THE UNIVERSITY OF MICHIGAN
DEPARTMENT OF MATHEMATICS

Qualifying Review examination in Algebraic Topology

January 2025

1. Let X, Y be two based CW-complees where $\pi_1(X)$ is the free group $F(x, y)$ on two free generators x, y and $\pi_1(Y)$ is the free group on two generators u, v. A based CW-complex Q is obtained from $X \amalg Y$ by identifying the base points and additionally identifying a CW-complex $Z \subset X$ with a CW-complex $T \subset Y$ (both disjoint from the basepoint) via a homeomorphism $Z \cong T \cong S^1$ so that the embeddings send a free generator of $\pi_1(S^1)$ to (a conjugate of) x^2 and (a conjugate of) u^2. Compute $\pi_1(Q)$ i terms of generators and relations.

   Solution: To apply the Seifert-VanKampen theorem, choose path ω in X from the image of $1 \in S^1$ to the base point of X and a path η in Y from the image of $1 \in S^1$ to the base point of Y. Form a space X' by attaching the domain of the path η by its endpoints to X and a space Y' by attaching the domain of the path ω by its endpoints to Y. This leads to a diagram of a wedge of two spheres mapping to X' and Y', to whose based homotopy pushout Q is equivalent. This leads to a presenttion of the form
$$\langle x, y, u, v, \beta \mid x^2 \beta u^2 \beta^{-1} \rangle$$
   where β is the generator of the fundamental group of the 1-spbere formed by the paths ω, η.

2. Let X be the covering space of $\mathbb{R}P^2 \times \mathbb{R}P^2$ corresponding to the diagonal $\mathbb{Z}/2$-subgroup of π_1. Compute the homology of X.

   Solution: The homology in degrees $0, 1, 2, 3, 4$ is $\mathbb{Z}, \mathbb{Z}/2, \mathbb{Z}/2, 0, \mathbb{Z}$. For example, consider first the space Y which is the pullback of the covering space to $S^1 \times \mathbb{R}P^2$ where S^1 maps to $\mathbb{R}P^2$ via the generator of π_1. Then Y is the homotopy coequalizer of the identity and the antipodal map on S^2, so its homology in degrees $0, 1, 2$ is $\mathbb{Z}, \mathbb{Z}, \mathbb{Z}/2$ by the long exact sequence in homology. Now the space X can be constructed as the homotopy pushout of the map $S^1 \times S^2$ to Y (the universal cover) and the projection $S^1 \times Sr \to S^2$, so the homology can be calculated by the Mayer-Vietoris sequence, giving the result.

3. (a) For what odd natural numbers k does there exists an n such that $\mathbb{Z}/k$ act freely on $\mathbb{C}P^n$? Prove your answer.

(b) (a) For what odd natural numbers k does there exists an n such that $\mathbb{Z}/k$ act freely on $\mathbb{R}P^n$? Prove your answer.

Soluton: (a) Only $k = 1$. Since k is odd, the generator would have to act trivially on all homology groups (since the non-trivial ones are $\mathbb{Z}$), and thus, since the non-trivial homology groups are in even degrees, the Lefschetz number would not be 0.

(b) Every odd natural number k, considering the action of $\mathbb{Z}/2k$ on S^1.

4. Prove or disprove the following statement:

   "Every continuous based map between based CW-complexes which induces 0 on reduced homology groups is homotopic to the constant map into the base point."

   Solution: It is not true. Consider the covering map

   $$f : S^2 \to \mathbb{R}P^2.$$

   The map f induces 0 in reduced homology since the two spaces do not share a non-zero homology group in the same positive dimension.

   On the other hand, if f was homotopic to 0, (base points don't matter), then the homotopy would lift to a homotopy of the identity $S^2 \to S^2$ with a constant map, which is not possible, since $H_2 S^2 \neq 0$.

5. Give an explicit example of non-normal subgroup of index 3 of the free group $F(a, b)$ on two elements.

   Solution: For example the free generators a, b^3, b^2ab^{-1}, bab^{-2} (coming from a non-regular covering of a wedge of two circles.