THE UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS

Qualifying Review examination in General and Differential Topology

January 2025

On this exam, "smooth" means "infinitely continuously differentiable."

1. For which $n \geq 1$ does there exist a smooth differential 2-form on $\mathbb{R}^n$ which is not an exterior product of two smooth 1-forms? Explain.

2. Let
$$X = \{(x, y, z) \in \mathbb{R}^3 \mid z^3 + z^2 = x^3 + x^2 + y^3 + y^2\}.$$
Is X a smooth submanifold of $\mathbb{R}^3$? Prove your answer rigorously.

3. Le S^2 be the unit sphere in $\mathbb{R}^3$. Define a map $S^2 \to S^2$ by
$$(x, y, z) \mapsto \frac{1}{\sqrt{3}}(1 - 2x(x + y + z), 1 - 2y(x + y + z), 1 - 2z(x + y + z)).$$
Verify that this map is well-defined, and compute its degree.

4. Let X be the set of all $n \times n$ real matrices A satisfing $AA^T = I$. Prove that M is a smooth submanifold of the Euclidean space $M_{n,n}$ of all $n \times n$ matrices, and compute the tangent space of X at a matrix $A \in X$ as a subspace of $M_{n,n}$ (which is identified with its own tangent space).

5. Let M be a smooth manifold and let $f : M \to M$ be a smooth map such that $f^2 = Id_M$. Let $M^f = \{x \in M \mid f(x) = x\}$.

   (a) Is M^f necessarily a smooth submanifold?

   (b) If M is oriented, is M^f necessarily an oriented smooth submanifold?

   Explain.