UNIVERSITY OF MICHIGAN
UNDERGRADUATE MATH COMPETITION 34
APRIL 8, 2017

SOLUTIONS

Solution 1. We have $P(x) - 2017 = Q(x)(x-a)(x-b)(x-c)(x-d)$ where a, b, c, d are distinct integers and Q is a polynomial with integer coefficients. Suppose that r is an integer and that $P(r) = 0$. Then

$$-2017 = Q(r)(r-a)(r-b)(r-c)(r-d)\,.$$

We know that 2017 is prime. Of the four numbers $r-a, r-b, r-c, r-d$ at most one of them is 1, at most one of them is -1, so the other two have absolute value > 1 and their product divides 2017, a contradiction.

Solution 2. We observe that

$$(1-z)\sum_{n=1}^{\infty}[n\beta]z^n = \sum_{n=1}^{\infty}[n\beta]z^n - \sum_{n=1}^{\infty}[n\beta]z^{n+1}.$$

We reindex the first sum to see that the above is

$$= \sum_{n=0}^{\infty}[(n+1)\beta]z^{n+1} - \sum_{n=1}^{\infty}[n\beta]z^{n+1}.$$

In the first sum, when $n = 0$ the summand is 0, so the value of the sum is unchanged when we delete that term, so the above is

$$= \sum_{n=1}^{\infty}\big([(n+1)\beta] - [n\beta]\big)z^{n+1}.$$

On dividing by z we see that the right hand side of the proposed identity is

$$\sum_{n=1}^{\infty}\big([(n+1)\beta] - [n\beta]\big)z^{n}.$$

Now $0 < \beta < 1$, so $[(n+1)\beta] - [n\beta]$ is 0 or 1 for all n, and it is 1 precisely when there is an integer, say m, between $n\beta$ and $(n+1)\beta$. Since β is irrational, this means that

$n\beta < m < (n+1)\beta$, which is to say $n < m/\beta < n+1$. We note that $\alpha = 1/\beta$. Hence $n = [m\alpha]$. Thus the power series above is exactly

$$\sum_{m=1}^{\infty} z^{[m\alpha]}.$$

Solution 3. Preliminary reasoning: If $a_i = a$ for all i, then the sum on the left is $n/(2a)$, and the sum on the right is $n/(3a)$, a ratio of 3/2. However, if the a_i alternate between a small value a and an enormously larger value b, then the sum on the left is approximately n/b, while the sum on the right is roughly $\frac{n}{2}(\frac{1}{b} + \frac{1}{2b}) = \frac{4}{3} \cdot \frac{n}{b}$. Thus 4/3 is the best possible constant, and we believe that we have identified the worst case, at least for even n. This motivates what follows.

For positive real numbers p and q we have

$$\frac{p+q}{2} \ge \frac{2}{\dfrac{1}{p} + \dfrac{1}{q}}\,.$$

(More generally, the harmonic mean does not exceed the arithmetic mean.) Take $p = 2a + b$ and $q = b + 2c$ to see that

$$\frac{1}{a+b+c} \le \frac{1}{2}\Big(\frac{1}{2a+b} + \frac{1}{b+2c}\Big).$$

Furthermore,

$$\begin{aligned}\frac{3}{a+b} - \frac{2}{2a+b} - \frac{2}{a+2b} &= \frac{3(2a+b)(a+2b) - 2(a+b)(2a+b) - 2(a+b)(a+2b)}{(a+b)(2a+b)(a+2b)} \\ &= \frac{3abc}{(a+b)(2a+b)(a+2b)} > 0\,.\end{aligned}$$

It is now convenient to assume that the a_i are periodic with period n. Thus

$$\begin{aligned}\sum_{i=1}^{n} \frac{1}{a_i + a_{i+1}} &> \frac{2}{3}\Big(\sum_{i=1}^{n} \frac{1}{2a_i + a_{i+1}} + \sum_{i=1}^{n} \frac{1}{a_i + 2a_{i+1}}\Big) \\ &= \frac{2}{3}\Big(\sum_{i=1}^{n} \frac{1}{2a_i + a_{i+1}} + \sum_{i=1}^{n} \frac{1}{a_{i+1} + 2a_{i+2}}\Big) \\ &\ge \frac{4}{3}\sum_{i=1}^{n} \frac{1}{a_i + a_{i+1} + a_{i+2}}\,.\end{aligned}$$

Solution 4. We observe that $2y^3 + 9y^2 - 27 = (2y-3)(y+3)^2$. The power of 2 dividing this number is even, but the power of 2 in $2x^2$ is odd, so these numbers are not equal, if they are nonzero. Therefore, the only solution in integers is $(x, y) = (0, -3)$.

Solution 5. Preliminary comment: Let $A(\mathbb{T})$ denote the set of continuous functions with period 1 whose Fourier series is absolutely convergent. Wiener's famous theorem assert that if $f \in A(\mathbb{T})$, and if $f(x) = 0$ has no solution, then $1/f \in A(\mathbb{T})$. It is trivial that if $f, g \in A(\mathbb{T})$, then $fg \in A(\mathbb{T})$. The same is true for trigonometric polynomials. The question is: Does Wiener's theorem also hold for trigonometric polynomials? The answer is a resounding "No!", as we see by this simple example. This makes a poor competition problem, since it can be solved in so many ways. Some example solutions follow. For reference, let $f(x) = 1/(2 - \cos 2\pi x)$.

(1) For a trigonometric polynomial T in the given form, put

$$d(T) = \max_{t_n \ne 0} n \; - \; \min_{t_n \ne 0} n \,.$$

Clearly, if T_1 and T_2 are trigonometric polynomials, then $d(T_1T_2) = d(T_1) + d(T_2)$. The function $T_1(x) = 2 - \cos 2\pi x$ is a trigonometric polynomial with $d(T_1) = 2$. Suppose that f is a trigonometric polynomial, which is to say that there is a trigonometric polynomial T_2 such that $T_1T_2 = 1$. But $d(1) = 0$, so we have a contradiction.

(2) Corresponding to a trigonometric polynomial in the given generic form, we may define a rational function

$$Q(z) = \sum_{n=-N}^{N} t_n z^n.$$

Thus $T(x)$ is the restriction to the unit circle of Q, in the sense that $T(x) = Q(e(x))$. We note that Q is a rational function, but of a rather special form: its poles are all at the origin. On the other hand, when we apply this change of variable to f, we find that

$$f(x) = \frac{1}{2 - \frac{1}{2}e(x) - \frac{1}{2}e(-x)} = \frac{1}{2 - \frac{1}{2}z - \frac{1}{2z}} = \frac{-2z}{z^2 - 4z + 1},$$

which has poles at $2 \pm \sqrt{3}$, not at the origin. For full points it should be noted that the rational function Q is the unique rational function $R(z)$ such that $R(e(x)) = T(x)$, since Q and R are equal on a continuum.

Solution 6. By a crude form of Stirling's formula we know that

$$\ln k! = k \ln k - k + O(\ln k)\,. \tag{1}$$

Put $\phi = (1 + \sqrt{5})/2$. We know that

$$F_n = \frac{\phi^n - (-1/\phi)^n}{\sqrt{5}}, \tag{2}$$

from which we deduce that

$$\ln F_n = n \ln \phi - \frac{1}{2}\ln 5 + O\big(\phi^{-2n}\big)\,. \tag{3}$$

Since $\ln F_n = O(n)$, it follows by three applications of (1) that

$$\begin{aligned}\ln\binom{F_{n+1}}{F_n} &= \ln F_{n+1}! - \ln F_n! - \ln F_{n-1}!\\ &= F_{n+1}\ln F_n - F_{n+1} - F_n \ln F_n + F_n - F_{n-1}\ln F_{n-1} + F_{n-1} + O(n)\,.\end{aligned}$$

By (3), this is

$$\begin{aligned}&= F_{n+1}\big((n+1)\ln\phi - \frac{1}{2}\ln 5 + O(\phi^{-2n})\big) - F_n\big(n\ln\phi - \frac{1}{2}\ln 5 + O(\phi^{-2n})\big)\\ &\qquad - F_{n-1}\big((n-1)\ln\phi - \frac{1}{2}\ln 5 + O(\phi^{-2n})\big) + O(n)\\ &= (F_{n+1} + F_{n-1})\ln\phi + O(n).\end{aligned}$$

Hence

$$\ln\ln\binom{F_{n+1}}{F_n} = \ln(F_{n+1} + F_{n-1}) + O(1) = \ln F_n + O(1) = n\ln\phi + O(1)\,.$$

Thus the proposed limit exists, and has the value $\ln\phi$.

Solution 7. If $n = 1$, this follows from the Euclidean algorithm. If $n = 2$, then we apply a linear transformation, invertible over the integers, and so assume that one point is $(1, 0)$, and that the other is (a, b). Since a is invertible modulo b, it has finite order. Choose $m > 1$ so that $a^m \equiv 1 \pmod{b}$, say $a^m = 1 + bh$. Since $s^{m-1}b$ and b^m have b as their greatest common divisor, we can write b as a linear combination of these numbers, say $b = ca^{m-1}b + db^m$. Then $x^m - h(cx^{m-1}y + dy^m)$ has the required property. We now induct on n. For $n \geq 3$ there exists an F_1 such that $F_1(a_i, b_i) = 1$ for $i = 2, 3, \ldots, n$, and an F_2 such that $F_2(a_i, b_i) = 1$ for $i = 1, 3, 4, 5, \ldots, n$. We can take powers so that these have the same degrees. Put $r = F_1(a_1, b_1$ and $s = F_2(a_2, b_2)$. It suffices to find a homogeneous polynomial $H(x, y)$ of positive degree whose values on $(r, 1)$, $(1, 1)$, and $(1, s)$ are all 1, for then $H(F_1, F_2)$ has the desired property. Choose a positive integer such that $s^k \equiv 1 \pmod{(1-s)(1-rs)}$, say $s^k = 1 + t(1-s)(1-rs)$. Then $y^k - tx^{k-2}(x-y)(x-ry)$ has the required property.

Solution 8. The statement is equivalent to the assertion that if S is partitioned into at most n subsets, then one of the subsets contains the difference of two distinct elements in it. Assume otherwise. One of the sets has at least $a_{n-1} + 1$ elements. Call one such set A_1. Subtract the least element of this set from the other members, which produces at least a_{n-1} differences. These must lie in the other $n-1$ sets, and so one of the other sets, call it A_2, must contain at least $a_{n-2} + 1$ of these differences. Iterate this procedure: Subtract the least element from the others. The result is at least a_{n-2} differences, and these are also differences of elements of A_1. Hence these elements must lie inthe remaining $n-2$ sets. Continuing by induction we produce an ordering of the sets $A_1, A_2, \ldots, A_k, \ldots$ such that A_k contains at least $a_{n-k}+1$ elements each of which is a difference of two elements in all of the sets A_i for $i < k$. The process shows that A_n contains at least two elements whose difference is not in any of the sets, a contradiction.

Solution 9. The determinant is 0, and 2017 can be replaced by any odd positive number, say $2h+1$. Let $T = BA^{-1}$, so that $B = TA$. Then $AB^{-1} = T^{-1} = T$, so $T^2 = I$. Hence all eigenvalues of T are 1 or -1. If all eigenvalues are -1, then, using the Jordan form, T is similar to $-I + N$ where N is upper triangular and nilpotent. Then $T^2 = I - 2N + N^2 = I$. Hence $N = N^2/2 = (N^2/2)^2/2 = N^4/8$. By induction we find that $N = N^{2^t}/2^{2^t-1}$. Since N is nilpotent, we deduce that $N = 0$. But then $B = -A$, so the identity $F_{2h+1}(A, B) = F_{2h+1}(B, A)$ implies that $(-1)^h A^{2h+1} = (-1)^{h+1} A^{2h+1}$. This in turn implies that $I = -I$, a contradiction. Thus T has at least one eigenvalue equal to 1. Hence $\det(I - T) = 0$. Since $A - B = A - TA = (I - T)A$, it follows that $\det(A - B) = 0$. The result is true over any field whose characteristic is not 2.

Solution 10. We may assume that P is the origin and that the edges are along rays emanating from the origin into the first orthant. We may assume that one edge is in the direction of the unit vector $(1, 0, 0)$, and that another edge lies in the first quadrant of the xy-plane and is in the direction $(a, s, 0)$ where $s^2 = 1 - a^2$. Let u be a unit vector in the direction of the third edge. Then the dot product of u with $(1, 0, 0)$ and with $(a, s, 0)$ can be assumed to be b and c, respectively. Thus $u = (b, (c - ab)/s, z)$. Since the sum of the squares is 1, we have $b^2 + (c - ab)^2/(1 - a^2) + z^2 = 1$. We multiply both sides by $1 - a^2$ and expand $(c - ab)^2$, to find that $sz = \sqrt{1 - a^2 - b^2 - c^2 - 2abc}$. If p, q, r are the lengths of the sides, then the volume is one sixth of the absolute value of the determinant of the matrix formed with the edges as rows. We factor p, q, r out of the respective rows, and we obtain $M/6$ times the determinant of a low triangular matrix with diagonal elements $1, s, z$. The volume is therefore $(M/6)\sqrt{1 - a^2 - b^2 - c^2 - 2abc}$.