

AIM Qualifying Review Exam in Differential Equations & Linear Algebra

August 2026

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. But if you use other sheets of paper, be sure to mark them clearly and staple them to the booklet. No credit will be given for answers without supporting work and/or reasoning.

Problem 1

Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

- (a) Find the singular values σ_1, σ_2 and an SVD factorization: $A = U\Sigma V^T$. Hint: consider the eigenvalues and eigenvectors of a related matrix.
- (b) Describe geometrically what happens to points on the unit circle $x_1^2 + x_2^2 = 1$ in $\mathbb{R}^2$ under multiplication by A , given as a sequence of multiplications by V^T , Σ , and U .
- (c) Find a polar decomposition of A in the form $A = QS$ where Q is orthogonal and S is symmetric positive semidefinite. Show that the polar decomposition is not unique.

Solution

- (a) Since $A = U\Sigma V^T$, $A^T A = V\Sigma^2 V^T$ is the eigenvalue decomposition of $A^T A$, with Σ^2 diagonal and non-negative and V an orthogonal matrix of eigenvectors. Thus the singular values σ_i are the nonnegative square roots of the eigenvalues of $A^T A$.

We have $A^T A = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$. To find the eigenvalues, solve the characteristic equation $\det(A^T A - \lambda I) = 0$:

$$\det \begin{pmatrix} 2 - \lambda & 2 \\ 2 & 2 - \lambda \end{pmatrix} = (2 - \lambda)^2 - 4 = \lambda^2 - 4\lambda = 0$$

This yields the eigenvalues:

$$\lambda_1 = 4, \quad \lambda_2 = 0$$

Next, find the unit eigenvectors corresponding to each eigenvalue:

- $(A^T A - 4I)v_1 = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} v_1 = 0 \implies v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
- $(A^T A)v_2 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} v_2 = 0 \implies v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$$\sigma_1 = \sqrt{4} = 2, \quad \sigma_2 = \sqrt{0} = 0$$

The right singular matrix V is formed by placing v_1 and v_2 in columns:

$$V = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

For $\sigma_1 = 2 \neq 0$, the left singular vector u_1 is computed via $u_1 = \frac{Av_1}{\sigma_1}$:

$$u_1 = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

To complete the orthogonal matrix U , choose u_2 orthogonal to u_1 :

$$u_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Combining these gives the full SVD factorization $A = U\Sigma V^T$:

$$A = \left(\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \right) \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \left(\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \right)^T$$

(b) The transformation $A = U\Sigma V^T$ acts on the unit circle $x_1^2 + x_2^2 = 1$ in three sequential steps:

- V^T rotates the unit circle in $\mathbb{R}^2$.
- Σ stretches the space along the x axis by a factor of $\sigma_1 = 2$, while squashing the y component down to 0 because $\sigma_2 = 0$.
- U rotates the flattened circle to align with the direction of $u_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

(c) Using the SVD, write $A = UV^T V\Sigma V^T$, and set $Q = UV^T$ and $S = V\Sigma V^T$. In this case $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $S = A$. To find another choice of Q , consider the equation $Q \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. So Q maps $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ to itself. It could map $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ to $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, making $Q = I$ as above, or it could map $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ to $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, making Q the reflection matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

Problem 2

Let V be a finite-dimensional real inner product space. We say a linear operator $B : V \rightarrow V$ has **Property A** if for every $\mathbf{v} \in V$, $B\mathbf{v} \in W$ and $(\mathbf{v} - B\mathbf{v}) \in W^\perp$, where W is some subspace of V .

- (a) Show that if P has Property A then $P^2 = P$ and $P^* = P$ (i.e. P is an orthogonal projection).
- (b) Prove that if P has Property A, then $\|P\mathbf{v}\| \leq \|\mathbf{v}\|$ for all $\mathbf{v} \in V$.
- (c) Suppose $T : V \rightarrow V$ satisfies $T^2 = T$ but $\text{range}(T)$ is not orthogonal to $\text{kernel}(T)$. I.e. T is an oblique projector. Prove that there exists some $\mathbf{v} \in V$ such that $\|T\mathbf{v}\| > \|\mathbf{v}\|$, and deduce that $\|T\|_2 > 1$.

Solution

- (a) Suppose P has Property A. Since $V = W \oplus W^\perp$, every $\mathbf{v} \in V$ decomposes uniquely as $\mathbf{v} = \mathbf{w} + \mathbf{w}^\perp$ with $\mathbf{w} \in W$ and $\mathbf{w}^\perp \in W^\perp$. Thus $P\mathbf{v} = \mathbf{w}$, the W -component of this decomposition. The decomposition of $P\mathbf{v}$ is $P\mathbf{v} + \mathbf{0}$, so $P^2\mathbf{v} = P(P\mathbf{v}) =$ the W -component of $P\mathbf{v}$ which is $P\mathbf{v}$ itself. This holds for all $\mathbf{v} \in V$, so $P^2 = P$.

To show $P^* = P$, take any $\mathbf{u}, \mathbf{v} \in V$ and decompose them as $\mathbf{u} = \mathbf{u}_W + \mathbf{u}_{W^\perp}$ and $\mathbf{v} = \mathbf{v}_W + \mathbf{v}_{W^\perp}$. Then:

$$\langle P\mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}_W, \mathbf{v}_W + \mathbf{v}_{W^\perp} \rangle = \langle \mathbf{u}_W, \mathbf{v}_W \rangle$$

$$\langle \mathbf{u}, P\mathbf{v} \rangle = \langle \mathbf{u}_W + \mathbf{u}_{W^\perp}, \mathbf{v}_W \rangle = \langle \mathbf{u}_W, \mathbf{v}_W \rangle$$

Since $\langle P\mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{u}, P\mathbf{v} \rangle$ for all $\mathbf{u}, \mathbf{v} \in V$, $P^* = P$.

- (b) For any $\mathbf{v} \in V$, write $\mathbf{v} = P\mathbf{v} + (\mathbf{v} - P\mathbf{v})$. Since $P\mathbf{v} \in W$ and $(\mathbf{v} - P\mathbf{v}) \in W^\perp$, these vectors are orthogonal. By the Pythagorean theorem:

$$\|\mathbf{v}\|^2 = \|P\mathbf{v}\|^2 + \|\mathbf{v} - P\mathbf{v}\|^2 \geq \|P\mathbf{v}\|^2$$

Taking square roots on both sides gives $\|P\mathbf{v}\| \leq \|\mathbf{v}\|$ for all $\mathbf{v} \in V$.

- (c) If $T^2 = T$ but the $\text{range}(T)$ and $\text{ker}(T)$ are not orthogonal subspaces, there exist non-zero vectors $\mathbf{r} \in \text{range}(T)$ and $\mathbf{k} \in \text{ker}(T)$ such that $\langle \mathbf{r}, \mathbf{k} \rangle \neq 0$.

Without loss of generality, scale $\mathbf{k}$ such that $\langle \mathbf{r}, \mathbf{k} \rangle < 0$. Set $\mathbf{v} = \mathbf{r} + \mathbf{k}$. Applying T :

$$T\mathbf{v} = T(\mathbf{r} + \mathbf{k}) = T\mathbf{r} + T\mathbf{k} = \mathbf{r} + \mathbf{0} = \mathbf{r}$$

Now compute $\|\mathbf{v}\|^2$:

$$\|\mathbf{v}\|^2 = \|\mathbf{r} + \mathbf{k}\|^2 = \|\mathbf{r}\|^2 + 2\langle \mathbf{r}, \mathbf{k} \rangle + \|\mathbf{k}\|^2$$

By shrinking $\mathbf{k}$ further (i.e., replacing $\mathbf{k}$ with $\epsilon\mathbf{k}$ for sufficiently small $\epsilon > 0$), we ensure:

$$2\epsilon\langle \mathbf{r}, \mathbf{k} \rangle + \epsilon^2\|\mathbf{k}\|^2 < 0 \iff \epsilon\|\mathbf{k}\|^2 < -2\langle \mathbf{r}, \mathbf{k} \rangle$$

Under this choice, $\|\mathbf{v}\|^2 < \|\mathbf{r}\|^2 = \|T\mathbf{v}\|^2$, so $\|T\mathbf{v}\| > \|\mathbf{v}\|$.

Finally, by definition of the operator norm:

$$\|T\|_2 = \max_{\mathbf{x} \neq \mathbf{0}} \frac{\|T\mathbf{x}\|}{\|\mathbf{x}\|} \geq \frac{\|T\mathbf{v}\|}{\|\mathbf{v}\|} > 1$$

Problem 3

Consider the non-linear autonomous two-dimensional system:

$$\begin{cases} \dot{x} = 2y - 2x^3 + xy^2 \\ \dot{y} = -x - y^3 - 2x^2y \end{cases}$$

- (a) Linearize the system around the critical point $(0,0)$. Show that linearization alone is inconclusive regarding stability.
- (b) Use a Lyapunov function $V(x, y)$ to prove the global asymptotic stability or instability of the critical point.
- (c) Show that the origin is the unique critical point.

Solution

- (a) The Jacobian matrix $J(x, y)$ of the system at the origin is:

$$J(0, 0) = \begin{bmatrix} 0 & 2 \\ -1 & 0 \end{bmatrix}$$

The characteristic equation is $\det(J(0, 0) - \lambda I) = \lambda^2 + 2 = 0$, which yields purely imaginary eigenvalues:

$$\lambda = \pm i\sqrt{2} \implies \operatorname{Re}(\lambda) = 0$$

Thus the origin is a non-hyperbolic critical point. The linearized system has a center there, and linearization cannot determine stability at such points.

- (b) Let $V(x, y) = \frac{1}{2}(ax^2 + by^2)$ with $a, b > 0$. The time derivative along system trajectories is:

$$\begin{aligned} \dot{V}(x, y) &= ax\dot{x} + by\dot{y} \\ &= ax(2y - 2x^3 + xy^2) + by(-x - y^3 - 2x^2y) \\ &= 2axy - 2ax^4 + ax^2y^2 - bxy - by^4 - 2bx^2y^2 \end{aligned}$$

Grouping terms yields:

$$\dot{V}(x, y) = (2a - b)xy - 2ax^4 + (a - 2b)x^2y^2 - by^4$$

To eliminate the indefinite cross-term $(2a - b)xy$, set its coefficient to zero:

$$2a - b = 0 \implies b = 2a$$

Choosing $a = 1$ gives $b = 2$, which defines our candidate Lyapunov function:

$$V(x, y) = \frac{1}{2}x^2 + y^2$$

and gives

$$\dot{V}(x, y) = -2x^4 - 3x^2y^2 - 2y^4.$$

$\dot{V}(x, y)$ is a sum of squared quantities multiplied by negative coefficients, meaning $\dot{V}(x, y) < 0$ for all $(x, y) \neq (0, 0)$, and $V(0, 0) = 0$. Hence, $\dot{V}$ is strictly negative definite and furthermore V is positive definite ($V > 0$ for all $(x, y) \neq (0, 0)$). By Lyapunov's direct method for global asymptotic stability, the origin $(0, 0)$ is globally asymptotically stable.

- (c) $\dot{V} = \partial_x V \dot{x} + \partial_y V \dot{y}$. Therefore at any critical point we must have $\dot{V} = 0$, and the origin is the only point where $\dot{V} = 0$.

Problem 4

(a) Find the general solution of the ODE

$$y'' + 2y' + y = 2e^{-x} + 2\cos(x).$$

(b) Consider the initial value problem:

$$\frac{dy}{dx} = \frac{x(y-1)^{2/3}}{\sqrt{1+x^2}}, \quad y(x_0) = y_0$$

where the real value of $(\cdot)^{2/3}$ is used.

- (i) Apply a standard existence and uniqueness theorem to determine all initial points $(x_0, y_0) \in \mathbb{R}^2$ where a unique local solution $y(x)$ is guaranteed to exist. By “local” we mean on some x -interval containing x_0 .
- (ii) Show that there are at least two distinct local solutions at points where the theorem’s assumptions do not apply.

Solution

(a) The characteristic equation for $y'' + 2y' + y = 0$ is:

$$r^2 + 2r + 1 = (r+1)^2 = 0 \implies r = -1 \quad (\text{repeated root of multiplicity 2})$$

Thus, the complementary solution is:

$$y_c(x) = c_1 e^{-x} + c_2 x e^{-x}.$$

Particular solution: The right-hand side forcing term is $g(x) = 2e^{-x} + 2\cos(x)$.

- For $2e^{-x}$: The base trial form is Ae^{-x} . Since both e^{-x} and xe^{-x} are present in $y_c(x)$, we multiply by x^2 to eliminate duplication:

$$y_{p1}(x) = Ax^2 e^{-x}$$

- For $2\cos(x)$: No terms duplicate $y_c(x)$. The trial form is:

$$y_{p2}(x) = B\cos(x) + C\sin(x)$$

Combining both gives the total trial form:

$$y_p(x) = Ax^2 e^{-x} + B\cos(x) + C\sin(x)$$

For $y_{p1}(x) = Ax^2 e^{-x}$:

$$y'_{p1}(x) = A(2x - x^2)e^{-x}$$

$$y''_{p1}(x) = A(2 - 4x + x^2)e^{-x}$$

Plugging into $y'' + 2y' + y$:

$$y''_{p1} + 2y'_{p1} + y_{p1} = 2Ae^{-x} \equiv 2e^{-x} \implies A = 1$$

For $y_{p2}(x) = B\cos(x) + C\sin(x)$:

$$\begin{aligned} y''_{p2} + 2y'_{p2} + y_{p2} &= (-B\cos x - C\sin x) + 2(-B\sin x + C\cos x) + (B\cos x + C\sin x) \\ &= 2C\cos(x) - 2B\sin(x) \equiv 2\cos(x) \end{aligned}$$

Equating coefficients gives $C = 1, B = 0$.

Therefore, $y_p(x) = x^2 e^{-x} + \sin(x)$, and the general solution is:

$$y(x) = c_1 e^{-x} + c_2 x e^{-x} + x^2 e^{-x} + \sin(x)$$

(b) (i):

1. $f(x, y)$ is continuous everywhere on $\mathbb{R}^2$ and $\frac{\partial f}{\partial y} = \frac{2x}{3\sqrt{1+x^2}(y-1)^{1/3}}$ for $y \neq 1$, so $\frac{\partial f}{\partial y}$ is continuous for all $y \neq 1$. By Picard's Existence and Uniqueness Theorem, a unique local solution is guaranteed for all initial points (x_0, y_0) with $y_0 \neq 1$.

(ii): Consider the initial condition $y(x_0) = 1$.

• **Solution 1:**

$$y_1(x) \equiv 1 \quad \text{for all } x \in \mathbb{R}$$

• **Solution 2:** For $y > 1$ near $y = 1$,

$$(y-1)^{-2/3} dy = \frac{x}{\sqrt{1+x^2}} dx \implies y(x) = 1 + \left(\frac{\sqrt{1+x^2}}{3} - \frac{\sqrt{1+x_0^2}}{3} \right)^3$$

Problem 5

An infinite elastic string satisfies the one-dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} = 4 \frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty, \quad t > 0$$

(a) Change variables using $\xi = x - ct$ and $\eta = x + ct$ to derive the general solution as a sum of two undetermined functions. Hint:

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} \quad ; \quad \frac{\partial u}{\partial t} = \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial t}.$$

(b) The string initially has a triangular profile with zero velocity:

$$u(x, 0) = f(x) = \begin{cases} 1 - |x|, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases} \quad ; \quad u_t(x, 0) = 0.$$

Use the initial conditions to complete the solution of $u(x, t)$. Describe the motion in words.

(c) Calculate the displacement $u(0, t)$ at $x = 0$ for $t \geq 0$. Describe it in words.

Solution

Solution

(a) Let $\xi = x - ct$ and $\eta = x + ct$. By the multivariable chain rule, the first-order partial derivatives are:

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial x} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} \\ \frac{\partial u}{\partial t} &= \frac{\partial u}{\partial \xi} \frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta} \frac{\partial \eta}{\partial t} = -c \frac{\partial u}{\partial \xi} + c \frac{\partial u}{\partial \eta} \end{aligned}$$

Applying the operators a second time to find second derivatives:

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta} \right) = \frac{\partial^2 u}{\partial \xi^2} + 2 \frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2} \\ \frac{\partial^2 u}{\partial t^2} &= \frac{\partial}{\partial t} \left(-c \frac{\partial u}{\partial \xi} + c \frac{\partial u}{\partial \eta} \right) = c^2 \frac{\partial^2 u}{\partial \xi^2} - 2c^2 \frac{\partial^2 u}{\partial \xi \partial \eta} + c^2 \frac{\partial^2 u}{\partial \eta^2} \end{aligned}$$

Substituting these expressions into the wave equation $\frac{\partial^2 u}{\partial t^2} - 4\frac{\partial^2 u}{\partial x^2} = 0$:

$$\left(c^2 \frac{\partial^2 u}{\partial \xi^2} - 2c^2 \frac{\partial^2 u}{\partial \xi \partial \eta} + c^2 \frac{\partial^2 u}{\partial \eta^2} \right) - 4 \left(\frac{\partial^2 u}{\partial \xi^2} + 2 \frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2} \right) = 0$$

Take $c = 2$ to simplify:

$$-16 \frac{\partial^2 u}{\partial \xi \partial \eta} = 0 \implies \frac{\partial^2 u}{\partial \xi \partial \eta} = 0$$

Integrating first with respect to ξ yields $\frac{\partial u}{\partial \eta} = g(\eta)$, and integrating a second time with respect to η gives:

$$u(x, t) = F(\xi) + G(\eta) = F(x - 2t) + G(x + 2t)$$

(b) Determine F and G by applying the initial conditions:

1. **Initial Position:**

$$u(x, 0) = F(x) + G(x) = f(x)$$

2. **Initial Velocity:** Differentiating $u(x, t)$ with respect to t via the chain rule gives $u_t(x, t) = -2F'(x - 2t) + 2G'(x + 2t)$. At $t = 0$:

$$u_t(x, 0) = -2F'(x) + 2G'(x) = 0 \implies F'(x) = G'(x) \implies F(x) = G(x) + C$$

Setting $C = 0$ (as any additive constant can be absorbed into the choice of arbitrary functions), we obtain $F(x) = G(x)$. Substituting this into the initial position equation:

$$2F(x) = f(x) \implies F(x) = G(x) = \frac{1}{2}f(x)$$

Substituting F and G back into the general solution yields D'Alembert's formula for a system released from rest:

$$u(x, t) = \frac{1}{2} [f(x - 2t) + f(x + 2t)]$$

The motion is a sum of two triangular profiles, one moving rightward and one moving leftward away from the origin, each moving at speed 2.

(c) Evaluating $u(x, t)$ at $x = 0$:

$$u(0, t) = \frac{1}{2} [f(-2t) + f(2t)]$$

Since $f(x) = 1 - |x|$ is an even function ($f(-2t) = f(2t)$), this simplifies to:

$$u(0, t) = f(2t)$$

Applying the piecewise definition of $f(2t)$:

$$u(0, t) = \begin{cases} 1 - |2t|, & |2t| \leq 1 \\ 0, & |2t| > 1 \end{cases}$$

For $t \geq 0$,

$$u(0, t) = \begin{cases} 1 - 2t, & 0 \leq t \leq \frac{1}{2} \\ 0, & t > \frac{1}{2} \end{cases}$$

The displacement goes from 1 to 0 linearly in time, and then remains at zero when the triangular waves have moved away.