UNIVERSITY OF MICHIGAN
UNDERGRADUATE MATH COMPETITION 34
APRIL 8, 2017

Instructions. Write on the front of your blue book your student ID number. Do not write your name anywhere on your blue book. Each question is worth 10 points. For full credit, you must **prove** that your answers are correct even when the question doesn't say "prove". There are lots of problems of widely varying difficulty. It is not expected that anyone will solve them all; look for ones that seem easy and fun. No calculators are allowed.

Problem 1. A polynomial $P(x)$ with integer coefficients takes on the value 2017 at four distinct integers. Show that P has no integer root.

Problem 2. Suppose that $\alpha = 2+\sqrt{3}$ and that $\beta = 2-\sqrt{3}$. Show that

$$\sum_{m=1}^{\infty} z^{[m\alpha]} = \frac{1-z}{z}\sum_{n=1}^{\infty}[n\beta]z^n$$

for $|z| < 1$. Here $[x]$ denotes the unique integer k such that $k \le x < k+1$.

Problem 3. Prove the following inequality for all integers $n \ge 4$ and all positive real numbers $a_1, a_2, \ldots, a_n$:

$$\begin{aligned} &\frac{1}{a_1+a_2}+\frac{1}{a_2+a_3}+\cdots+\frac{1}{a_{n-1}+a_n}+\frac{1}{a_n+a_1} \\ > \ &\frac{4}{3}\left(\frac{1}{a_1+a_2+a_3}+\frac{1}{a_2+a_3+a_4}+\cdots+\frac{1}{a_{n-1}+a_n+a_1}+\frac{1}{a_n+a_1+a_2}\right). \end{aligned}$$

Problem 4. Find all solutions in integers of the equation

$$2x^2 = 2y^3 + 9y^2 - 27\,.$$

Problem 5. A *trigonometric polynomial* $T(x)$ is a finite sum of the form

$$T(x) = \sum_{n=-N}^{N} t_n e^{2\pi i n x}$$

where the numbers t_n are real or complex constants. Show that

$$\frac{1}{2 - \cos 2\pi x}$$

cannot be expressed as a trigonometric polynomial.

Problem 6. Let F_n be the sequence of Fibonacci numbers determined recursively by the rules $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$. Determine whether

$$\lim_{n\to\infty} \frac{\ln\ln\binom{F_{n+1}}{F_n}}{n}$$

exists, and if it does, find its value.

Problem 7. Let (a_1, b_1), …, (a_n, b_n) be n pairs of integers such that for every i, the numbers a_i and b_i are relatively prime. Show that there is a homogeneous polynomial $F(x, y)$ in two variables of positive degree with integer coefficients such that for all i, $F(a_i, b_i) = 1$.

Problem 8. Let $a_1 = 2$ and define a_n recursively by the formula $a_n = na_{n-1} + 1$ for $n \geq 2$. Prove that if a function f from the set of positive integers $S = \{s\colon 1 \leq s \leq a_n\}$ takes on n values, then there are integers $a < b$ in S such that $f(a) = f(b) = f(b - a)$.

Problem 9. Let A and B be $n \times n$ matrices over the complex numbers. Let $F_s(A, B)$ be the product of s matrices that are alternately A and B, beginning with A. Thus $F_1(A, B) = A$, $F_2(A, B) = AB$, $F_3(A, B) = ABA$, while $F_4(B, A) = BABA$. Suppose that A and B are invertible such that $AB^{-1} = BA^{-1}$ and $F_{2017}(A, B) = F_{2017}(B, A)$. Prove that the determinant of $A - B$ must be 0, or give a counterexample.

Problem 10. The three triangular faces of tetrahedron $PABC$ meeting at P have acute angles at P with cosines a, b, c. The product of the lengths of the edges meeting at P is M. Express the volume of the tetrahedron in terms of M, a, b, and c.

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