# AIM Qualifying Review Exam in Differential Equations & Linear Algebra

*January 2026*

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet. No credit will be given for answers without supporting work and/or reasoning.

**Problem 1**

Suppose $\mathbf{A}$ is a real $n$-by-$n$ matrix with rank one.

(a) Show there exists a unique number $c$ such that $\mathbf{A}^2 = c\mathbf{A}$.

(b) Show that if $c \neq 1$, then $\mathbf{I} - \mathbf{A}$ has an inverse, where $\mathbf{I}$ is the $n$-by-$n$ identity matrix.

**Problem 1**

**Problem 1**

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**Problem 2**

For the following matrices $M_1$ and $M_2$, find the corresponding Jordan forms $J_1$ and $J_2$. As a reminder, for each $k$, $M_k$ and $J_k$ are similar matrices and $J_k$ has only zero entries except possibly on the main diagonal and the first diagonal above it, where ones may occur. Hint: think about the eigenvectors.

(a) $M_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

(b) $M_2 = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

**Problem 2**

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**Problem 2**

**Problem 3**

(a) Can $y = t^3$ be a solution to $y'' + p(t)y' + q(t)y = 0$ on an interval that contains $t = 0$ and throughout which $p(t)$ and $q(t)$ are continuous? Explain your answer.

(b) Assume that $p$ and $q$ are continuous and that the functions $y_1$ and $y_2$ are solutions of the differential equation $y'' + p(t)y' + q(t)y = 0$ on an open interval I. Prove that if $y_1(x_0) = y_2(x_0) = y_1'(x_0) = y_2'(x_0)$ at some $x_0$ in I, then $\{y_1, y_2\}$ cannot be a fundamental set of solutions on I.

(c) Determine a positive lower bound for the radius of convergence of a series solution $y = \sum_{n=0}^{\infty} a_n t^n$ for the ODE

$$(1 + \cos t)y'' + (\sin t)y' + e^{t^2} y = 0. \qquad (1)$$

**Problem 3**

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**Problem 4**

(a) Compute $\mathrm{e}^{At}$ for $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$.

(b) If $B$ is a skew-symmetric matrix, i.e. $B^T = -B$, show that $\mathrm{e}^{Bt}$ is an orthogonal matrix.

(c) Solve $2x^2y'' + 3xy' - y = 0$ for $x > 0$.

**Problem 4**

**Problem 4**

**Problem 4**

**Problem 5**

Solve the PDE

$$\frac{\partial u}{\partial t} + 4\frac{\partial^2 u}{\partial x^2} + u = \cos x \tag{2}$$

for $u(x,t)$ with boundary conditions

$$u(0,t) = 0 \ , \ \frac{\partial u}{\partial x}(2\pi, t) = 0$$

and initial condition

$$u(x,0) = \sin\left(\frac{5x}{4}\right) + \frac{1}{3}\cos\left(\frac{x}{2}\right) - \frac{1}{3}\cos x.$$

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**Problem 5**

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