Algebra II QR — January 2025

Problem 1. Let G be a finite group of order mn, where m and n are relatively prime integers, and assume that there exists a subgroup $M \subset G$ of order m and a subgroup $N \subset G$ of order n. Prove that G is isomorphic to a subgroup of the symmetric group S_{m+n} on $m+n$ elements.

Problem 2. Let D_8 be the dihedral group of order 8, i.e. the group of symmetries of a square. Prove that there is an isomorphism $\mathrm{Aut}(D_8) \cong D_8$, where $\mathrm{Aut}(D_8)$ is the group of automorphisms of D_8.

Hint: It may be useful to consider an embedding $D_8 \hookrightarrow D_{16}$.

Problem 3. Let p be a prime. Compute the number of Sylow p-subgroups of $\mathrm{GL}_2(\mathbf{F}_p)$.

Problem 4. Let $F/\mathbf{Q}$ be a field extension such that F is contained in the ring $M_n(\mathbf{Q})$ of $n \times n$ matrices over $\mathbf{Q}$ (i.e. there is an injective ring homomorphism $F \to M_n(\mathbf{Q})$). Prove that the degree of the extension $F/\mathbf{Q}$ satisfies $[F : \mathbf{Q}] \leq n$.

Problem 5. Let p be a prime, let $a \in \mathbf{F}_p$ be a nonzero element, and consider the polynomial $f(x) = x^p - x + a \in \mathbf{F}_p[x]$. Let L be the splitting field of $f(x)$. Prove that the field extension $L/\mathbf{F}_p$ is Galois, and determine the Galois group.

You may use without proof that $f(x)$ is an irreducible polynomial.