Department of Mathematics, University of Michigan
Complex Analysis Qualifying Review Exam

May 5, 2025, 9:00am-noon

1. Let an analytic function $f : B_r(0) \smallsetminus \{0\} \to \mathbb{C}$ have an essential singularity at 0.
(a) Prove that the function $f(z) + f(z^2)$ *cannot* have a removable singularity at 0.
(b) Can it happen that the function $f(z) + f(z^2)$ has a pole at 0?

2. Let a function $f : \overline{\mathbb{D}} \smallsetminus \{0\} \to \mathbb{C}$ be analytic in the punctured unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and continuous in $\overline{\mathbb{D}} \smallsetminus \{0\}$. Assume that f has a simple pole at 0 and that $\operatorname{Im} f(z) \geq 0$ for all for all $z \in \partial\mathbb{D}$ (i.e., for all $z \in \mathbb{C}$ with $|z| = 1$). Prove that there exists $z \in \mathbb{D}$ such that $f(z) = -\mathrm{i}$.

3. Let $U \subsetneq \mathbb{C}$ be an open set such that $0 \in U$ and $f : U \to U$ be an analytic function such that $f(0) = 0$ and $f'(0) = 1$. Prove that $f(z) = z$ for all $z \in U$
(a) assuming that U is simply connected;
(b) assuming that U is bounded (but not necessarily simply connected).
[*Hint:* consider iterations $f \circ f \circ \ldots \circ f$ of f.]

4. Let $f : \mathbb{D} \to \mathbb{C}$ be an analytic function in the unit disc $\mathbb{D}$ such that $f(0) = 1$ and $|f(z)| \leq 2025$ for all $z \in \mathbb{D}$. Assume that this function has $n \geq 1$ zeroes $z_1, \ldots, z_n$ (listed with multiplicities) in $\mathbb{D}$. Prove that $\prod_{k=1}^{n} |z_k| \geq \frac{1}{2025}$.

5. Compute the integral $\displaystyle\int_{-\infty}^{+\infty} \frac{x}{\sinh x} dx$ via residue calculus.