AIM Qualifying Review Exam in Probability and Discrete Mathematics

Jan 2026

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet. No credit will be given for answers without supporting work and/or reasoning.

Problem 1 Prove that the number of permutations of $1, 2, \ldots, n$ such that i is not in position i for each $i = 1, 2, \ldots, n$ is given by

$$n!\Big(1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^n}{n!}\Big).$$

Solution outline

The number of permutations with $i_1, \ldots, i_k$ fixed in positions $i_1, \ldots, i_k$ is $(n-k)!$. Those k numbers can be chosen in $\binom{n}{k}$ different ways.

By the inclusion-exclusion principle the number of permutations (derangements) with no number in its position is

$$n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \cdots ,$$

which may be simplified to the stated form.

Problem 2 Suppose x and y are independent and normally distributed with mean 0 and variance 1, which means that their joint density function is

$$\frac{1}{2\pi} e^{-x^2/2} e^{-y^2/2}.$$

(a) Find the density function of $x + 2y$ conditional upon $2x - y = 1$.

(b) Find the expectation of $x + 2y$ conditional upon $2x - y = 4$.

Solution outline

This problem may be solved using the change of variables $u = x + 2y$ and $v = 2x - y$, whose inverse is given by $x = (u + 2v)/5$ and $y = (2u - v)/5$.

A simpler method is to notice that the columns of the matrix that map x, y to u, v are orthogonal. Thus u, v must be independent and normally distributed. Obviously, the expectation of u is zero and its variance is $\sigma^2 = 5$. Therefore, its density function must be

$$\frac{1}{\sqrt{10\pi}} \exp\left(-\frac{u^2}{10}\right)$$

regardless of v and the conditional expectation must be zero, because u, v are independent.

Problem 3

A single throw of a fair die comes up $1, 2, 3, 4, 5$, or 6 with equal probability. Find the probability that the sum of n numbers that come up in n throws of that fair die is divisible by 7.

Solution outline

Let the sum of the faces after k throws be s_k. We will assumed that the probability that $s_k \mod 7 = j$ is the same for each of $j = 1, 2, \ldots, 6$ and equal to p_k. This assumption will be proved inductively. The probability that $s_k \mod 7 = 0$ is then $1 - 6p_k$.

Obviously, $p_1 = 1/6$.

Given p_k we may obtain an expression for p_{k+1} as follows. If the $(k+1)$st throw is x_{k+1} and $s_k \mod 7 = r_k$, then for $(r_k + x_{k+1}) \mod 7$ is equally likely to be one of the six numbers

$$\{1, 2, \ldots, 6\} - \{r_k\}.$$

Thus the probability that $s_{k+1} \mod 7$ is equal to j for each of $j = 1, 2, \ldots, 6$ is given by

$$p_{k+1} = \frac{5p_k}{6} + \frac{1 - 6p_k}{6} = \frac{1 - p_k}{6}.$$

The sequence of p_ks is then given by

$$\frac{1}{6}, \frac{1}{6} - \frac{1}{6^2}, \frac{1}{6} - \frac{1}{6^2} + \frac{1}{6^3}, \ldots$$

The probability that the number after n throws is divisible by 7 is equal to

$$1 - 6p_n = 1 - 6\left(\frac{1}{6} - \frac{1}{6^2} + \cdots + (-1)^{n-1}\frac{1}{6^n}\right),$$

which is in fact equal to p_{n-1}.

Problem 4

Suppose $x_1, \ldots, x_n$ is a given sequence of integers (negative integers are allowed). Find an $O(n)$ algorithm to find a contiguous subsequence $x_i, \ldots, x_{i+j}$, with $j \geq 0$, whose sum is maximum.

Solution outline

Let $p_k = (i, j, s)$ if $x_i + \cdots + x_j$, $k \geq j \geq i$, is the maximum sum in $x_1, \ldots, x_k$, with s denoting the maximum sum. Similarly, let $q_k = (i, s)$ if $x_i + \cdots + x_k$ is maximum for $i \in \{1, \ldots, k\}$, with s denoting the maximum sum.

Given p_k, q_k, we may obtain p_{k+1}, q_{k+1} as follows. If $q_k = (i, s)$, then $q_{k+1} = (i, s + x_{k+1})$ if $s > 0$. If $s \leq 0$, $q_{k+1} = (k+1, x_{k+1})$.

If $p_k = (i, j, s)$, then $p_{k+1} = p_k$ if the sum in q_{k+1} is less than the sum in p_k. If the sum in q_{k+1} is greater than or equal to the sum in p_k, then

$$p_{k+1} = (i, k+1, s)$$

assuming $q_{k+1} = (i, s)$.

An $O(n)$ solution is obtained from p_n using these recurrences and the initialization $p_1 = (1, 1, x_1)$, $q_1 = (1, x_1)$.

Problem 5 Suppose that all n^2 numbers in an $n \times n$ matrix are distinct, that each row is increasing (from left to right), and that each column is increasing (from top to bottom). The reverse diagonal consists of entries in row i and column j for (i, j) equal to

$$(n, 1), (n-1, 2), \ldots, (1, n)$$

and in that order.

(a) Explain why the n entries on the reverse diagonal can be in any order.

(b) Derive an $O(n \log_2 n)$ algorithm to search for a given number x in this matrix.

Solution outline

Begin by placing some n numbers along the reverse diagonal in any order. Choose $n(n-1)/2$ numbers, each smaller than any number along the reverse diagonal, and arrange them in increasing order above the reverse diagonal, starting in row 1 followed by row 2 and so on. Similarly, $n(n-1)/2$ numbers, each larger than any number along the reverse diagonal, may be arranged below the reverse diagonal. The matrix will then have each row and column increasing.

For an $O(n \log_2 n)$ search algorithm, simply do binary search along each of the $2n - 1$ diagonals.