Department of Mathematics, University of Michigan
Complex Analysis Qualifying Review Exam
January 6, 2025 (9am-noon)

Problem 1. Find the number of solutions (counted with multiplicities) of the equation $\sin z = z + 2025z^3$ that belong to the horizontal strip $\{z \in \mathbb{C} : |\operatorname{Im} z| < 1\}$.

Problem 2. Let $A = \{z \in \mathbb{C} : 5 < |z| < 10\}$ and $f : A \to \mathbb{C}$ be an analytic function such that $|f(z)| \le 1 + 2025|z|^{-4}$ for all $z \in A$. Let $f(z) = \sum_{n=-\infty}^{+\infty} a_n z^n$ be the Laurent expansion of f in the annulus A. Prove that $|a_{-2}| \le 90$.

Problem 3. Let $f_n : \mathbb{D} \to \mathbb{H}$ be analytic functions, where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ stands for the unit disc and $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ is the upper half-plane. Assume that the sequence $f_n(1/k)$ converges (to a finite number or to ∞) as $n \to \infty$ for each $k \in \mathbb{N}$. Prove that the sequence $f_n(z)$ converges for each $z \in \mathbb{D}$.

Problem 4. Let $P(z) = c_0 + c_1 z + c_2 z^2 + \ldots + c_n z^n$ be a polynomial of degree n with *real* coefficients $c_k \in \mathbb{R}$ and assume that $P(\mathrm{i}) = \mathrm{i}$. Denote by $z_1, z_2, \ldots, z_n \in \mathbb{C}$ the zeroes of P (counted with multiplicity). Express $\sum_{k=1}^{n} \frac{1}{z_k^2+1}$ as a contour integral and prove that this integral equals $\operatorname{Re} P'(\mathrm{i})$.

Problem 5. Let $S := \{z \in \mathbb{C} : |\operatorname{Im} z| < 1\}$ and $f : S \to \mathbb{D}$ be an analytic function such that $f(0) = 0$, where $\mathbb{D}$ stands for the unit disc. Prove that $|f(1)| \le \tanh \frac{\pi}{4}$ (here, $\tanh a := (e^a - e^{-a})/(e^a + e^{-a})$).