AIM Qualifying Review Exam in Advanced Calculus & Complex Variables

January 2026

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. If you use other sheets of paper, be sure to mark them clearly and staple them to the booklet. No credit will be given for answers without supporting work and/or reasoning.

Problem 1

Identify whether each of the following statements are true or false, and provide a proof or a counterexample as appropriate:

(a) A uniformly continuous function $f : (0, 1) \to \mathbb{R}$ is bounded.

(b) If $f : \mathbb{R} \to \mathbb{R}$ maps Cauchy sequences to Cauchy sequences, then f is continuous.

(c) The pointwise limit of a sequence of uniformly continuous functions on $\mathbb{R}$ is uniformly continuous.

(d) If $f, g : \mathbb{R} \to \mathbb{R}$ are uniformly continuous, then their product fg is uniformly continuous.

Solution

(a) True. Since f is uniformly continuous on $(0, 1)$ it extends to a continuous function on $[0, 1]$, which is bounded by the extreme value theorem.

(b) True. Let $x_n \to x$ and define a new sequence $\tilde{x}_n$ that alternates between x_n and x, i.e., let $\tilde{x}_n = x_{n/2}$ if n is even and $\tilde{x}_n = x$ if n is odd. Then $\tilde{x}_n \to x$ and is Cauchy; hence $f(\tilde{x}_n)$ is Cauchy and therefore converges. Since $f(\tilde{x}_n) = f(x)$ for odd n, $f(\tilde{x}_n) \to f(x)$. Examining even n we get that $f(x_n) \to f(x)$.

(c) False. For a counterexample take $f_n : \mathbb{R} \to \mathbb{R}$ to be given by $f_n(x) = \min(x^2, n)$ and note that $f_n \to x^2$ pointwise. The latter is not uniformly continuous.

(d) False. For a counterexample let $f(x) = g(x) = x$. Both f and g are uniformly continuous, but $fg = x^2$ is not uniformly continuous.

Problem 2

Let

$$f(x) = \begin{cases} 1 + x & x > 0 \\ 0 & x = 0 \\ -1 + x & x < 0 \end{cases}$$

and consider the minimization problems

$$\text{Problem I:} \quad \min_{c \in \mathbb{R}} \int_{-1}^{1} |f(x) - c|^2 \, dx \qquad \text{and} \qquad \text{Problem II:} \quad \min_{c \in \mathbb{R}} \int_{-1}^{1} |f(x) - c| \, dx.$$

(a) Produce an optimal c for each problem. Be sure to justify your steps, e.g., if you decide to commute two limits as part of a calculation you must explain why that is possible. (Hint: it is not necessary to commute limits to solve this problem; if you are stuck, try drawing a graph.)

(b) One of these problems has more than one minimizer. Identify which problem and find all of its minimizers.

Solution

For problem one, write

$$\int_{-1}^{1} |f - c|^2 = \int_{-1}^{1} f^2 - 2c \int_{-1}^{1} f + 2c^2.$$

The expression is quadratic in c and convex, so its unique minimizer can be found by the critical point test. Compute

$$\frac{d}{dc} \int_{-1}^{1} |f - c|^2 = -2 \int_{-1}^{1} f + 4c.$$

The unique critical c is

$$c = \frac{1}{2} \int_{-1}^{1} f = 0.$$

This answers (a) and (b) for problem one.

For problem two, we evaluate the integral

$$I(c) = \int_{-1}^{1} |f - c|$$

as a piecewise function of c. First, simplify by noting that

$$I(c) = I(-c)$$

by the odd symmetry $f(x) = -f(-x)$ and a change of variables:

$$\int_{-1}^{1} |f(x) - c| \, dx = \int_{-1}^{1} | - f(-x) - c| \, dx = \int_{-1}^{1} |f(x) + c| \, dx.$$

Now consider the cases (i) $c \in [0,1]$, (ii) $c \in (1,2)$, and (iii) $c \in [2,\infty)$. In case (i),

$$\begin{aligned} I(c) &= \left(\int_{-1}^{0} + \int_{0}^{1}\right) |f - c| = \int_{-1}^{0} c - f + \int_{0}^{1} f - c \\ &= -\int_{-1}^{0} f + \int_{0}^{1} f. \end{aligned}$$

In case (ii),

$$\begin{aligned} I(c) &= \left(\int_{-1}^{c-1} + \int_{c-1}^{1}\right) |f - c| = \int_{-1}^{c-1} c - f + \int_{c-1}^{1} f - c \\ &= c(c-1+1) - \int_{-1}^{c-1} f + \int_{c-1}^{1} f - c(1-(c-1)) \\ &= 2c^2 - 2c - \int_{-1}^{c-1} f + \int_{c-1}^{1} f. \end{aligned}$$

In case (iii),

$$\begin{aligned} I(c) &= \int_{-1}^{1} |f - c| = \int_{-1}^{1} c - f \\ &= 2c - \int_{-1}^{1} f. \end{aligned}$$

Altogether,

$$I(c) = \begin{cases} 2c - \int_{-1}^{1} f & c \in [2,\infty) \\ 2c^2 - 2c - \int_{-1}^{c-1} f + \int_{c-1}^{1} f & c \in (1,2) \\ -\int_{-1}^{0} f + \int_{0}^{1} f & c \in [0,1] \end{cases}.$$

Since $I(c) = I(-c)$ we have the formula for I.

We are to minimize. Differentiating the formula above,

$$\frac{d}{dc} I(c) = \begin{cases} 2 & c \in (2,\infty) \\ 2c - 2 & c \in (1,2) \\ 0 & c \in (0,1) \end{cases}.$$

Note I is strictly increasing on $[1,\infty)$, strictly decreasing on $(-\infty,-1]$, and constant on $[-1,1]$. Thus I is minimized for $c \in [-1,1]$. This answers (a) and (b) for problem two.

<u>Problem 3</u>

Evaluate

$$\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)(3+x^2)^2}\, dx$$

Solution

Write $z = x + iy$. Introduce the contour Γ_R made of the segment on the real axis from $x = -R$ to $x = R$ and the semicircle from $x = R$ to $x = -R$ in the upper half plane. First, note that

$$\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)(3+x^2)^2}\,dx = \lim_{R\to\infty} \oint_{\Gamma_R} \frac{z^2}{(1+z^2)(3+z^2)^2}\,dz$$

since on the semicircle the integrand is $\lesssim R^{2-2-4} = R^{-4}$ and the length of the semicircle is $\sim R$, so the contribution from the semicircle to the integral is $\lesssim R^{-3} \to 0$ as $R \to \infty$.

Next, use residue calculus to evaluate the integral over Γ_R for large R. The integrand has poles at $\pm i$ and $\pm\sqrt{3}i$. The relevant poles are i and $\sqrt{3}i$. By the residue theorem,

$$\oint_{\Gamma_R} \frac{z^2}{(1+z^2)(3+z^2)^2}\,dz = 2\pi i \left(\mathrm{Res}(z=i) + \mathrm{Res}(z=\sqrt{3}i)\right)$$

for large R. The pole at i is simple, and its residue $i/8$ is found by writing

$$\frac{z^2}{(1+z^2)(3+z^2)^2} = \frac{i^2}{(z-i)(i+i)(3+i^2)^2} + \text{h.o.t.} = \frac{i}{8}\frac{1}{z-i} + \text{h.o.t.}$$

The pole at $\sqrt{3}i$ is a double pole, and its residue $-\sqrt{3}i/12$ is found by taking $z = \sqrt{3}i + \zeta$ and collecting the $1/\zeta$ terms:

$$\begin{aligned}
&\frac{z^2}{(1+z^2)(3+z^2)^2} \\
&= \frac{(\sqrt{3}i+\zeta)^2}{(1+(\sqrt{3}i+\zeta)^2)(3+(\sqrt{3}i+\zeta)^2)^2} = \frac{-3+2\sqrt{3}i\zeta+\zeta^2}{(-2+2\sqrt{3}i\zeta+\zeta^2)(-12\zeta^2+4\sqrt{3}i\zeta^3+\zeta^4)} \\
&= \frac{-3+2\sqrt{3}i\zeta+\zeta^2}{24\zeta^2}\frac{1}{(1-\sqrt{3}i\zeta+\frac{1}{2}\zeta^2)(1-\frac{\sqrt{3}i}{3}\zeta-\frac{1}{12}\zeta^2)} = \left(-\frac{1}{8}\frac{1}{\zeta^2}+\frac{\sqrt{3}i}{12}\frac{1}{\zeta}+\frac{1}{24}\right)\left(1+\frac{4\sqrt{3}i}{3}\zeta+\text{h.o.t.}\right) \\
&= -\frac{1}{8}\frac{1}{\zeta^2} - \frac{\sqrt{3}i}{12}\frac{1}{\zeta} + \text{h.o.t.}
\end{aligned}$$

Therefore,

$$\int_{-\infty}^{\infty} \frac{x^2}{(1+x^2)(3+x^2)^2}\,dx = 2\pi i\left(\frac{i}{8} - \frac{\sqrt{3}i}{12}\right) = \pi\left(\frac{\sqrt{3}}{6} - \frac{1}{4}\right).$$

Problem 4

Consider the complex polynomial

$$p(z) = 2z^4 + z^3 + 8z - 4.$$

How many zeros counting multiplicity does p have in the annulus $1 < |z| < 2$?

Solution

The answer is three. To prove it, use Rouche's theorem twice. First, take $|z| = 2$ and note that

$$|2z^4| > |z^3 + 8z - 4|$$

because

$$|z^3 + 8z - 4| \le |z|^3 + 8|z| + 4 = 2^3 + 8 \cdot 2 + 4 = 2^3 + 2^4 + 2^2$$

and

$$|2z^4| = 2^5$$

and $2^5 > 2^2 + 2^3 + 2^4$. Thus, by Rouche, p has the same number of zeros in $|z| < 2$ as does $2z^4$, i.e., four zeros.

Next take $|z| = 1$ and use instead that

$$|8z - 4| > |2z^4 + z^3|$$

since

$$|8z - 4| = 8|z - \frac{1}{2}| \ge 4$$

whereas

$$|2z^4 + z^3| \le 2|z|^4 + |z|^3 = 3.$$

Again, by Rouche, p has the same number of zeros in $|z| < 1$ as does $8z - 4$, i.e., one. The same conclusion holds for $|z| \le 1$, and subtracting gives the result.

Problem 5

Recall that the real and imaginary parts of a complex analytic function are harmonic. Using this, find a harmonic function $v = v(x, y)$ on the unit disc $x^2 + y^2 < 1$ such that $v = 0$ for $x^2 + y^2 = 1$ and $x > 0$, and $v = 1$ for $x^2 + y^2 = 1$ and $x < 0$. Your answer may involve the complex variable $z = x + iy$.

Solution

The Möbius transformation $z \mapsto \frac{1}{z-i} + \frac{1}{2i}$ takes the unit disc to the upper half plane, and sends $i \mapsto \infty$ and $-i \mapsto 0$. Consequently, it takes the right half of the unit circle to the positive real axis, and the left half of the unit circle to the negative real axis. Writing $\log z = \log|z| + i \arg z$ for the complex logarithm with branch cut on the negative imaginary axis, so that $\arg z \in (-\pi/2, 3\pi/2)$, we see that

$$w(x, y) = \frac{1}{\pi} \arg(x + iy)$$

is harmonic and satisfies $w = 0$ for $x > 0$ and $y = 0$, and $w = 1$ for $x < 0$ and $y = 0$. Thus

$$v(x, y) = \frac{1}{\pi} \arg\left(\frac{1}{x + iy - i} + \frac{1}{2i}\right)$$

is the desired function.