Algebra 1 QR — May 2025

Problem 1. Let $A \subset B$ be rings, with B a principal ideal domain, and let I be an ideal of B. Show, by example, that $I \cap A$ need not be a principal ideal of A. Be sure to justify why your example works.

Problem 2. Let M be a finitely generated $\mathbb{Z}$-module. Show that there exists an integer $c \geq 0$ such that for all positive integers n we have

$$\#(M/nM) - \#(M[n]) = n^c.$$

Here $M[n]$ denotes the set of elements $x \in M$ such that $nx = 0$.

Problem 3. Let V be a two dimensional $\mathbb{Q}$-vector space with basis e_1 and e_2. Write down four vectors $v_1, \ldots, v_4$ such that $v_1^{\otimes 3}, \ldots, v_4^{\otimes 3}$ are linearly independent in $V^{\otimes 3}$. Here $v^{\otimes 3}$ denotes the vector $v \otimes v \otimes v$ in $V^{\otimes 3} = V \otimes V \otimes V$. Be sure to justify your answer.

Problem 4. Let A and B be 6×6 matrices with complex entries. Suppose that

$$\begin{aligned} 3 &= \dim(\ker(A)) = \dim(\ker(B)) \\ 5 &= \dim(\ker(A^2)) = \dim(\ker(B^2)) \\ 6 &= \dim(\ker(A^3)) = \dim(\ker(B^3)). \end{aligned}$$

Show that A and B are conjugate.

Problem 5. Show that $\mathbb{Z}[x]/(x^3 - 1)$ has exactly two idempotents, and $\mathbb{Q}[x]/(x^3 - 1)$ has exactly four idempotents. Recall that an *idempotent* is an element a such that $a^2 = a$.