# AIM Qualifying Review Exam in Differential Equations & Linear Algebra

*January 2026*

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet. No credit will be given for answers without supporting work and/or reasoning.

## Problem 1

Suppose $\mathbf{A}$ is a real $n$-by-$n$ matrix with rank one.

(a) Show there exists a unique number $c$ such that $\mathbf{A}^2 = c\mathbf{A}$.

(b) Show that if $c \neq 1$, then $\mathbf{I} - \mathbf{A}$ has an inverse, where $\mathbf{I}$ is the $n$-by-$n$ identity matrix.

**Solution outline**

(a) Since $\mathbf{A}$ has rank one, all the columns of $\mathbf{A}$ are multiples of a single nonzero vector $a$, and thus $\mathbf{A} = ab^T$ with the multiples given in $b$. So $\mathbf{A}^2 = (b^T a)ab^T = c\mathbf{A}$ where $c = b^T a$.

(b) $\mathbf{I} - \mathbf{A}$ has an inverse iff its nullspace is just the zero vector. We have $c = b^T a \neq 1$. Case 1: assume $c = b^T a \neq 0$. Then the whole space $\mathbb{R}^n$ can be written as a direct sum of the $n-1$ dimensional space $\langle b \rangle^\perp$ and the 1-dimensional space $\langle a \rangle$. $\mathbf{I} - \mathbf{A}$ is the identity on $\langle b \rangle^\perp$ and is nonzero on nonzero multiples of $a$ (since $c \neq 1$), so the nullspace is trivial. Case 2: assume $c = b^T a = 0$. Then we decompose $\mathbb{R}^n$ into $\langle b \rangle^\perp$ and the 1-dimensional space $\langle b \rangle$. Again, $\mathbf{I} - \mathbf{A}$ is the identity on $\langle b \rangle^\perp$. $(\mathbf{I} - \mathbf{A})b = b - a(b^T b) \neq 0$ since $b^T a = 0$, so again the nullspace is trivial. Thus $\mathbf{I} - \mathbf{A}$ has an inverse.

## Problem 2

For the following matrices $M_1$ and $M_2$, find the corresponding Jordan forms $J_1$ and $J_2$. As a reminder, for each $k$, $M_k$ and $J_k$ are similar matrices and $J_k$ has only zero entries except possibly on the main diagonal and the first diagonal above it, where ones may occur. Hint: think about the eigenvectors.

(a) $M_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

(b) $M_2 = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

**Solution outline**

(a) This $M_1$ is diagonalizable with eigenvalues 0 and 2 and corresponding eigenvectors $(1, -1)$ and $(1, 1)$. Thus the Jordan form is $J_1 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$ or $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$.

(b) By inspection, two eigenvectors are (1, 0, 0) and (0, 2, -1), both with eigenvalues 0. If there is a third eigenvector, its eigenvalue must be nonzero, since a matrix with a full set of eigenvectors with zero eigenvalues must be the zero matrix. $M_2$ maps any vector to a multiple of (1, 0, 0). Therefore any eigenvector with a nonzero eigenvalue would be proportional to (1, 0, 0), but we already know this is an eigenvector with eigenvalue zero. Thus there are only two eigenvectors, so the Jordan form is $J_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

## Problem 3

(a) Can $y = t^3$ be a solution to $y'' + p(t)y' + q(t)y = 0$ on an interval that contains $t = 0$ and throughout which $p(t)$ and $q(t)$ are continuous? Explain your answer.

(b) Assume that $p$ and $q$ are continuous and that the functions $y_1$ and $y_2$ are solutions of the differential equation $y'' + p(t)y' + q(t)y = 0$ on an open interval I. Prove that if $y_1(x_0) = y_2(x_0) = y_1'(x_0) = y_2'(x_0)$ at some $x_0$ in I, then $\{y_1, y_2\}$ cannot be a fundamental set of solutions on I.

(c) Determine a positive lower bound for the radius of convergence of a series solution $y = \sum_{n=0}^{\infty} a_n t^n$ for the ODE

$$(1 + \cos t)y'' + (\sin t)y' + e^{t^2} y = 0. \qquad (1)$$

**Solution outline**

(a) The uniqueness theorem for linear equations tells us that for $y$ and $y'$ given at a point in the interval, we have a unique solution throughout the interval. $y \equiv 0$ throughout the interval is a solution with $y(0) = y'(0) = 0$. $y = t^3$ also obeys these initial conditions, so if it were a solution to the ODE also, we would have nonuniqueness, a contradiction. Thus the answer is no, $y = t^3$ cannot be a solution.

(b) If $\{y_1, y_2\}$ were a fundamental set of solutions on I, we would have that the Wronskian $W = y_1y_2' - y_2y_1' \neq 0$ throughout I. But $W(x_0) = 0$, so $\{y_1, y_2\}$ cannot be a fundamental set of solutions on I.

(c) The radius of convergence is at least as large as the distance from the origin to the nearest zero of $1 + \cos t$, which occurs at $t = \pm\pi$. So $\pi$ is a lower bound.

## Problem 4

(a) Compute $e^{At}$ for $A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$.

(b) If $B$ is a skew-symmetric matrix, i.e. $B^T = -B$, show that $\mathrm{e}^{Bt}$ is an orthogonal matrix.

(c) Solve $2x^2y'' + 3xy' - y = 0$ for $x > 0$.

**Solution outline**

(a) We have $A = A^2 = A^3 = \ldots$. We write the series $e^{At} = I + At + \frac{A^2t^2}{2} + \ldots = I + A\left(t + \frac{t^2}{2} + \ldots\right) = I + A(e^t - 1)$. Thus $e^{At} = \begin{bmatrix} e^t & 0 \\ e^t - 1 & 1 \end{bmatrix}$.

(b) $(e^{Bt})^T = e^{B^Tt}$ using the series form of the exponential. $e^{B^Tt} = e^{-Bt}$ and $e^{-Bt}e^{Bt} = e^{-Bt+Bt} = e^{\mathbf{0}t} = I$ since $-Bt$ and $Bt$ commute. Thus $\mathrm{e}^{Bt}$ is an orthogonal matrix.

(c) Substitute $y = x^r$, obtain $r = 1/2$ and -1, so $y = c_1x^{1/2} + c_2x^{-1}$.

## Problem 5

Solve the PDE

$$\frac{\partial u}{\partial t} + 4\frac{\partial^2 u}{\partial x^2} + u = \cos x \tag{2}$$

for $u(x,t)$ with boundary conditions

$$u(0,t) = 0\ ,\ \frac{\partial u}{\partial x}(2\pi, t) = 0$$

and initial condition

$$u(x,0) = \sin\left(\frac{5x}{4}\right) + \frac{1}{3}\cos\left(\frac{x}{2}\right) - \frac{1}{3}\cos x.$$

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**Solution outline**

We decompose the solution as $u(x,t) = u_s(x) + u_1(x,t)$ where $u_s(x)$ is a steady solution to the inhomogeneous PDE and $u_1(x,t)$ solves the homogeneous PDE with the initial condition $u_1(x,0) = u(x,0) - u_s(x)$.

The first step is to solve $4u_s'' + u_s = \cos x$ for $u_s(x)$, such that $u_s(0) = u_s'(2\pi) = 0$. The homogeneous ODE has solution $u_h = A\sin\left(\frac{x}{2}\right) + B\cos\left(\frac{x}{2}\right)$. The method of undetermined coefficients gives the particular solution $u_p = -\frac{1}{3}\cos x$. The boundary conditions give $A = 0$ and $B = 1/3$. So $u_s = \frac{1}{3}\cos\left(\frac{x}{2}\right) - \frac{1}{3}\cos x$.

Now we solve for $u_1$. We write $u_1(x,t) = X(x)T(t)$ and obtain

$$4\frac{X''}{X} = -\frac{T'}{T} - 1 = -4\left(\frac{2k+1}{4}\right)^2.$$

We have placed the constant -1 in the $T$ equation, to simplify the $X$ equation. It has sinusoidal eigenfunctions with quarter-integer wavenumbers (1/4, 3/4, 5/4, ...), which satisfy the boundary conditions $X(0) = X'(2\pi) = 0$. I.e. $X_k = \sin\left(\frac{2k+1}{4}x\right)$, $k = 1, 2, 3, \ldots$. The corresponding $T_k(t) = e^{(-1+4\left(\frac{2k+1}{4}\right)^2)t}$. The initial condition for $u_1$ is $u_1(x,0) = \sin\left(\frac{5x}{4}\right)$. In general we have a series solution $u_1(x,t) = \sum_k A_kX_k(x)T_k(t)$, and we set the coefficients to match the initial condition. So we have $A_2 = 1$ and all other coefficients zero.

Thus

$$u_1 = e^{\left(\frac{21}{4}\right)t}\sin\left(\frac{5x}{4}\right).$$