Algebra II QR — May 2025

Problem 1. Let G be a finite group, let p be the smallest prime dividing the order of G, and suppose that $H \subset G$ is a subgroup of index p. Prove that H is a normal subgroup of G.

Problem 2. Let p be a prime. Prove that there exists a nonabelian group G which can be expressed as a semidirect product $(\mathbf{Z}/p)^2 \rtimes \mathbf{Z}/p$, and that any two such nonabelian semidirect products are isomorphic.

Problem 3. Let $K/\mathbf{Q}$ be a Galois extension of fields of degree 49. Suppose that there there exist two distinct intermediate fields $\mathbf{Q} \subset F_1 \subset K$ and $\mathbf{Q} \subset F_2 \subset K$ which are both of degree 7 over $\mathbf{Q}$. Determine the group $\mathrm{Gal}(K/\mathbf{Q})$ (up to isomorphism).

Hint: Recall that a group whose order is the square of a prime number is necessarily abelian.

Problem 4. Let $K/\mathbf{Q}$ be a Galois extension of fields with $\mathrm{Gal}(K/\mathbf{Q}) = A_5$, the group of even permutations of 5 elements. Let $f(x) \in \mathbf{Q}[x]$ be a degree 3 polynomial that has a root in K. Prove that $f(x)$ has a root in $\mathbf{Q}$.

Problem 5. Suppose that n is a positive integer such that $\cos(\frac{2\pi}{n})$ is a rational number. Prove that $n \in \{1, 2, 3, 4, 6\}$.

Hint: The identity $\cos(\frac{2\pi}{n}) = \frac{\zeta_n + \zeta_n^{-1}}{2}$ and the isomorphism $\mathrm{Gal}(\mathbf{Q}(\zeta_n)/\mathbf{Q}) \cong (\mathbf{Z}/n)^\times$, where $\zeta_n = e^{\frac{2\pi i}{n}} = \cos(\frac{2\pi}{n}) + i\sin(\frac{2\pi}{n})$, may be useful.