Thirty-Fifth
University of Michigan
Undergraduate Mathematics Competition
April 7, 2018

Instructions. Write on the front of your blue book your student ID number. Do not write your name anywhere on your blue book. Each question is worth 10 points. For full credit, you must **prove** that your answers are correct even when the question doesn't say "prove". There are lots of problems of widely varying difficulty. It is not expected that anyone will solve them all; look for ones that seem easy and fun. No calculators are allowed.

Problem 1. A certain number,

when divided by 10 leaves a remainder of 8;
when divided by 9 leaves a remainder of 7;
when divided by 8 leaves a remainder of 6;
when divided by 7 leaves a remainder of 5;
when divided by 6 leaves a remainder of 4;
when divided by 5 leaves a remainder of 3;
when divided by 4 leaves a remainder of 2;
when divided by 3 leaves a remainder of 1;
when divided by 2 leaves a remainder of 0.

What is the smallest positive number satisfying these conditions?

Problem 2. A triangle has integer base and height lengths, b and h. A square of side length 2 is inscribed in the triangle with one side on the given base, and the other two vertices on the other sides of the triangle. What are the possible values of the pair (b, h)?

Problem 3. Do there exist polynomials $P_1, P_2 \in \mathbb{C}[z]$ such that

$$\pm\sqrt{2z + 1 \pm 2\sqrt{z^2 + z + 1}} = \pm\sqrt{P_1(z)} \pm \sqrt{P_2(z)}\,?$$

Problem 4. Call a set $\mathcal{A}$ of positive integers *affable* if it has the following three properties:

(a) $1 \in \mathcal{A}$;
(b) $n \in \mathcal{A}$ implies that $2n \in \mathcal{A}$;
(c) Whenever $\mathcal{A}$ contains three distinct integers whose average m is an integer, $m \in \mathcal{A}$. (For example, if $5 \in \mathcal{A}$, $6 \in \mathcal{A}$, $10 \in \mathcal{A}$, then $21/3 = 7 \in \mathcal{A}$.)

Show that the only affable set is the set of all positive integers.

Problem 5. Suppose that 64 teams are competing in a basketball tournament. Each team is given a number, with different numbers for different teams, and all numbers from the set $\{1, 2, \ldots, 64\}$. The numbers are assigned randomly, so that each of the possible 64! assignments is equally likely. In the first round, team numbered $2k - 1$ plays against the team numbered $2k$, for $k = 1, 2, \ldots, 32$. The losing teams are eliminated. In the second round, the winner of the 1vs2 game plays the winner of the 3vs4 game, and so on, making 16 games. This continues until in the sixth round the single surviving team with a number in $\{1, 2, \ldots, 32\}$ plays the single surviving team with a number in $\{33, 34, \ldots, 64\}$. Assume that the teams are evenly matched, so that each team has an even chance of winning a game, and that these outcomes are independent from game to game. Assume also that of the 64 teams, one is known as Michigan, and another one is known as Ohio State. What is the probability that these two teams play a game against each other in the course of this tournament?

Problem 6. Let f be a differentiable real-valued function on the interval $[0, 2\pi]$ with derivative f'. Show that for every real number a there exists a real number $b \in (0, 2\pi)$ such that the unit vector $(\cos b, \sin b)$ is orthogonal to $\big(f(b), f'(b) - a\big)$.

Problem 7. Let r be a real number such that $\cos 2\pi r = 3/5$. Show that r is irrational.

Problem 8. Suppose that f is a real-valued function with a continuous first derivative on the interval $[0, 1]$. Show that

$$\max_{0 \le x \le 1} |f(x)| \le \int_0^1 |f(u)|\, du + \int_0^1 |f'(u)|\, du\,.$$

Problem 9. Let S and T be linear transformations from V to W, which are 5-dimensional vector spaces over the complex numbers $\mathbb{C}$. The kernels of S and T are disjoint, and for all complex scalars a, b, not both 0, $aS + bT$ has rank 3. Prove that there is that there is a four-dimensional subspace of W that contains the images of both S and T.

Problem 10. Show that there is a function $g : \mathbb{Q}^2 \to \mathbb{Q}$ such that $g(p, y) \in \mathbb{Q}[y]$ whenever $p \in \mathbb{Q}$, and $g(x, q) \in \mathbb{Q}[x]$ whenever $q \in \mathbb{Q}$, but g is not a polynomial.

Contributors: Hyman Bass, Yiwang Chen, Angus Chung, Edward Dunne, Mel Hochster, Igor Kriz, Hugh Montgomery, Mark Rudelson, Wijit Yangjit