Department of Mathematics, University of Michigan
Complex Analysis Qualifying Review Exam
January 6, 2024 (9am-noon)

Problem 1. Find the number of solutions (counted with multiplicities) of the equation $\sin z = z + 2025z^3$ that belong to the horizontal strip $\{z \in \mathbb{C} : |\operatorname{Im} z| < 1\}$.

Solution: Note that $|\sin z| \le e^{|\operatorname{Im} z| \le e}$ for all $z \in \mathbb{C}$ such that $|\operatorname{Im} z| \le 1$. Hence,

$$2025|z|^3 > |z| + e \ge |z - \sin z| \quad \text{for all } z \in \mathbb{C} \text{ such that } |z| \ge 1 \text{ and } |\operatorname{Im} z| \le 1.$$

This allows one to apply the Rouché theorem to the functions $f(z) := 2025z^3$ and $g(z) := z - \sin z$ in this horizontal strip. Namely, let $R \ge 1$ and consider a rectangle $\{z \in \mathbb{C} : |\operatorname{Im} z| < 1, |\operatorname{Re} z| < R\}$. One has $|f(z)| > |g(z)|$ everywhere on the boundary of such a rectangle and hence the functions $f(z)$ and $f(z) + g(z)$ have the same number of zeroes, counted with multiplicities, inside it. It follows that the equation $2025z^3 + z - \sin z = 0$ has the same number of solutions in the horizontal strip $\{z \in \mathbb{C} : |\operatorname{Im} z| < 1\}$ as the equation $2025z^3 = 0$, i.e., three solutions.

Problem 2. Let $A = \{z \in \mathbb{C} : 5 < |z| < 10\}$ and $f : A \to \mathbb{C}$ be an analytic function such that $|f(z)| \le 1 + 2025|z|^{-4}$ for all $z \in A$. Let $f(z) = \sum_{n=-\infty}^{+\infty} a_n z^n$ be the Laurent expansion of f in the annulus A. Prove that $|a_{-2}| \le 90$.

Solution: For each $5 < r < 10$ one has

$$|a_{-2}| \;=\; \left| \frac{1}{2\pi \mathrm{i}} \oint_{|z|=r} f(z) z dz \right| \;\le\; r^2 \max_{|z|=r} |f(z)| \;\le\; r^2 + 2025r^{-2}.$$

Let us now optimize the right-hand side in r. It is easy to see that the minimum is attained when $r^2 = \sqrt{2025} = 45$, which leads to the desired bound $|a_{-2}| \le 90$. (Note also that $5 < \sqrt{45} < 10$.)

Problem 3. Let $f_n : \mathbb{D} \to \mathbb{H}$ be analytic functions, where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ stands for the unit disc and $\mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ is the upper half-plane. Assume that the sequence $f_n(1/k)$ converges (to a finite number or to ∞) as $n \to \infty$ for each $k \in \mathbb{N}$. Prove that the sequence $f_n(z)$ converges for each $z \in \mathbb{D}$.

Solution: Recall that the upper half-plane $\mathbb{H}$ and the unit disc $\mathbb{D}$ are conformally equivalent: Möbius mapping $\phi : \mathbb{H} \to \mathbb{D}$, $z \mapsto \frac{z-\mathrm{i}}{z+\mathrm{i}}$ is a bijection. Let us consider the functions $g_n := \phi \circ f_n : \mathbb{D} \to \mathbb{D}$, it is clear that the sequence $g_n(1/k)$ converges as $n \to \infty$ for each $k \in \mathbb{N}$. The functions g_n are uniformly bounded and hence form a normal family due to Montel's theorem. (In fact, a general form of this theorem applies directly to f_n but we do not use it here.)

Therefore, each subsequence of g_n contains a sub-subsequence that converges to an analytic function g in $\mathbb{D}$. It is clear that $g(1/k) = \phi(\lim_{n\to\infty} f_n(1/k))$. Using uniqueness theorem for analytic functions, this implies that all such subsequential limits have to be the same. In particular, this implies that the sequence $g_n(z)$ converges for each $z \in \mathbb{D}$ as otherwise, one could find two different subsequential limits. It remains to note that the convergence of $g_n(z)$ implies the convergence (to a finite number or to ∞) of $f_n(z) = (\phi^{-1} \circ g_n)(z)$.

Problem 4. Let $P(z) = c_0 + c_1 z + c_2 z^2 + \ldots + c_n z^n$ be a polynomial of degree n with *real* coefficients $c_k \in \mathbb{R}$ and assume that $P(\mathrm{i}) = \mathrm{i}$. Denote by $z_1, z_2, \ldots, z_n \in \mathbb{C}$ the zeroes of P (counted with multiplicity). Express $\sum_{k=1}^{n} \frac{1}{z_k^2+1}$ as a contour integral and prove that this integral equals $\operatorname{Re} P'(\mathrm{i})$.

Solution: Note that $P(\mathrm{i}) = \mathrm{i} \neq 0$ and $P(-\mathrm{i}) = \overline{P(\mathrm{i})} = -\mathrm{i} \neq 0$. The logarithmic derivative $P'(z)/P(z)$ has simple poles at each of the zeroes of P with the residues equal to their multiplicities. Therefore, applying the Cauchy formula to the function $P'(z)/(P(z)(z^2+1))$ one obtains the identity

$$\sum_{k=1}^{n} \frac{1}{z_k^2+1} + \operatorname*{res}_{z=\mathrm{i}} \frac{P'(z)}{P(z)(z^2+1)} + \operatorname*{res}_{z=-\mathrm{i}} \frac{P'(z)}{P(z)(z^2+1)} = \frac{1}{2\pi\mathrm{i}} \oint_{|z|=R} \frac{P'(z)}{P(z)(z^2+1)} dz\,,$$

provided that R is large enough so that all the poles are inside the circle $|z| = R$. It is clear that $P'(z)/(P(z)(z^2+1)) = O(|z|^{-3})$ as $|z| \to \infty$, which implies that the right-hand side tends to zero as $R \to \infty$. This gives the identity

$$\sum_{k=1}^{n} \frac{1}{z_k^2+1} + \operatorname*{res}_{z=\mathrm{i}} \frac{P'(z)}{P(z)(z^2+1)} + \operatorname*{res}_{z=-\mathrm{i}} \frac{P'(z)}{P(z)(z^2+1)} = 0\,.$$

(A slightly different way of proving this identity is to say that the residue of the meromorphic function $P'(z)/(P(z)(z^2+1))$ at ∞ equals 0.) Finally, $P(\mathrm{i}) = \mathrm{i} \neq 0$ gives

$$\operatorname*{res}_{z=\mathrm{i}} \frac{P'(z)}{P(z)(z^2+1)} = \frac{P'(\mathrm{i})}{\mathrm{i}\cdot 2\mathrm{i}} = -\frac{P'(\mathrm{i})}{2} \quad \text{and similarly} \quad \operatorname*{res}_{z=-\mathrm{i}} \frac{P'(z)}{P(z)(z^2+1)} = -\frac{P'(-\mathrm{i})}{2}\,.$$

Therefore, $\sum_{k=1}^{n} \frac{1}{z_k^2+1} = \frac{1}{2}(P'(\mathrm{i}) + P'(-\mathrm{i})) = \frac{1}{2}(P'(\mathrm{i}) + \overline{P'(\mathrm{i})}) = \operatorname{Re} P'(\mathrm{i})$.

Problem 5. Let $S := \{z \in \mathbb{C} : |\operatorname{Im} z| < 1\}$ and $f : S \to \mathbb{D}$ be an analytic function such that $f(0) = 0$, where $\mathbb{D}$ stands for the unit disc. Prove that $|f(1)| \leq \tanh \frac{\pi}{4}$ (here, $\tanh a := (e^a - e^{-a})/(e^a + e^{-a})$).

Solution: Due to the Riemann uniformization theorem, the strip S is conformally equivalent to the unit disc $\mathbb{D}$. In order to construct a uniformization, one can, e.g., first apply the map $z \mapsto \exp(\frac{\pi}{2} z)$, which maps S onto the right-half plane, and then use a Möbius map $z \mapsto \frac{z-1}{z+1}$. The composition of these two maps reads as

$$\phi : S \to \mathbb{D}, \quad z \mapsto \frac{e^{\frac{\pi}{2} z} - 1}{e^{\frac{\pi}{2} z} + 1} = \tanh(\tfrac{\pi}{4} z)\,.$$

We can now apply the Schwarz–Pick lemma to the function $f \circ \phi^{-1} : \mathbb{D} \to \mathbb{D}$. (Note that $f(\phi^{-1}(0)) = f(0) = 0$.) It gives

$$|f(1)| = |(f \circ \phi^{-1})(\tanh \tfrac{\pi}{4})| \leq \tanh \tfrac{\pi}{4}\,.$$