Department of Mathematics, University of Michigan
Real Analysis Qualifying Review Exam
January 7, 2025 (9am-noon)

Problem 1. Prove that $\lim_{n\to\infty} \int_{\mathbb{R}} \frac{(\cos \pi x)^n}{(x-n)^2+1} dx$ exists and find it.

Problem 2. Let $f_n : X \to \mathbb{R}$, $n \geq 1$, be a sequence of measurable functions on a measure space $(X, \mathcal{A}, \nu)$. Assume that $\nu(X) < +\infty$. Prove that $f_n \to 0$ in measure if and only if each subsequence $(f_{n_k})_{k=1}^{\infty}$ of this sequence $(f_n)_{n=1}^{\infty}$ contains a sub-subsequence $(f_{n_{k_m}})_{m=1}^{\infty}$ such that $f_{n_{k_m}} \to 0$ almost everywhere (as $m \to \infty$).

Problem 3. Let $E \subset \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq x \leq 1\}$ be a measurable set (with respect to the two-dimensional Lebesgue measure). Given $x, y \in [0, 1]$, denote

$$E_x := \{y \in [0, x] : (x, y) \in E\} \qquad \text{and} \qquad E^y := \{x \in [y, 1] : (x, y) \in E\}.$$

Assume that $\lambda(E_x) \geq x^3$ for almost all $x \in [0, 1]$.

(a) Prove that there exists $y \in [0, 1]$ such that $\lambda(E^y) \geq \frac{1}{4}$.

(b) Prove that there exists $y \in [0, 1]$ such that $\lambda(E^y) \geq 1 - \frac{1}{\sqrt{2}}$.

Problem 4. Let $\mu_1 \leq \mu_2 \leq \ldots$ be a sequence of positive absolutely continuous measures on a measure space $(X, \mathcal{A}, \rho)$. Assume that there exists a finite positive measure ν such that $\mu_n \leq \nu$ for all $n \in \mathbb{N}$. Set $\mu(A) := \lim_{n\to\infty} \mu_n(A)$ for $A \in \mathcal{A}$. Prove that μ is an absolutely continuous measure.

Problem 5. Find all $p \geq 1$ for which the following statement holds: for each function $f \in L^p(0, 1)$ the function $g(x) := f(x)f(x^2)$, $x \in (0, 1)$, belongs to $L^1(0, 1)$.