

AIM Qualifying Review Exam: Probability and Discrete Mathematics

August 20, 2026

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet.

Problem 1

Recall that a random variable X is *Poisson*-distributed with parameter $\lambda > 0$ if

$$\Pr(X = i) = e^{-\lambda} \frac{\lambda^i}{i!}$$

and that Y is *binomial* distributed with parameters n, p if $\Pr(Y = i) = \binom{n}{i} p^i (1-p)^{n-i}$.

- (a) Given n, p , put $\lambda = np$. Show that, if X is λ -Poisson-distributed and Y is (n, p) -binomial-distributed, then, under conditions on i, n, p , $\Pr(X = i)$ is approximated by $\Pr(Y = i)$. State weak general assumptions on i, n, p , of the form i, n, p large or small compared with 1 or with each other, etc., that guarantee the approximation.
- (b) Suppose each math lesson contains an error, independently, with probability 0.1. What is the probability that ten lessons contain at most one error? Write an unsimplified expression using binomial symbols and show work.
- (c) Approximate the above probability by a suitable Poisson approximation. Write an unsimplified expression using exponential and show work.

Solution outline

Ross p. 143ff.

(a)

$$\begin{aligned}\Pr(X = i) &= \frac{n!}{(n-i)!i!} p^i (1-p)^{n-i} \\ &= \frac{n(n-1)\cdots(n-i+1)}{n^i} \cdot \frac{\lambda^i}{i!} \cdot \frac{(1-\lambda/n)^n}{(1-\lambda/n)^i} \\ &\approx 1 \cdot \frac{\lambda^i (1-\lambda/n)^n}{i! \cdot 1} \\ &\approx 1 \cdot \frac{\lambda^i e^{-\lambda}}{i! \cdot 1}.\end{aligned}$$

This requires n large, $i \ll n$, and $i\lambda \ll n$.

(b) $\binom{10}{0} \cdot 1^0 \cdot 9^{10} + \binom{10}{1} \cdot 1^1 \cdot 9^9$. (This is approximately .7361.)

(c) Here $\lambda = np = 1$. We have probability $e^{-\lambda} \left(\frac{\lambda^0}{0!} + \frac{\lambda^1}{1!} \right) = 2e^{-1}$.

Mathematical concepts: Binomial distribution, Poisson distribution, Poisson approximation to Binomial, discrete distributions

Problem 2

Let X be a random variable with probability density function

$$f(x) = \begin{cases} c(1-x^2), & -1 < x < 1; \\ 0, & \text{otherwise.} \end{cases}$$

- (a) What is the value of c ?
- (b) What is the associated cumulative distribution function, $F(x)$?
- (c) Sketch f , and label minima, maxima, increase, decrease, and asymptotes.
- (d) Sketch F , and label minima, maxima, increase, decrease, and asymptotes.

Solution outline

Ross p. 224, problem 5.1

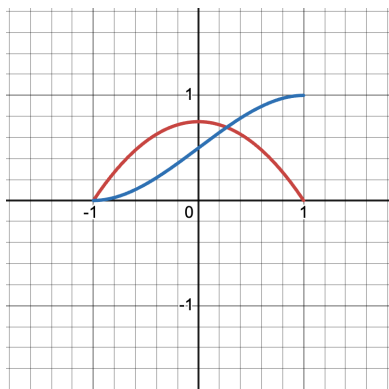
(a) Find c to make $\int_{-\infty}^{+\infty} f(x) dx = 1$. We have $c(x - \frac{x^3}{3})|_{-1}^1 = \frac{4}{3}c$, so $c = \frac{3}{4}$.

$$(b) F(x) = \int_{-\infty}^x f(t) dt = \begin{cases} 0, & x < -1; \\ \frac{3}{4}(x - \frac{x^3}{3}), & -1 \leq x \leq 1; \\ 1, & 1 < x. \end{cases}$$

(The function is continuous, so the parts agree at the seams, and we can be a bit cavalier with $<$ versus $\leq$.)

(c) In the parabola below, all density functions are non-negative. This has a maximum at $c = \frac{3}{4}$ and asymptotes (not shown) at $f(\pm\infty) = 0$. This function happens to increase to the unique maximum and then decrease.

- (d) All distribution functions have asymptotes $f(-\infty) = 0$ and $f(\infty) = 1$ and increase in height from 0 to 1.



Mathematical concepts: Foundations of probability; continuous distribution

Problem 3

How many permutations of the letters M, A, T, H, I, S, F, U, N are there in which none of the three words MATH, IS, FUN appear as consecutive substrings?

Show work, and give an unsimplified expression using factorial, binomial symbols, parentheses, $+$, $-$, $\times$, $\div$.

Solution outline

Brualdi, p. 166, Example.

Use Inclusion-Exclusion.

Let S be the set of $9!$ permutations. Let A_1 denote the set of permutations in S such that MATH appears, A_2 for IS, and A_3 for FUN. We want $|\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}|$, complementing with respect to S .

The permutations in A_1 are the $6!$ permutations of the six symbols MATH, I, S, F, U, N, etc. The permutations in $A_1 \cap A_2$ are permutations of the five symbols MATH, IS, F, U, N; etc.

We want $|S| - (|A_1| + |A_2| + |A_3|) + (|A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3|) - |A_1 \cap A_2 \cap A_3|$.

This is $9! - (6! + 8! + 7!) + (5! + 4! + 6!) - 3!$.

Mathematical concepts: Inclusion-Exclusion, permutations, generalized permutations.

Problem 4

Determine the generating function for the sequence of squares, i.e., find a closed form for $0^2 + 1^2x + 2^2x^2 + 3^2x^3 \dots$. You need not address radius of coverage.

Solution outline

Brualdi pp. 234–235. (See text for variation of proof.)

We have $f(x) = \frac{1}{1-x} = 1 + x + x^2 + \dots$. Differentiating, we get $f'(x) = (1-x)^{-2} = 1 + 2x + 3x^2 + \dots$ and, differentiating again, we get $f''(x) = 2(1-x)^{-3} = 2 + 6x + 12x^2 + \dots + n(n-1)x^{n-2} + \dots$

We need to check that $xf'(x) = 0x^0 + 1x^1 + 2x^2 + 3x^3 + \dots$ and $x^2f''(x) = 0 \cdot (-1)x^0 + 1 \cdot 0 \cdot x^1 + 2x^2 + 6x^3 + \dots$ fit the pattern for the x^0 and x^1 terms. They do, because xf' and x^2f'' , which must have one and two leading zeros, match the the data, which happens to have a leading zero. This is analogous to fiddling with initial conditions for recurrence or differential equations.

We want $x^2f''(x) + xf'(x) = \frac{2x^2}{(1-x)^3} + \frac{x}{(1-x)^2} = \frac{x(1+x)}{(1-x)^3}$.

Note: Substituting $x = \frac{1}{100}$ gives $\frac{.01 \cdot 1.01}{.99^3} = .0104091625 \dots$, which is a sanity check.

Mathematical concepts: Generating functions, linear recurrences

Problem 5

A Boolean circuit on TRUE/FALSE values is an arrangement of input gates, 2-input AND gates, 2-input OR gates, 1-input NOT gates, and a single output gate, organized and evaluated in the natural way. For example $x \vee \neg(y \wedge z)$. Here $\neg$ is not, $\wedge$ is AND, and $\vee$ is OR.

A circuit is satisfiable if there is some setting of TRUE/FALSE to the input gates that makes the output gate TRUE.

A 3-CNF (for Conjunctive Normal Form) is a giant AND of 3-wise ORs of boolean variables and their negations, often written using overlines for negation, e.g., $(x \vee \overline{y} \vee z) \wedge (\overline{u} \vee \overline{x} \vee \overline{z}) \wedge \dots$. Each 3-wise OR is called a clause.

Show how, given a circuit ψ , to quickly build a 3-CNF φ , such that ψ is satisfiable iff φ is satisfiable.

Illustrate by giving **representatives** of the clauses that arise in $x \vee \neg(y \wedge z)$.

Solution outline

Kleinberg and Tardos, p. 468ff. (and background)

Given a circuit, label each input and each output wire of each gate with a boolean variable. A clause $x \vee y \vee z$ can be regarded as an implication $(\neg x \wedge \neg y) \rightarrow z$. Thus saying that the ≤ 2 -input gate g on inputs α and β produces output γ is expressed as $\neg\alpha \vee \neg\beta \vee \gamma$.

Label the output of $\wedge$ as $w_\wedge$, the output of $\vee$ as $w_\vee$, and the output of $\neg$ as $w_\neg$. Label the overall output as w .

Produce the following clauses.

For the $\wedge$ -gate, we need that if the inputs are TRUE, FALSE, say, then the output is FALSE. This becomes $(y \wedge \neg z) \rightarrow w_\wedge$, whence $\neg y \vee z \vee w_\wedge$. Repeat for the rest of the $2^2 = 4$ possible input vectors. For the unary NOT gate, we get $w_\wedge \rightarrow \neg w_\neg$, i.e., $\neg w_\wedge \vee \neg w_\neg$, and, similarly, $w_\wedge \vee w_\neg$ “one is true and one is false.” We can make these proper 3-clauses by repeating a literal, e.g., $w_\wedge \vee w_\neg \vee w_\neg$.

Finally, the output wire is labeled w , so there is a clause $(w \vee w \vee w)$. (Here, we could just use $w_\vee$ as the overall output, or also assert that $w \leftrightarrow w_\vee$ with $w_\vee \rightarrow w$ and $\neg w_\vee \rightarrow \neg w$, i.e., $(w_\vee \vee \neg w), (\neg w_\vee \vee w)$.)

We’ve produced a 3-CNF φ by construction. If ψ is satisfiable, there is a TRUE/FALSE assignment to input wires which implies, through natural evaluation, a TRUE/FALSE assignment to all the wires. That assignment will satisfy all the relevant clauses generated by the gates because of the semantics of the gates; the irrelevant clauses are satisfied because an antecedent is false. For example, in the gate $s \vee t$, if s and t are both true, then so is the output u , and the clause $(s \wedge t) \rightarrow u$, i.e., $(\neg s \vee \neg t \vee u)$, is relevant and satisfied because that’s how $\vee$ gates work. The irrelevant clause $(\neg s \wedge t) \rightarrow u$, i.e., $(s \vee \neg t \vee u)$, is satisfied because s is true, so the antecedent $\neg s$ is false and the implication is therefore true. Finally, the overall output singleton clause v is satisfied by this assignment because ψ is satisfied.

Conversely, if ψ is not satisfiable, then any assignment to the inputs of ψ makes the output false. That is, there is no assignment that makes all the gates work properly (even restricted to relevant settings) and makes the output true. So φ is not satisfiable.

(There’s not more to prove at the level of (in-)formality with which the problem is posed, e.g., without formally saying what it means for a gate to work properly.)

Mathematical concepts: NP-completeness and reductions