

AIM Qualifying Review Exam in Advanced Calculus & Complex Variables

August 2026

There are five (5) problems in this examination.

There should be sufficient room in this booklet for all your work. If you use other sheets of paper, be sure to mark them clearly and staple them to the booklet. No credit will be given for answers without supporting work and/or reasoning.

Problem 1

Assume f is a **strictly decreasing and continuous** function on $\mathbb{R}$ and $\{x_n\}$ is a bounded sequence.

Prove: $\liminf_{n \rightarrow \infty} f(x_n) = f(\limsup_{n \rightarrow \infty} x_n)$.

Solution

Since $\{x_n\}$ is a bounded sequence, $\limsup x_n$ exists and is finite. Let $M = \limsup x_n$. According to the definition of $\limsup x_n$, for any given $\epsilon > 0$, there exists $N > 0$ such that for any $n > N$, $x_n < M + \epsilon$. This implies

$$f(x_n) > f(M + \epsilon), \quad n > N.$$

Thus, we have $\liminf_{n \rightarrow \infty} f(x_n) \geq f(M + \epsilon)$ for any $\epsilon > 0$. Letting ϵ go to 0 and using f is continuous, we have

$$\liminf_{n \rightarrow \infty} f(x_n) \geq f\left(\limsup_{n \rightarrow \infty} x_n\right).$$

Also, according to the definition of $\limsup x_n$, for any given $\epsilon > 0$ and $N > 0$, there exists $n > N$, $x_n > M - \epsilon$. This implies

$$\inf_{n \geq N} f(x_n) < f(M - \epsilon),$$

for any N . This implies

$$\liminf_{n \rightarrow \infty} f(x_n) < f(M - \epsilon)$$

for any $\epsilon > 0$. Letting ϵ go to 0 and using f is continuous, we have

$$\liminf_{n \rightarrow \infty} f(x_n) \leq f\left(\limsup_{n \rightarrow \infty} x_n\right).$$

Problem 2

Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a **smooth** function. Prove the the following two statements are equivalent:

- For any $x, \xi \in \mathbb{R}^n$,

$$\sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(x) \xi_i \xi_j \geq 0.$$

- For any $x, y \in \mathbb{R}^n$,

$$(\nabla f(x) - \nabla f(y)) \cdot (x - y) \geq 0$$

Solution

Define $\mathcal{H}f(x)$ for the Hessian matrix of f at x . We have

$$\sum_{i,j=1}^n \frac{\partial^2 f}{\partial x_i \partial x_j}(x) \xi_i \xi_j = \xi^T \mathcal{H}f(x) \xi.$$

Assume first that

$$\xi^T \mathcal{H}f(x) \xi \geq 0$$

for every $x, \xi \in \mathbb{R}^n$. Fix $x, y \in \mathbb{R}^n$ and define

$$\gamma(t) = y + t(x - y), \quad 0 \leq t \leq 1.$$

By the fundamental theorem of calculus,

$$\begin{aligned} (\nabla f(x) - \nabla f(y)) \cdot (x - y) &= \int_0^1 \frac{d}{dt} (\nabla f(\gamma(t))) \cdot (x - y) dt \\ &= \int_0^1 (x - y)^T \mathcal{H}f(\gamma(t)) (x - y) dt \\ &\geq 0. \end{aligned}$$

Conversely, assume that

$$(\nabla f(x) - \nabla f(y)) \cdot (x - y) \geq 0$$

for every $x, y \in \mathbb{R}^n$. Fix $x, \xi \in \mathbb{R}^n$ and let $h > 0$. Apply the monotonicity condition with $x + h\xi$ in place of x and x in place of y . Then

$$(\nabla f(x + h\xi) - \nabla f(x)) \cdot h\xi \geq 0.$$

for any $h > 0$. Dividing by $h^2 > 0$ gives

$$\left(\frac{\nabla f(x + h\xi) - \nabla f(x)}{h} \right) \cdot \xi \geq 0.$$

Letting $h \rightarrow 0^+$ and using smoothness of f ,

$$\xi^T \mathcal{H}f(x) \xi \geq 0.$$

Problem 3

Let E be the solid region in $\mathbb{R}^3$ lying above the cone

$$z = \sqrt{x^2 + y^2}$$

and below the sphere

$$x^2 + y^2 + z^2 = 16.$$

- (a) Describe the region E using spherical coordinates $x = \rho \cos \theta \sin \phi$, $y = \rho \sin \theta \sin \phi$, $z = \rho \cos \phi$.
- (b) Evaluate the integral

$$\iiint_E z \, dV.$$

- (c) Rewrite the same integral in cylindrical coordinates $x = r \cos \theta$, $y = r \sin \theta$, $z = z$, including all limits of integration.

Solution

In spherical coordinates, $x = \rho \cos \theta \sin \phi$, $y = \rho \sin \theta \sin \phi$, $z = \rho \cos \phi$. The cone $z = \sqrt{x^2 + y^2}$ is the surface $\phi = \pi/4$. Thus E is described by

$$0 \leq \theta \leq 2\pi, \quad 0 \leq \phi \leq \frac{\pi}{4}, \quad 0 \leq \rho \leq 4.$$

Therefore

$$\begin{aligned}
\iiint_E z \, dV &= \int_0^{2\pi} \int_0^{\pi/4} \int_0^4 (\rho \cos \phi) \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta \\
&= 2\pi \left(\int_0^4 \rho^3 \, d\rho \right) \left(\int_0^{\pi/4} \sin \phi \cos \phi \, d\phi \right) \\
&= 2\pi(64) \left(\frac{1}{4} \right) = 32\pi.
\end{aligned}$$

In cylindrical coordinates the sphere is $z = \sqrt{16 - r^2}$ and the cone is $z = r$. They intersect when $r = \sqrt{16 - r^2}$, so $r = 2\sqrt{2}$. Hence the same integral is

$$\int_0^{2\pi} \int_0^{2\sqrt{2}} \int_r^{\sqrt{16-r^2}} z \, r \, dz \, dr \, d\theta.$$

Problem 4

Evaluate the real integral

$$\int_{-\infty}^{\infty} \frac{\cos(3x)}{(x^2 + 1)^2(x^2 + 4)} \, dx.$$

Solution

Consider

$$F(z) = \frac{e^{3iz}}{(z^2 + 1)^2(z^2 + 4)}$$

and integrate F over the upper-half-plane semicircle contour C_R . Since

$$|e^{3iz}| = e^{-3\operatorname{Im} z},$$

we can justify that the integral over the semicircular vanishes. Let

$$\Gamma_R = \{Re^{i\theta} : 0 \leq \theta \leq \pi\}.$$

For $z \in \Gamma_R$ and $R > 3$,

$$|z^2 + 1| \geq R^2 - 1, \quad |z^2 + 4| \geq R^2 - 4,$$

and also $|e^{3iz}| \leq 1$ on the upper half-plane. Hence

$$\begin{aligned}
\left| \int_{\Gamma_R} F(z) \, dz \right| &\leq \int_{\Gamma_R} \frac{|e^{3iz}|}{|z^2 + 1|^2 |z^2 + 4|} |dz| \\
&\leq \frac{\pi R}{(R^2 - 1)^2 (R^2 - 4)} \rightarrow 0 \quad \text{as } R \rightarrow \infty.
\end{aligned}$$

Therefore

$$\int_{C_R \cap \{|z|=R, \operatorname{Im} z \geq 0\}} F(z) \, dz \rightarrow 0 \quad \text{as } R \rightarrow \infty.$$

Thus

$$\int_{-\infty}^{\infty} \frac{e^{3ix}}{(x^2 + 1)^2(x^2 + 4)} dx = 2\pi i \sum \text{Res}(F; z_k),$$

where the poles in the upper half-plane are $z = i$, a double pole, and $z = 2i$, a simple pole. At $z = 2i$,

$$\text{Res}(F; 2i) = \frac{e^{3i(2i)}}{((2i)^2 + 1)^2(2i + 2i)} = \frac{e^{-6}}{9 \cdot 4i} = -\frac{ie^{-6}}{36}.$$

For the double pole at $z = i$, write

$$F(z) = \frac{1}{(z - i)^2} \frac{e^{3iz}}{(z + i)^2(z^2 + 4)}.$$

Hence

$$\text{Res}(F; i) = \frac{d}{dz} \left(\frac{e^{3iz}}{(z + i)^2(z^2 + 4)} \right) \Big|_{z=i}.$$

Let

$$H(z) = \frac{e^{3iz}}{(z + i)^2(z^2 + 4)}.$$

Differentiate H directly. If

$$D(z) = (z + i)^2(z^2 + 4),$$

then

$$H(z) = \frac{e^{3iz}}{D(z)}$$

and

$$H'(z) = \frac{3ie^{3iz}D(z) - e^{3iz}D'(z)}{D(z)^2}.$$

Now

$$D'(z) = 2(z + i)(z^2 + 4) + (z + i)^2(2z).$$

At $z = i$,

$$D(i) = (2i)^2(3) = -12$$

and

$$D'(i) = 2(2i)(3) + (2i)^2(2i) = 12i - 8i = 4i.$$

Therefore

$$\begin{aligned} \text{Res}(F; i) &= H'(i) \\ &= \frac{3ie^{-3}(-12) - e^{-3}(4i)}{(-12)^2} \\ &= -\frac{40ie^{-3}}{144} = -\frac{5ie^{-3}}{18}. \end{aligned}$$

So

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{e^{3ix}}{(x^2 + 1)^2(x^2 + 4)} dx &= 2\pi i \left(-\frac{5ie^{-3}}{18} - \frac{ie^{-6}}{36} \right) \\ &= \frac{5\pi e^{-3}}{9} + \frac{\pi e^{-6}}{18}. \end{aligned}$$

Taking real parts gives

$$\int_{-\infty}^{\infty} \frac{\cos(3x)}{(x^2 + 1)^2(x^2 + 4)} dx = \frac{\pi}{18} (10e^{-3} + e^{-6}).$$

Problem 5

1. Let F be an analytic function on $\mathbb{C}$ satisfying $|F(z)| \leq \sqrt{|z|}$. Prove $F(z) = 0$.

Hint: Write a bound for the first derivative of F .

2. Let U be a bounded and connected open set in $\mathbb{C}$ and let F be continuous on $\overline{U}$ and analytic on U . Assume F has no zeros in U and $|F(z)| = 1$ on the boundary of U . Prove that F is a constant function.

Hint: Consider $1/F(z)$ and open mapping theorem.

Solution

1. First,

$$|F(0)| \leq \sqrt{|0|} = 0,$$

so $F(0) = 0$.

We now show that $F'(z) = 0$ for every $z \in \mathbb{C}$. Fix $z \in \mathbb{C}$. For any $R > |z|$, apply Cauchy's estimate to F on the circle $|\zeta - z| = R$. On this circle,

$$|\zeta| \leq |\zeta - z| + |z| = R + |z|,$$

so

$$|F(\zeta)| \leq \sqrt{R + |z|}.$$

Therefore Cauchy's estimate for the first derivative gives

$$|F'(z)| \leq \frac{\max_{|\zeta - z| = R} |F(\zeta)|}{R} \leq \frac{\sqrt{R + |z|}}{R}.$$

Letting $R \rightarrow \infty$ gives $F'(z) = 0$. Since z was arbitrary, $F' \equiv 0$ on $\mathbb{C}$. Hence F is constant on $\mathbb{C}$. Since $F(0) = 0$, the constant is 0, so $F \equiv 0$.

2. Because F is continuous on $\overline{U}$ and analytic on U , the maximum modulus principle gives

$$|F(z)| \leq \sup_{z \in \partial U} |F(z)| = 1, \quad z \in U,$$

Because F has no zeros in U , the function $1/F$ is analytic on U and continuous on $\overline{U}$. On the boundary,

$$\left| \frac{1}{F(z)} \right| = 1, \quad z \in \partial U.$$

Applying the maximum modulus principle to $1/F$ gives

$$\left| \frac{1}{F(z)} \right| \leq 1, \quad z \in U.$$

Hence $|F(z)| \geq 1$ for $z \in U$. Combining the two inequalities,

$$|F(z)| = 1, \quad z \in U.$$

Thus $F(U)$ is contained in the unit circle. If F were nonconstant, the open mapping theorem would imply that $F(U)$ is open in $\mathbb{C}$, which is impossible because the unit circle has empty interior. Therefore F is constant on U .