

AIM Qualifying Review Exam in Differential Equations & Linear Algebra

August 2026

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet. No credit will be given for answers without supporting work and/or reasoning.

Problem 1

Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

- (a) Find the singular values σ_1, σ_2 and an SVD factorization: $A = U\Sigma V^T$. Hint: consider the eigenvalues and eigenvectors of a related matrix.
- (b) Describe geometrically what happens to points on the unit circle $x_1^2 + x_2^2 = 1$ in $\mathbb{R}^2$ under multiplication by A , given as a sequence of multiplications by V^T , Σ , and U .
- (c) Find a polar decomposition of A in the form $A = QS$ where Q is orthogonal and S is symmetric positive semidefinite. Show that the polar decomposition is not unique.

Problem 1

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Problem 2

Let V be a finite-dimensional real inner product space. We say a linear operator $B : V \rightarrow V$ has **Property A** if for every $\mathbf{v} \in V$, $B\mathbf{v} \in W$ and $(\mathbf{v} - B\mathbf{v}) \in W^\perp$, where W is some subspace of V .

- (a) Show that if P has Property A then $P^2 = P$ and $P^* = P$ (i.e. P is an orthogonal projection).
- (b) Prove that if P has Property A, then $\|P\mathbf{v}\| \leq \|\mathbf{v}\|$ for all $\mathbf{v} \in V$.
- (c) Suppose $T : V \rightarrow V$ satisfies $T^2 = T$ but $\text{range}(T)$ is not orthogonal to $\text{kernel}(T)$. I.e. T is an oblique projector. Prove that there exists some $\mathbf{v} \in V$ such that $\|T\mathbf{v}\| > \|\mathbf{v}\|$, and deduce that $\|T\|_2 > 1$.

Problem 2

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Problem 3

Consider the non-linear autonomous two-dimensional system:

$$\begin{cases} \dot{x} = 2y - 2x^3 + xy^2 \\ \dot{y} = -x - y^3 - 2x^2y \end{cases}$$

- (a) Linearize the system around the critical point $(0,0)$. Show that linearization alone is inconclusive regarding stability.
- (b) Use a Lyapunov function $V(x,y)$ to prove the global asymptotic stability or instability of the critical point.
- (c) Show that the origin is the unique critical point.

Problem 3

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Problem 4

- (a) Find the general solution of the ODE

$$y'' + 2y' + y = 2e^{-x} + 2\cos(x).$$

- (b) Consider the initial value problem:

$$\frac{dy}{dx} = \frac{x(y-1)^{2/3}}{\sqrt{1+x^2}}, \quad y(x_0) = y_0$$

where the real value of $(\cdot)^{2/3}$ is used.

- (i) Apply a standard existence and uniqueness theorem to determine all initial points $(x_0, y_0) \in \mathbb{R}^2$ where a unique local solution $y(x)$ is guaranteed to exist. By “local” we mean on some x -interval containing x_0 .
- (ii) Show that there are at least two distinct local solutions at points where the theorem’s assumptions do not apply.

Problem 4

Problem 4

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Problem 5

An infinite elastic string satisfies the one-dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} = 4 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0$$

- (a) Change variables using $\xi = x - ct$ and $\eta = x + ct$ to derive the general solution as a sum of two undetermined functions. Hint:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} \quad ; \quad \frac{\partial u}{\partial t} = \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial t}.$$

- (b) The string initially has a triangular profile with zero velocity:

$$u(x, 0) = f(x) = \begin{cases} 1 - |x|, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases} \quad ; \quad u_t(x, 0) = 0.$$

Use the initial conditions to complete the solution of $u(x, t)$. Describe the motion in words.

- (c) Calculate the displacement $u(0, t)$ at $x = 0$ for $t \geq 0$. Describe it in words.

Problem 5

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