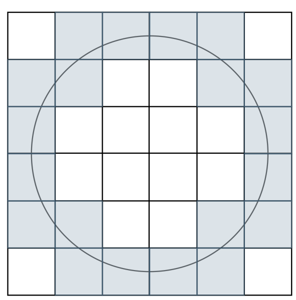
37TH UNIVERSITY OF MICHIGAN
UNDERGRADUATE MATHEMATICS COMPETITION

APRIL 4, 2020

Rules:

1. **The competition starts on April 4 at 1pm and ends at 4pm.**
2. **Between 4pm and 4:30pm you can scan your written solutions and upload it to the canvas website. The 4:30pm deadline is strict. (If you have issues with uploading your solutions and you cannot resolve it, then you can also email your solutions to `hderksen@umich.edu` before the deadline.)**
3. **It is probably not possible to solve all problems. Work on the problems that seem most inviting to you.**
4. **Prove all your answers to get credit.**
5. **Participants are not allowed to consult books, the internet, other people or other resources.**
6. **Calculators are not allowed.**

Problem 1. Consider the unit grid in $\mathbb{R}^2$ and let $f(t)$ be the number of 1×1 squares in the grid that intersect with the circle defined by $x^2 + y^2 = t$. (A circle C intersects a square $S = [p, p+1] \times [q, q+1]$ if $C \cap S \neq \emptyset$.) For example, $f(6) = 20$ because $x^2 + y^2 = 6$ intersects 10 squares:



Determine $f(999999)$.

Problem 2. For which positive integers n do we have

$$\int_0^\pi \cos(x)\cos(2x)\cos(3x)\cdots\cos(nx)\,dx = 0?$$

Problem 3. Suppose that k and n are positive integers, $a_1, a_2, \ldots, a_k \in \{1, 2, \ldots, n\}$ and $b_1, b_2, \ldots, b_n \in \{1, 2, \ldots, k\}$. Show that the sequences $a_1, a_2, \ldots, a_k$ and $b_1, b_2, \ldots, b_n$ have nonempty subsequences with the same sum.

Problem 4. Define a sequence $C_0, C_1, C_2, \ldots$ by $C_0 = 0$, $C_1 = 1$ and $C_{n+1} = C_n - 2C_{n-1}$ for $n \geq 1$. Suppose that a, b, g, n are positive integers such that a and b divide n and $g = \gcd(a, b)$ is the greatest common divisor. Show that $C_a C_b$ divides $C_g C_n$.

Problem 5. In tetrahedron $PABC$, face ABC is an equilateral triangle and P is equally distant from A, B and C, so that the other three faces are congruent isosceles triangles. In each of these three triangles, the cosine of the angle ϕ at P is $\frac{3}{5}$. Find the cosine of the (smaller) angle θ between the planes of PAB and PAC. (Note that for any xed point Q on the line L through P and A, then angle between the planes is the angle between a line in the plane of PAB perpendicular to L at Q and a line in the place of PAC perpendicular to L at Q.) Your answer should be an explicit rational number.

Problem 6. Find explicitly a complex root of $x^{2021} = x^{2020} + 1$ that satisfies a quadratic equation with integer coefficients.

Problem 7. Consider a chess board with a coin on each of the 64 squares. Each coin shows either head or tail. In each move you can choose a 3×3 or a 4×4 square area and flip all the coins in this square area. Is it possible to make a sequence of moves to get the whole board showing head for any starting configuration of heads and tails?

Problem 8. The sequence a_n is defined recursively so that $a_1 = 2$, $a_2 = 4$ and $a_{n+1} = \binom{a_n}{a_{n-1}}$ for all integers $n \geq 2$. For example, $a_3 = \binom{4}{2} = 6$ and $a_4 = \binom{6}{4} = 15$. Determine, with proof,

$$\lim_{n \to \infty} \frac{a_{n-1}}{a_n} a_n^{1/a_{n-1}}.$$

Problem 9. Suppose we have several identical dice that are biased (so the probability of each of the numbers 1,2,3,4,5,6 facing up could be different from $\frac{1}{6}$). If we roll three dice the probability that all three are the same is p. If we roll four dice then the probability that all are the same is q and the probability that all are distinct is r. If we roll two dice, what is the probability that they are distinct? Express your answer in p, q and r.

Problem 10. For which integers n does there exists a nonzero rational function $f(x)$ that satisfies

$$f'(x)^2 = f(x) x^n (x - 1)?$$

(Here $f'(x)$ is the derivative of $f(x)$.)