Applied Functional Analysis QR Exam

January 7, 2025

Problem 1

Consider the boundary-value problem on $[0,1]$:

$$u''(x) + \pi^2 u(x) = f(x), \quad u(0) + u(1) = 0, \quad u'(0) + u'(1) = 0.$$

Determine the adjoint problem on $L^2(0,1)$, and use the result to characterize the quadratic polynomials $f(x) = Ax^2 + Bx + C$ for which this problem has a solution. Characterize the ambiguity, if any, in the corresponding solution.

Problem 2

Let f and g be given elements of the sequence space $\ell^1(\mathbb{N})$, and consider the equation

$$u_j = f_j + \left(\sum_{i=1}^{\infty} |u_i| \right)^2 g_j, \quad j \in \mathbb{N}.$$

Let $\|\cdot\|$ denote the $\ell^1(\mathbb{N})$ norm. Show that if $\|f\| \cdot \|g\| < \frac{1}{4}$, then there exists a unique solution $u \in \ell^1(\mathbb{N})$ with the additional property that $\|u\| \leq (1 - \sqrt{1 - 4\|f\| \cdot \|g\|})/(2\|g\|)$.

Problem 3

Let $0 < p < 1$. Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined for $x \neq 0$ by $f(x) := |x|^{-p}$ and with $f(0) = 0$ defines a distribution in $\mathcal{D}'(\mathbb{R})$. Then find its derivative, and write it without derivatives or any limits.

Problem 4

Consider the inner product defined for complex-valued functions f, g defined on $(0, \infty)$ by

$$\langle f, g \rangle := \int_0^{\infty} f(x) g(x)^* \mathrm{e}^{-x} \, \mathrm{d}x.$$

This defines an inner-product space on functions $f : (0, \infty) \to \mathbb{C}$ for which the corresponding norm is finite. Let $\mathcal{H}$ be the Hilbert space obtained by completion. Show that the monomials $\{x^n\}_{n=0}^{\infty}$ all lie in $\mathcal{H}$, and apply the Gram-Schmidt process to explicitly compute an orthonormal basis of $\mathrm{span}\{1, x\}$. Then show that $\mathrm{e}^{x/4} \in \mathcal{H}$ and find the linear function that best approximates it in $\mathcal{H}$.

Problem 5

Consider the sequence space $\ell^2(\mathbb{N})$ and the linear operator A with action

$$A(u_1, u_2, u_3, u_4, \dots) := \left(0, u_1, \frac{u_2}{2^2}, \frac{u_3}{3^2}, \dots \right).$$

Determine whether A is bounded, and if so, compute its norm. Determine whether A is compact, and prove or disprove. Determine the kernel and range of A and whether they are closed.