Department of Mathematics, University of Michigan
Complex Analysis Qualifying Review Exam

May 5, 2025, 9:00am-noon

1. Let an analytic function $f : B_r(0) \setminus \{0\} \to \mathbb{C}$ have an essential singularity at 0.
(a) Prove that the function $f(z) + f(z^2)$ *cannot* have a removable singularity at 0.
(b) Can it happen that the function $f(z) + f(z^2)$ has a pole at 0?

Solution: **(a)** Let $\sum_{n=-\infty}^{+\infty} a_n z^n$ be the Laurent series of f in the punctured disc $B_r(0) \setminus \{0\}$ and recall that this series converges uniformly on compacts. Hence,

$$f(z) + f(z^2) = \sum_{n=-\infty} c_n z^n + \sum_{n=-\infty}^{+\infty} c_n z^{2n} = \sum_{k=-\infty}^{+\infty} c_{2k+1} z^{2k+1} + \sum_{k=-\infty}^{+\infty} (c_{2k} + c_k) z^{2k}$$

and these series converge uniformly on compact subsets of the punctured disc $B_{\min(r,\sqrt{r})}(0) \setminus \{0\}$. Suppose that this function has a removable singularity at 0. Then, $c_{2k+1} = 0$ and $c_{2k} = -c_k$ for all $k < 0$. It is easy to see that these conditions imply $c_n = 0$ for all $n < 0$, which means that the function f also has a removable singularity at 0, a contradiction.

(b) Suppose now that the function $f(z) + f(z^2)$ has a pole at 0. This means that there is $N \in \mathbb{N}$ such that $c_{2k+1} = 0$ and $c_{2k} = -c_k$ for all $k \le -N$. As f has an essential singularity at 0 we know that infinitely many coefficients c_n with $n < 0$ do not vanish. Hence, there exists $m < 0$ such that $c_m \neq 0$ and $c_{2^k m} = (-1)^k c_m$. In particular, this means that the series $\sum_{n=-\infty}^{+\infty} c_n z^n$ *cannot* converge for $|z| < 1$, which is a contradiction. Therefore, the function $f(z) + f(z^2)$ cannot have a pole at 0 and has to have an essential singularity.

2. Let a function $f : \overline{\mathbb{D}} \setminus \{0\} \to \mathbb{C}$ be analytic in the punctured unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and continuous in $\overline{\mathbb{D}} \setminus \{0\}$. Assume that f has a simple pole at 0 and that $\operatorname{Im} f(z) \ge 0$ for all for all $z \in \partial\mathbb{D}$ (i.e., for all $z \in \mathbb{C}$ with $|z| = 1$). Prove that there exists $z \in \mathbb{D}$ such that $f(z) = -\mathrm{i}$.

Solution: It follows from the argument principle that the number of zeroes of the meromorphic function $f + \mathrm{i}$ minus the number of poles (both counted with multiplicities) in the unit disc $\mathbb{D}$ equals the winding number $W_{f(\partial\mathbb{D})}(0)$ of the closed curve $\gamma(\theta) = f(e^{\mathrm{i}\theta}) + \mathrm{i}$, $\theta \in [0, 2\pi]$ around 0. As $\operatorname{Im}(f(e^{\mathrm{i}\theta}) + \mathrm{i}) \ge 1$, this number is 0. The function $f + \mathrm{i}$ has exactly one simple pole in $\mathbb{D}$ and thus must have exactly one zero, which means that the equation $f(z) + \mathrm{i}$ has a unique solution in $\mathbb{D}$.

Remark. One can write this solution as follows; note that this is nothing but reproving of the argument principle in the given context. Let $\mathbb{D}_r := r\mathbb{D} = B(0, r)$ be the disc of radius r centered at 0 and $0 < \varepsilon \ll 1$. The number of solutions (counted with multiplicities) of the equation $f(z) = -\mathrm{i}$ in the annulus $\mathbb{D}_{1-\varepsilon} \setminus \overline{\mathbb{D}_\varepsilon}$ equals

$$\frac{1}{2\pi \mathrm{i}} \left(\oint_{\partial\mathbb{D}_{1-\varepsilon}} \frac{f'(z)dz}{f(z) + \mathrm{i}} - \oint_{\partial\mathbb{D}_\varepsilon} \frac{f'(z)dz}{f(z) + \mathrm{i}} \right).$$

The first term equals 0 since $\operatorname{Im}(f(z) + \mathrm{i}) \ge \frac{1}{2}$ for all $z \in \partial\mathbb{D}_{1-\varepsilon}$ (provided that ε is small enough) and hence the increment of $\log(f(z) + \mathrm{i})$ along $\partial\mathbb{D}_{1-\varepsilon}$ is zero. To compute the second term (again, for small enough ε), note that $f(z)$ cannot attain the value $-\mathrm{i}$ in a small vicinity of 0 and that the function $f'(z)/(f(z)+\mathrm{i})$ has a simple pole at 0 with residue -1. Therefore, $\frac{1}{2\pi\mathrm{i}}\oint_{\partial\mathbb{D}_\varepsilon}\frac{f'(z)dz}{f(z)+\mathrm{i}}=-1$ and hence the equation $f(z)+\mathrm{i}=0$ has a (unique) solution in the punctured disc $\mathbb{D}\smallsetminus\{0\}$.

Alternative solution: Suppose that $f(z)\neq-\mathrm{i}$ for all $z\in\mathbb{D}\smallsetminus\{0\}$. Consider a Möbius mapping $\phi(z):=\frac{z-\mathrm{i}}{z+\mathrm{i}}$ and note that ϕ maps the upper half-plane $\mathbb{H}$ onto the unit disc $\mathbb{D}$. Denote $g:=\phi\circ f:\overline{\mathbb{D}}\smallsetminus\{0\}\to\mathbb{C}$. This function is holomorphic in the punctured disc and has a *removable* singularity at 0: if we denote $g(z):=1$, then g becomes continuous at 0. However, we know that $\operatorname{Im} f(z)\geq 0$ and hence $|g(z)|\leq 1$ for $z\in\partial\mathbb{D}$. This contradicts to the maximum principle: it follows that $g(z)=1$ for all $z\in\mathbb{D}$, which is impossible since $f(z)\neq\infty$ for $z\neq 0$.

3. Let $U\subsetneq\mathbb{C}$ be an open set such that $0\in U$ and $f:U\to U$ be an analytic function such that $f(0)=0$ and $f'(0)=1$. Prove that $f(z)=z$ for all $z\in U$

(a) assuming that U is simply connected;

(b) assuming that U is bounded (but not necessarily simply connected).
[*Hint:* consider iterations $f\circ f\circ\ldots\circ f$ of f.]

Solution: **(a)** This is an easy corollary of the Schwarz–Pick lemma and the Riemann uniformization theorem. Namely, let $\phi:U\to\mathbb{D}$ be a conformal mapping such that $f(0)=0$. Then, the function $g:=\phi\circ f\circ\phi^{-1}:\mathbb{D}\to\mathbb{D}$ satisfies $g(0)=0$ and $g'(0)=\phi'(0)f'(0)(\phi^{-1})'(0)=f'(0)=1$. This implies $g(z)=z$ for all $z\in\mathbb{D}$ and hence $f(z)=z$ for all $z\in U$.

(b) It is easy to see that all the iterations $f_n:=f\circ\ldots\circ f:U\to U$ satisfy the same conditions $f_n(0)=0$ and $f_n'(0)=0$. Suppose that f is not the identity function and let $m\geq 2$ be the minimal non-zero coefficient in the Taylor expansion of f around the origin, i.e., $f(z)=z+c_mz^m+O(z^{m+1})$ as $z\to 0$. A simple computation shows (e.g., by induction) that one has $f_n(z)=z+nc_mz^m+O(z^{m+1})$ as $z\to 0$. This easily leads to a contradiction: e.g., Cauchy's formula for a small enough (but fixed) ε gives the identity

$$nc_m=\frac{1}{2\pi\mathrm{i}}\oint_{|z|=\varepsilon}\frac{f_n(z)}{z^{m+1}}dz$$

and these integrals are uniformly (in m) bounded as $n\to\infty$ as the values $f_n(z)\in U$ are uniformly bounded. (Alternatively, the concluding part of the argument can be replaced by Montel's theorem on normal families of analytic functions.)

4. Let $f:\mathbb{D}\to\mathbb{C}$ be an analytic function in the unit disc $\mathbb{D}$ such that $f(0)=1$ and $|f(z)|\leq 2025$ for all $z\in\mathbb{D}$. Assume that this function has $n\geq 1$ zeroes $z_1,\ldots,z_n$ (listed with multiplicities) in $\mathbb{D}$. Prove that $\prod_{k=1}^n|z_k|\geq\frac{1}{2025}$.

Solution: Recall that the fractional-linear functions $z\mapsto\frac{z-z_n}{1-\overline{z}_nz}$ send the unit disc $\mathbb{D}$ onto itself. Denote $g(z):=\prod_{k=1}^n\frac{z-z_n}{1-\overline{z}_nz}$ and consider the ratio f/g. This function is analytic in $\mathbb{D}$ (since the zeroes of g cancel with those of f) and satisfies the estimate $|f(z)/g(z)|\leq 2025/|g(z)|$. The maximal principle implies that

$$\frac{|f(0)|}{|g(0)|}\;=\;\frac{1}{\prod_{k=1}^n|z_k|}\;\leq\;\frac{2025}{\min_{|z|=r}|g(z)|}\quad\text{for all } r<1.$$

It remains to note that $\min_{|z|=r}|g(z)|\to 1$ as $r\uparrow 1$ (recall that each factor in the definition of g sends the unit circle onto the unit circle) and hence $\prod_{k=1}^n|z_k|\geq\frac{1}{2025}$.

5. Compute the integral $\displaystyle\int_{-\infty}^{+\infty} \frac{x}{\sinh x} dx$ via residue calculus.

Solution: Note that $f(z) := z/\sinh(z)$ is a meromorphic function with poles at the points $\pi i m$, $m \in \mathbb{Z} \smallsetminus \{0\}$ and that $\sinh(z+\pi \mathrm{i}) = -\sinh(z)$. (Note that the singularity at $z = 0$ is removable.) For $R \gg 1$ and $0 < \varepsilon \ll 1$ consider the closed contour $\gamma = \gamma(\varepsilon, R)$ in the complex plane formed by the horizontal segment $\gamma_1 := [-R; R]$, vertical segment $\gamma_2 := [R; R+\pi \mathrm{i}]$, horizontal segment $\gamma_3 := [R+\pi \mathrm{i}, \varepsilon+\pi \mathrm{i}]$, half-circle $\gamma_4 := \{z = \pi \mathrm{i} + \varepsilon e^{\mathrm{i}\theta}, \theta \in [0, -\pi]\}$, horizontal segment $\gamma_5 := [-\varepsilon+\pi \mathrm{i}; -R+\pi \mathrm{i}]$ and vertical segment $\gamma_6 := [-R+\pi \mathrm{i}; -R]$. The function $z/\sinh z$ does not have singularities inside γ, hence

$$\oint_{\gamma} \frac{z dz}{\sinh(z)} = 0\,.$$

As $R \to +\infty$ (first) and $\varepsilon \to 0$ (after that), we have the following:

- $\int_{\gamma_1} f(z)dz \to I := \int_{-\infty}^{+\infty} \frac{x dx}{\sinh x}$, the quantity that we want to compute;
- contribution of vertical segments vanishes: $|\int_{\gamma_2} f(z)dz| + |\int_{\gamma_6} f(z)dz| \to 0$ since $|\sinh(z)| = \frac{1}{2}|e^z - e^{-z}| \geq \frac{1}{2}(e^{|\operatorname{Re} z|} - e^{-|\operatorname{Re} z|})$;
- $\int_{\gamma_3 \cup \gamma_5} f(z)dz \underset{R\to+\infty}{\to} -\int_{\mathbb{R}\smallsetminus[-\varepsilon;\varepsilon]} \frac{x+\pi i}{-\sinh x} dx = \int_{\mathbb{R}\smallsetminus[-\varepsilon;\varepsilon]} \frac{x}{\sinh(x)} dx$ since the contribution of the term $\pi \mathrm{i}$ in the numerator vanishes due to the symmetry, and the latter integral converges to I as $\varepsilon \to 0$.

Therefore,

$$2I = -\lim_{\varepsilon\to 0} \int_{\gamma_4} f(z)dz \;=\; \lim_{\varepsilon\to 0} \int_{-\pi}^{0} f(\pi \mathrm{i} + \varepsilon e^{\mathrm{i}\theta}) \cdot \mathrm{i}\varepsilon e^{\mathrm{i}\theta} d\theta$$

$$\stackrel{(\star)}{=} \pi \mathrm{i} \operatorname*{res}_{z=\pi \mathrm{i}} \frac{z}{\sinh(z)} = \pi \mathrm{i} \cdot \frac{\pi \mathrm{i}}{\cosh(\pi \mathrm{i})} = \pi^2\,.$$

(Recall that the equality $(\star)$ follows from the fact that f has a *simple* pole at $z = \pi \mathrm{i}$ and hence has the expansion $f(\pi \mathrm{i} + \varepsilon e^{\mathrm{i}\theta}) = \varepsilon^{-1} e^{-\mathrm{i}\theta} \cdot \operatorname{res}_{z=\pi \mathrm{i}} f(z) + O(1)$ as $\varepsilon \to 0$.) This concludes the computation: $I = \frac{\pi^2}{2}$.

Remark: Many other standard ideas can be used: e.g., change of the variable $w = e^z$ leads to the integral $2\int_0^{+\infty} \frac{\log x}{x^2-1} dx$, which can be evaluated by similar methods.