k+1}(-1)^i\binom{k}{i}i^m + \sum_{i=0}^{k+1}(-1)^i\binom{k}{i-1}i^m = 0$$

which implies that

$$\sum_{i=0}^{k+1}(-1)^{i+1}\binom{k}{i-1}i^m = \sum_{i=0}^{k+1}(-1)^i\binom{k}{i}i^m = \sum_{i=0}^{k}(-1)^i\binom{k}{i}i^m$$

since $\binom{k}{k+1} = 0$ by convention. Now $\sum_{i=0}^{k}(-1)^i\binom{k}{i}i^m = 0$ for $0 \leq m \leq k-1$ by the identity in (34), so $\sum_{i=0}^{k+1}(-1)^{i+1}\binom{k}{i-1}j^m = 0$ for $0 \leq m \leq k-1$. When $m > 0$ we have that $0^m = 0$, so we conclude $\sum_{i=1}^{k+1}(-1)^{i+1}\binom{k}{i-1}i^m = 0$ for $0 \leq m \leq k-1$ as desired. On the other hand, identity #4 in §0.154 of [3] gives

$$\sum_{j=0}^{k}(-1)^j\binom{k}{j}j^k = (-1)^kk!\,, \tag{35}$$

which in combination with (34) and the fact that $\binom{k+1}{j} = \binom{k}{j} + \binom{k}{j-1}$ gives

$$\sum_{j=0}^{k+1}(-1)^j\binom{k+1}{j}j^k = \sum_{j=0}^{k+1}(-1)^j\binom{k}{j}j^k + \sum_{j=0}^{k+1}(-1)^j\binom{k}{j-1}j^k$$
$$\implies \sum_{j=1}^{k+1}(-1)^{j+1}\binom{k}{j-1}j^k = (-1)^k k!$$

which is precisely what we wished to show. Hence the coefficient of x^m in c_{k+1} is $\frac{a_m}{k!}\cdot 0 = 0$ for $0 \le m \le k-1$ while the coefficient of x^k is $\frac{(-1)^k}{k!}\cdot(-1)^k k! = (-1)^{2k} = 1$, completing the proof that $c_{k+1} = (\frac{a}{d})^k$, and therefore that $\begin{bmatrix} a & 0 \\ 0 & d\end{bmatrix}\cdot\sum_{j=0}^{p-(k+1)}\binom{j+k}{j}\overline{\delta_j} = \chi(\begin{bmatrix} a & 0 \\ 0 & d\end{bmatrix})\cdot(\frac{a}{d})^k\sum_{j=0}^{p-(k+1)}\binom{j+k}{j}\overline{\delta_j}$. □

Recall we wish to show that $\sigma^{(k+1)}/\sigma^{(k)}$ is equivalent to $\mathrm{Inf}_{B_{r-1}}^{I_r^{r-1}}(\chi\cdot(\frac{a}{d})^k)$ as I_r^{r-1} representations. Let $T : \langle\sum_{j=0}^{p-k}\binom{j+k-1}{j}\overline{\delta_j}\rangle \to \mathbb{F}_p$ be the isomorphism sending $\sum_{j=0}^{p-k}\binom{j+k-1}{j}\overline{\delta_j} \mapsto 1$. For all $\begin{bmatrix} a & b \\ ct^{r-1} & d\end{bmatrix} \in I_r^{r-1}$, we have

$$T(\begin{bmatrix} a & b \\ ct^{r-1} & d\end{bmatrix}\cdot\sum_{j=0}^{p-k}\binom{j+k-1}{j}\overline{\delta_j}) = T(\begin{bmatrix} 1 & 0 \\ ca^{-1}t^{r-1} & 1\end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & -ca^{-1}bt^{r-1}+d\end{bmatrix}\begin{bmatrix} 1 & ba^{-1} \\ 0 & 1\end{bmatrix}\cdot\sum_{j=0}^{p-k}\binom{j+k-1}{j}\overline{\delta_j})$$
$$(36)\qquad = T(\begin{bmatrix} 1 & 0 \\ ca^{-1}t^{r-1} & 1\end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & -ca^{-1}bt^{r-1}+d\end{bmatrix}\cdot\sum_{j=0}^{p-k}\binom{j+k-1}{j}\overline{\delta_j}).$$

Now $(\begin{bmatrix} a & 0 \\ 0 & -ca^{-1}bt^{r-1}+d\end{bmatrix}\cdot\delta_j)(i) \neq 0$ if and only if $i\begin{bmatrix} a & 0 \\ 0 & -ca^{-1}bt^{r-1}+d\end{bmatrix} \in B_r\begin{bmatrix} 1 & 0 \\ jt^{r-1} & 1\end{bmatrix}$, which holds if and only if $i \in B_r\begin{bmatrix} 1 & 0 \\ jt^{r-1} & 1\end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & -ca^{-1}bt^{r-1}+d\end{bmatrix}^{-1} = B_r\begin{bmatrix} 1 & 0 \\ \frac{d}{a}jt^{r-1} & 1\end{bmatrix}$. A similar argument as the one for $\begin{bmatrix} a & 0 \\ 0 & d\end{bmatrix}\cdot\delta_j = \chi(\begin{bmatrix} a & 0 \\ 0 & d\end{bmatrix})\delta_{\frac{d}{a}j}$ reveals that $\begin{bmatrix} a & 0 \\ 0 & -ca^{-1}bt^{r-1}+d\end{bmatrix}\cdot\delta_j = \chi(\begin{bmatrix} a & 0 \\ 0 & d\end{bmatrix})\delta_{\frac{d}{a}j}$, and therefore Lemma 3.9 applies to (36) to give $\chi(\begin{bmatrix} a & 0 \\ 0 & d\end{bmatrix})(\frac{a}{d})^k\cdot T(\sum_{j=0}^{p-k}\binom{j+k-1}{j}\overline{\delta_j}) = \chi(\begin{bmatrix} a & 0 \\ 0 & d\end{bmatrix})(\frac{a}{d})^k$. On the other hand, we have that

$$(37)\quad \mathrm{Inf}_{B_{r-1}}^{I_r^{r-1}}(\chi\cdot(\frac{a}{d})^k)(\begin{bmatrix} a & b \\ ct^{r-1} & d\end{bmatrix})(T(\sum_{j=0}^{p-k}\binom{j+k-1}{j}\overline{\delta_j})) = (\chi\cdot(\frac{a}{d})^k)(\begin{bmatrix} a & 0 \\ 0 & d\end{bmatrix})(T(\sum_{j=0}^{p-k}\binom{j+k-1}{j}\overline{\delta_j}))$$
$$= \chi(\begin{bmatrix} a & 0 \\ 0 & d\end{bmatrix})(\frac{a}{d})^k$$

which shows that $T\circ\sigma^{(k+1)}/\sigma^{(k)}(\begin{bmatrix} a & b \\ ct^{r-1} & d\end{bmatrix}) = \mathrm{Inf}_{B_{r-1}}^{I_r^{r-1}}(\chi\cdot(\frac{a}{d})^k)(\begin{bmatrix} a & b \\ ct^{r-1} & d\end{bmatrix})\circ T$, and hence that $\sigma^{(k+1)}/\sigma^{(k)}$ and $\mathrm{Inf}_{B_{r-1}}^{I_r^{r-1}}(\chi\cdot(\frac{a}{d})^k)$ are isomorphic as I_r^{r-1}-representations.

Now because the diagram

$$\begin{array}{ccc} I_r^{r-1} & \xrightarrow{t^{r-1}\mapsto 0} & B_{r-1} \\ \downarrow & & \downarrow \\ G_r & \xrightarrow{t^{r-1}\mapsto 0} & G_{r-1} \end{array}$$

commutes, we have by commutativity of inflation and induction that $\mathrm{Ind}_{I_r^{r-1}}^{G_r}\mathrm{Inf}_{B_{r-1}}^{I_r^{r-1}}(\chi\cdot(\frac{a}{d})^k) \cong \mathrm{Inf}_{G_{r-1}}^{G_r}\mathrm{Ind}_{B_{r-1}}^{G_{r-1}}(\chi\cdot(\frac{a}{d})^k)$. But this implies $\mathrm{Ind}_{I_r^{r-1}}^{G_r}(\sigma^{(k+1)}/\sigma^{(k)}) \cong \mathrm{Inf}_{G_{r-1}}^{G_r}\mathrm{Ind}_{B_{r-1}}^{G_{r-1}}(\chi\cdot(\frac{a}{d})^k)$, completing the proof of Theorem 1.1.

4. Semisimplifications

From Theorem 1.1 we deduce that

$$(\mathrm{Ind}_{B_r}^{G_r}(\chi))^{ss} = (\mathrm{Ind}_{B_{r-1}}^{G_{r-1}}(\chi))^{ss} \oplus \cdots \oplus (\mathrm{Ind}_{B_{r-1}}^{G_{r-1}}(\chi \cdot (\frac{a}{d})^k))^{ss} \oplus \cdots \oplus (\mathrm{Ind}_{B_{r-1}}^{G_{r-1}}(\chi \cdot (\frac{a}{d})^{p-1}))^{ss}$$
$$= (\mathrm{Ind}_{B_{r-1}}^{G_{r-1}}(\chi))^{ss} \oplus \cdots \oplus (\mathrm{Ind}_{B_{r-1}}^{G_{r-1}}(\chi \cdot (\frac{a}{d})^k))^{ss} \oplus \cdots \oplus (\mathrm{Ind}_{B_{r-1}}^{G_{r-1}}(\chi))^{ss} \quad (38)$$

In particular, we see that $(\mathrm{Ind}_{B_{r-1}}^{G_{r-1}}(\chi))^{ss}$ appears twice in the direct sum of (38), while $(\mathrm{Ind}_{B_{r-1}}^{G_{r-1}}(\chi \cdot (\frac{a}{d})^k))^{ss}$ appears once in the direct sum for every $1 \leq k \leq p-2$. Hence we may express

$$(\mathrm{Ind}_{B_r}^{G_r}(\chi))^{ss} = ((\mathrm{Ind}_{B_{r-1}}^{G_{r-1}}(\chi))^{ss})^2 \oplus \bigoplus_{k=1}^{p-2} (\mathrm{Ind}_{B_{r-1}}^{G_{r-1}}(\chi \cdot (\frac{a}{d})^k))^{ss}. \quad (39)$$

Since the semisimplifications of $\mathrm{Ind}_{B_1}^{G_1}(\chi)$ are well known for all characters $\chi : B(GL_2(\mathbb{F}_p)) \to \overline{\mathbb{F}_p}^{\times}$, it is desirable to express (39) explicitly in terms of $(\mathrm{Ind}_{B_1}^{G_1}(\chi))^{ss}$ for various χ. We claim that we may continue simplifying (39) inductively to obtain:

Corollary 4.1. *For a prime p, $(\mathrm{Ind}_{B_r}^{G_r}(\chi))^{ss} = ((\mathrm{Ind}_{B_1}^{G_1}(\chi))^{ss})^{\frac{p^{r-1}+p-2}{p-1}} \oplus \bigoplus_{k=1}^{p-2}((\mathrm{Ind}_{B_1}^{G_1}(\chi \cdot (\frac{a}{d})^k))^{ss})^{\frac{p^{r-1}-1}{p-1}}$.*

Proof. We prove the corollary by induction on r. When $r = 1$, the claim is that

$$(\mathrm{Ind}_{B_1}^{G_1}(\chi))^{ss} = ((\mathrm{Ind}_{B_1}^{G_1}(\chi))^{ss})^{\frac{p^0+p-2}{p-1}} \oplus \bigoplus_{k=1}^{p-2}((\mathrm{Ind}_{B_1}^{G_1}(\chi \cdot (\frac{a}{d})^k))^{ss})^{\frac{p^0-1}{p-1}}$$

which is easily seen to be true when one simplifies the exponents on the right hand side of the equality. Suppose the claim in the proposition holds for some $r \in \mathbb{N}$. We wish to show it holds for $r+1$. As a corollary of Theorem 1.1, we have that

$$(\mathrm{Ind}_{B_{r+1}}^{G_{r+1}}(\chi))^{ss} = ((\mathrm{Ind}_{B_r}^{G_r}(\chi))^{ss})^2 \oplus \bigoplus_{k=1}^{p-2} (\mathrm{Ind}_{B_r}^{G_r}(\chi \cdot (\frac{a}{d})^k))^{ss}.$$

Utilizing the inductive hypothesis on $(\mathrm{Ind}_{B_r}^{G_r}(\chi))^{ss}$ and on each $(\mathrm{Ind}_{B_r}^{G_r}(\chi \cdot (\frac{a}{d})^k))^{ss}$ gives

$$(\mathrm{Ind}_{B_{r+1}}^{G_{r+1}}(\chi))^{ss} = \left(((\mathrm{Ind}_{B_1}^{G_1}(\chi))^{ss})^{\frac{p^{r-1}+p-2}{p-1}} \oplus \bigoplus_{k=1}^{p-2}((\mathrm{Ind}_{B_1}^{G_1}(\chi \cdot (\frac{a}{d})^k))^{ss})^{\frac{p^{r-1}-1}{p-1}} \right)^2$$
$$\oplus \left(\bigoplus_{k=1}^{p-2} \left[((\mathrm{Ind}_{B_1}^{G_1}(\chi \cdot (\frac{a}{d})^k))^{ss})^{\frac{p^{r-1}+p-2}{p-1}} \oplus \bigoplus_{m \neq k} ((\mathrm{Ind}_{B_1}^{G_1}(\chi \cdot (\frac{a}{d})^m))^{ss})^{\frac{p^{r-1}-1}{p-1}} \right] \right). \quad (40)$$

Counting how many times $(\mathrm{Ind}_{B_1}^{G_1}(\chi))^{ss}$ appears in the direct sum of (40) yields that $(\mathrm{Ind}_{B_1}^{G_1}(\chi))^{ss}$ appears

$$2(\frac{p^{r-1}+p-2}{p-1}) + (p-2)\frac{p^{r-1}-1}{p-1} = \frac{p^r+p-2}{p-1}$$

times, whereas counting how many times $(\mathrm{Ind}_{B_1}^{G_1}(\chi \cdot (\frac{a}{d})^n))^{ss}$ appears in (40) for a given $1 \leq n \leq p-2$ yields that $(\mathrm{Ind}_{B_1}^{G_1}(\chi \cdot (\frac{a}{d})^n))^{ss}$ appears

$$2(\frac{p^{r-1}-1}{p-1}) + \frac{p^{r-1}+p-2}{p-1} + (p-3)\frac{p^{r-1}-1}{p-1} = \frac{p^r-1}{p-1}$$

times. Therefore

$$(\mathrm{Ind}_{B_{r+1}}^{G_{r+1}}(\chi))^{ss} = ((\mathrm{Ind}_{B_1}^{G_1}(\chi))^{ss})^{\frac{p^r+p-2}{p-1}} \oplus \bigoplus_{k=1}^{p-2}((\mathrm{Ind}_{B_1}^{G_1}(\chi \cdot (\frac{a}{d})^k))^{ss})^{\frac{p^r-1}{p-1}}. \quad (41)$$

proving the inductive claim. □

A complete semisimplification expresses the given representation as a direct sum of its unique set of composition factors, which are each irreducible representations. Hence giving the semisimplification $\mathrm{Ind}_{B_r}^{G_r}(\chi)$ requires knowing the irreducible characteristic p representations of $GL_2(\mathbb{F}_p[t]/(t^r))$.

4.1. **Classifying Modular Irreps of** $GL_2(\mathbb{F}_p[t]/(t^r))$**.** We give a complete characterization of the irreducible characteristic p representations of G_r for $r \geq 2$. For $r = 1$ we have that $\mathbb{F}_p[t]/(t) \cong \mathbb{F}_p$, and the characteristic p irreducible representations of $GL_2(\mathbb{F}_p)$ are fully classified; see [1]. Consider the surjective homomorphism

$$\pi : GL_2(\mathbb{F}_p[t]/(t^r)) \twoheadrightarrow GL_2(\mathbb{F}_p)$$

$$\begin{bmatrix} a_0 + \cdots + a_{r-1}t^{r-1} & b_0 + \cdots + b_{r-1}t^{r-1} \\ \\ c_0 + \cdots + c_{r-1}t^{r-1} & d_0 + \cdots + d_{r-1}t^{r-1} \end{bmatrix} \mapsto \begin{bmatrix} a_0 & b_0 \\ c_0 & d_0 \end{bmatrix} \tag{42}$$

and notice that $G_1 = GL_2(\mathbb{F}_p)$ may be viewed as a subgroup of G_r, as it respects multiplication in G_r. By the first isomorphism theorem for groups we know that $\ker \pi \trianglelefteq G_r$, and since the matrix

$$\begin{bmatrix} a_0 + \cdots + a_{r-1}t^{r-1} & b_0 + \cdots + b_{r-1}t^{r-1} \\ \\ c_0 + \cdots + c_{r-1}t^{r-1} & d_0 + \cdots + d_{r-1}t^{r-1} \end{bmatrix}$$

belongs to $\ker \pi$ if and only if $a_0 = d_0 = 1, b_0 = c_0 = 0$, and $a_i, b_i, c_i, d_i \in \mathbb{F}_p$ for $1 \leq i \leq r-1$, then $|\ker \pi| = |\mathbb{F}_p|^{4(r-1)} = p^{4(r-1)}$.

We wish to show that every irreducible characteristic p representation of G_r is of the form $\rho \circ \pi$, where π is as in (42) and ρ is an irreducible characteristic p representation of $GL_2(\mathbb{F}_p)$. To prove this fact we need the following two lemmas, which then establish the result as a quick corollary. The fact that $\ker \pi$ is a p-group is essential.

Lemma 4.2. *Let G be a finite group and let $H \trianglelefteq G$ be a p-group. If V is an irreducible characteristic p representation of G, then $V^H = V$, that is, H acts trivially on all elements of V.*

Proof. Let $V^H = \{v \in V : h \cdot v = v\}$, with the action of H on V given by the action of G on V. We wish to show that V^H is a nonzero subrepresentation of V, such that V being irreducible implies that $V^H = V$.

We claim that there exists a nonzero element of V which is fixed by all $h \in H$. By the Orbit-Stabilizer theorem, we have that for any $v \in V$, $H/H_v \cong \operatorname{Orb}_H(v)$, where $H_v = \{h \in H : h \cdot v = v\}$ and $\operatorname{Orb}_H(v) = \{h \cdot v : h \in H\}$. In particular this tells us that $|\operatorname{Orb}_H(v)| \mid |H|$ for every $v \in V$, so if $|H| = p^k$ for some k we must have $|\operatorname{Orb}_H(v)| = p^\ell$ for some $0 \leq \ell \leq k$. Notice that V can be assumed to be finite; otherwise let $v \neq 0 \in V$, and consider the $\mathbb{F}_q$ span of $\operatorname{Orb}(v)$, where q is a power of p and this orbit is considered over all $g \in G$. Let this finite vector space be denoted W. Since G (and thus H) acts on V via the irreducible characteristic p representation $\rho : G \to GL(V)$, H also acts on W, and we get that

$$|W| = |\{0\}| + \sum_{w \neq 0} |\operatorname{Orb}_H(w)|. \tag{43}$$

Because W is a finite vector space over a field of characteristic p, then $|W| = p^m$ for some m. But then $|W| - |\{0\}| = p^m - 1$ which is not divisible by p, and hence on the right hand side of (43) there must exist some nonzero w for which $|\operatorname{Orb}_H(w)| = 1$, that is, some nonzero w which is fixed by the action of all $h \in H$. Now the $\overline{\mathbb{F}_p}$ span of w is a subspace of V which is fixed by all $h \in H$, and hence $V^H \neq \{0\}$. To see that V^H is a subrepresentation of V, notice that V^H is invariant under the action by G, since if $v \in V^H$, then $h \cdot v = v$ for all $h \in H$, and thus $h \cdot (g \cdot v) = hg \cdot v = gh' \cdot v = g \cdot v$, where $hg = gh'$ for some $h' \in H$ by the fact that $H \trianglelefteq G$, and where $h' \cdot v = v$ since $v \in V^H$. Finally, since V is irreducible, this gives $V^H = V$. □

In particular Lemma 4.2 tells us that if G is a finite group, $H \trianglelefteq G$ is a p-group, and V is an irreducible characteristic p representation of G, then V must be the trivial representation on H. We claim that this implies V factors through G/H.

Lemma 4.3. *A representation of a finite group G is trivial on a normal subgroup H if and only if it factors through G/H.*

Proof. Suppose $\rho : G \to GL(V)$ is trivial on a normal subgroup H. Let $\pi : G \to G/H$ be the natural projection. We wish to show that there exists some group homomorphism $\psi : G/H \to GL(V)$ such that $\rho = \psi \circ \pi$. Define $\psi(gH) = \rho(g)$. Then $\psi(g_1Hg_2H) = \psi(g_1g_2H) = \rho(g_1g_2) = \rho(g_1)\rho(g_2)$, and

$\psi(H) = \rho(e) = I \in GL(V)$, so ψ is indeed a representation of G/H. In addition, we have that $\rho(g) = \psi \circ \pi(g)$ by definition.

Suppose a representation $\tilde{\rho} : G \to GL(V)$ factors through G/H where $H \trianglelefteq G$. We wish to show that $\tilde{\rho}$ is trivial on H. Express $\tilde{\rho} = \rho \circ \pi$. Then for all $h \in H$, we have $\tilde{\rho}(h) = \rho(\pi(h)) = \rho(H) = I \in GL(V)$ since H is the identity of G/H and ρ is a representation of G/H. □

The preceding lemmas allow us to prove the claim established at the beginning of this section:

Corollary 4.4. *Any irreducible modular representation of $GL_2(\mathbb{F}_p[t]/(t^r))$ is the inflation of an irreducible modular representation of $GL_2(\mathbb{F}_p)$.*

Proof. The surjective homomorphism π in (42) gives us $H = \ker \pi \trianglelefteq G_r$. Since H is a p-group, we know by Lemma 4.2 that any irreducible modular representation of G_r must be trivial on H. But by Lemma 4.3, we know that a representation of G_r is trivial on a normal subgroup H if and only if it factors through G_r/H. Since $G_r/H \cong GL_2(\mathbb{F}_p)$, then every irreducible characteristic p representation $\tilde{\rho}$ of G_r must be of the form $\rho \circ \pi$ where π is the map given in (42) and ρ is an irreducible characteristic p representation of $GL_2(\mathbb{F}_p)$. □

Fortunately the irreducible characteristic p representations ρ of $GL_2(\mathbb{F}_p)$ are fully classified (see [1] or [4] for the proofs). Given $0 \leq n \leq p-1$ and $0 \leq \ell \leq p-2$, let P_n be the $\overline{\mathbb{F}_p}$ span of the basis $\{x^n, x^{n-1}y, \dots, xy^{n-1}, y^n\}$. Define

$$\rho_{n,\ell} : GL_2(\mathbb{F}_p) \to GL(P_n) \tag{44}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot P(x, y) = P(ax+cy, bx + dy) \cdot \left(\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)^\ell$$

Then $\{\rho_{n,\ell}\}$ give a complete set of irreducible characteristic p representations of $GL_2(\mathbb{F}_p)$ up to equivalence. Hence every irreducible characteristic p representation of G_r is given by $\rho_{n,\ell} \circ \pi$, where π is the map in (42).

Acknowledgements

The author would like to acknowledge professors Charlotte Chan and Karol Koziol for both posing the problem and providing tremendous guidance. The author would also like to thank Andy Gordon for answering questions, the University of Michigan for running the REU, and the NSF for providing funding.

References

[1] LAURENT BERGER and SANDRA ROZENSZTAJN, *Modular representations of gl2(fp).*
[2] Henry W. Gould, *Combinatorial identities: A standardized et of tables, listing 500 binomial coefficient summations*, West Virginia University, 1972.
[3] Gradstejn IzrailS., Ryzik Iosif M., and Daniel Zwillinger, *Table of integrals, series, and products*, 2015.
[4] Niels Ketelaars, *The irreducible mod p representations of gl2(fp)*, Jun 2021.

Univ. of California- Santa Barbara
Email address: gahlshemy@ucsb.edu