THE UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS

Qualifying Review examination in Algebraic Topology

January 2025

1. Let X, Y be two connected based CW-complexes where $\pi_1(X)$ is a free group $F(x,y)$ on two free generators x, y and $\pi_1(Y)$ is a free group on two free generators u, v. A based CW-complex Q is obtained from the disjoint union $X \amalg Y$ by identifying the base points and additionally identifying a CW-subcomplex $Z \subset X$ with a CW-subcomplex $T \subset Y$, both disjoint from the basepoint, via a homeomorphism $Z \cong T \cong S^1$. Assume that the resulting embeddings $S^1 \subset X$, $S^1 \subset Y$ send a given free generator of $\pi_1(S^1)$ to (a conjugate of) $x^2 \in \pi_1(X)$ and (a conjugate of) $u^2 \in \pi_1(Y)$. Compute $\pi_1(Q)$.

2. Let X be the covering space of $\mathbb{R}P^2 \times \mathbb{R}P^2$ corresponding to the diagonal $\mathbb{Z}/2$-subgroup of $\pi_1(\mathbb{R}P^2 \times \mathbb{R}P^2)$, identified with $\mathbb{Z}/2 \times \mathbb{Z}/2$ via the two projections
$$\mathbb{R}P^2 \times \mathbb{R}P^2 \to \mathbb{R}P^2.$$
Compute the homology of X.

3. (a) For what odd natural numbers k does there exists an $n > 0$ such that $\mathbb{Z}/k$ acts freely on $\mathbb{C}P^n$? Prove your answer.

   (b) For what odd natural numbers k does there exists an $n > 0$ such that $\mathbb{Z}/k$ acts freely on $\mathbb{R}P^n$? Prove your answer.

4. Prove or disprove the following statement:

   "Every continuous based map between based CW-complexes which induces 0 on reduced homology groups is homotopic to the constant map into the base point."

   [By *homology* we mean singular homology with coefficients in $\mathbb{Z}$, as covered in Math 592.]

5. Give an explicit example of a non-normal subgroup of index 3 of the free group $F(a,b)$ on two elements.