UNIVERSITY OF MICHIGAN
UNDERGRADUATE MATH COMPETITION 31
MARCH 29, 2014

Instructions. Write on the front of your blue book your student ID number. Do not write your name anywhere on your blue book. Each question is worth 10 points. For full credit, you must **prove** that your answers are correct even when the question doesn't say "prove". There are lots of problems of widely varying difficulty. It is not expected that anyone will solve them all; look for ones that seem easy and fun. No calculators are allowed.

Problem 1. An island has a population of chameleons of which, currently, 21 are blue, 34 are maize, and 47 are green. Whenever two chameleons of different colors meet, they both change to the third color. There are no color changes otherwise. Is it possible that at some future time a sequence of meetings will lead to their being all the same color? Prove your answer.

Solution. Work modulo 3. The difference between the number of maize chameleons and the number of blue chameleons never changes (the argument works for any two of the colors). Since this is not zero initially it is impossible for the numbers to become 102, 0, 0 (up to order), since these are 0, 0, 0 mod 3.

Problem 2. A point traverses one edge AB of a regular tetrahedron $ABCD$ with constant speed, taking one minute to travel from A to B. Another point, starting at the same time, traverses the opposite edge CD at constant speed, taking thirty seconds to travel from C to D, and then remains stationary. How many seconds after starting are the two points closest to one another?

Solution. Consider the tetrahedron with vertices $A = (1, 1, 1)$, $B = (1, -1, -1)$,$C = (-1, 1, -1)$ and $D = (-1, -1, 1)$. The first point goes A to B and its trajectory is given by the parameterization $(1, 1 - 2t, 1 - 2t)$, where the time t is measured in minutes. The second point is given by the parameterization $(-1, 1 - 4t, -1 + 4t)$. The square of the distance is

$$2^2 + (2t)^2 + (2 - 6t)^2 = 40t^2 - 24t + 8.$$

If we differentiate we get $80t - 24$. To find the minimum we set this equal to zero and obtain $t = 0.3$ min $= 18$ s.

Problem 3. A *stylish partition* of n is a collection of positive integers, not necessarily all different, which sum to n, such that, for each integer $1 \leq m \leq n$, there exists a unique subcollection which sums to m. For example, the stylish partitions of 9 are $5+1+1+1+1$, $2+2+2+2+1$, and $1+1+1+\cdots+1$. Notice that $1+1+\cdots+1$ is always a stylish partition for any n, called the trivial stylish partition. Find, with proof, the smallest $n \geq 2014$ such that n has exactly one nontrivial stylish partitition.

Solution. Suppose that

$$n = \underbrace{a_r + a_r + \cdots + a_r}_{k_r} + \underbrace{a_{r-1} + a_{r-1} + \cdots + a_{r-1}}_{k_{r-1}} + \cdots + \underbrace{a_1 + a_1 + \cdots + a_1}_{k_1}$$

is a stylish partition, where $a_r > a_{r-1} > \cdots > a_1$. Since 1 can be written as a subsum, we must have $a_1 = 1$. Since $a_j - 1$ can be written as a subsum, we have $\sum_{i=1}^{j-1} k_i a_i \geq a_j - 1$. We claim that we must have equality. Suppose that $\sum_{i=1}^{j-1} k_i a_i \geq a_j$. We can find nonnnegative integers $l_1, l_2, \ldots, l_{j-1}$ such that $a_j \leq \sum_{i=1}^{j-1} l_i a_i < 2a_j$. Let $m = \sum_{i=1}^{j-1} a_i - a_j$. We can write $m = \sum_{i=1}^{j-1} c_i a_i$ for some nonnegative integers $c_1, c_2, \ldots, c_{j-1}$. We have

$$m + a_j = a_j + \sum_{i=1}^{j-1} c_i a_i = \sum_{i=1}^{j-1} l_i a_i.$$

So we have two different representations of $m + a_j$ as a subsum. This contradicts the stylishness of n. We conclude that $a_j = \sum_{i=1}^{j-1} k_i a_i + 1$ for all j. It follows that $a_1 = 1$, $a_2 = k_1 + 1$, $a_3 = k_2(k_1+1) + k_1 \cdot 1 + 1 = (k_1+1)(k_2+1)$. By induction one shows that $a_j = (k_1+1)(k_2+1)\cdots(k_{j-1}+1)$ and $n = (k_1+1)(k_2+1)\cdots(k_r+1) - 1$. The number of stylish partitions of n is equal to the number of factorizations of n into integers ≥ 2. For example 7 has 4 stylish partitions, because 8 has the factorizations 8, $4 \cdot 2$, $2 \cdot 4$, $2 \cdot 2 \cdot 2$. These correspond to the stylish partitions $1+1+1+1+1+1+1+1$, $2+2+2+1+1$, $4+1+1+1$ and $4+2+1$. The only numbers that have exactly two stylish partitions are of the form $p^2 - 1$ where p is a prime. The number $\sqrt{2014}$ lies between 44 and 45. The smallest $n \geq 2014$ that has exactly 2 stylish partitions is $47^2 - 1 = 2208$.

Problem 4. Suppose that $P_1, P_2, \ldots, P_6$ are points in $\mathbb{R}^3$. Let $D = (d_{i,j})$ be a 6×6 matrix, where $d_{i,j}$ is the square of the distance between P_i and P_j. Show that $\det(D) = 0$.

Solution. Let (x_i, y_i, z_i) be the coordinates of P_i. We can write

$$D = \begin{pmatrix} l_1 \\ l_2 \\ \vdots \\ l_6 \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & \cdots & l_6 \end{pmatrix} - 2 \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_6 \end{pmatrix} \begin{pmatrix} x_1 & x_2 & \cdots & x_6 \end{pmatrix}$$
$$= 2 \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_6 \end{pmatrix} \begin{pmatrix} y_1 & y_2 & \cdots & y_6 \end{pmatrix} - 2 \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_6 \end{pmatrix} \begin{pmatrix} z_1 & z_2 & \cdots & z_6 \end{pmatrix}$$

where $l_i = x_i^2 + y_i^2 + z_i^2$. It is now clear that rank of D is at most 5, because it is the sum of 5 rank 1 matrices. Therefore, $\det(D) = 0$.

Problem 5. Define a_n recursively by $a_1 = 2$ and $a_{n+1} = (n+2)^{a_n}$ for $n \geq 1$. Thus, $a_2 = 3^2$, $a_3 = 4^{3^2}$, and so forth. Let $\ln^{[n]}$ denote the n-fold iterated composition of the natural logarithm function, so $\ln^{[2]}(x) = \ln(\ln(x))$, $\ln^{[3]}(x) = \ln(\ln(\ln(x)))$, etc. Show that $\lim\limits_{n\to\infty} \ln^{[n]}(a_n)$ exists.

Solution. Let b_n be $\ln^{[n]}(a_n)$. Then we have $a_n = \exp^{[n]}(b_n)$ and

$$\exp^{[n+1]}(b_{n+1}) = a_{n+1} = (n+2)^{a_n} = \exp(a_n \ln(n+2)) = (\exp^{[n+1]}(b_n))^{\ln(n+2)}.$$

Taking logs yields

$$\exp^{[n]}(b_{n+1}) = \ln(n+2) \exp^{[n]}(b_n)$$

and

$$\exp^{[n-1]}(b_{n+1}) = \exp^{n-1}(b_n) + \ln^{[2]}(n+2).$$

We then have

$$\exp^{[n-1]}(b_{n+1}) = \exp^{[n-1]}(b_n)(1 + \ln^{[2]}(n+2)/\exp^{[n-1]}(b_n)).$$

If we apply $\ln(\cdot)$ to this equation, and use the inequality $\ln(1+x) \leq x$, we get

$$\exp^{[n-2]}(b_n) \leq \exp^{[n-2]}(b_{n+1}) \leq \exp^{[n-2]}(b_n) + \frac{\ln^{[2]}(n+2)}{\exp^{[n-1]}(b_n)}.$$

This shows that

$$b_n \leq b_{n+1} \leq b_n + \frac{\ln^{[2]}(n+2)}{\ln(a_n)} = b_n + \frac{\ln^{[2]}(n+2)}{\ln(n+2)a_{n-1}} \leq b_n + \frac{1}{a_{n-1}}$$

for $n \geq 3$. Then $b_n \leq b_3 + (1/a_2 + 1/a_3 + \cdots + 1/a_{n-2})$. The term in parentheses converges very rapidly.

Problem 6. Suppose that A and B are two opposite sides of an icosahedron. A frog starts at A. Every second, the frog jumps to one of the 3 adjacent sides. The probability for each direction is 1/3. The frog stops when it arrives at side B. What is the expected number of jumps it takes for the frog to move from A to B?

Solution. Suppose that A and B are two arbitrary sides. The jump distance between A and B is the smallest number of jumps needed to go from A to B. Let d_i be the expected number of jumps it takes for a frog to jump from A to B. The problem asks to determine d_5. If A and B have distance 1, then the frog jumps from A to B with probability $\frac{1}{3}$ and it jumps to a side that has distance 2 to B with probability $\frac{2}{3}$. This yields the equation $d_1 = \frac{1}{3} \cdot 0 + \frac{2}{3}d_2 + 1$. Similar reasoning gives the equations $d_2 = \frac{1}{3}d_1 + \frac{1}{3}d_2 + \frac{1}{3}d_3 + 1$, $d_3 = \frac{1}{3}d_2 + \frac{1}{3}d_3 + \frac{1}{3}d_4 + 1$, $d_4 = \frac{2}{3}d_3 + \frac{1}{3}d_5 + 1$ and $d_5 = d_4 + 1$. Solving the equations yields $d_1 = 19, d_2 = 27$, $d_3 = 32$, $d_4 = 34$ and $d_5 = 35$.

Problem 7. On a certain island, $n \geq 3$ villages are arranged in a circle. Between each pair of villages, there is a single straight dirt road. Each dirt road is considered to be the property of one of the two rival clans. Is it always possible to pave a certain set of roads so that the following conditions are all met?

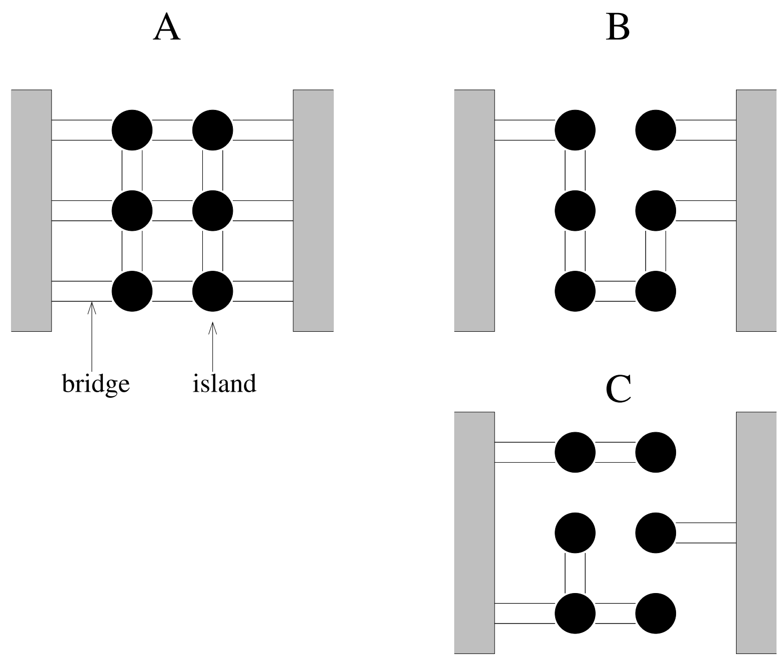
- It is possible to go from any village to any other using paved roads only (passing through other villages as needed along the way).
- The paved roads do not cross (except at villages).
- All the paved roads belong to the same clan.

Solution. We prove the statement by induction on n. It is easily checked for $n = 3$. Suppose that $n \geq 4$. The villages $V_0, V_1, \ldots, V_{n-1}$ are arranged a circle. If all the roads from V_i to V_{i+1} ($i = 0, 1, \ldots, n-1$, index modulo n) belong to the same clan then we are done. So assume that they do not all belong to the same clan. Without loss of generality we may assume that the road from V_{n-1} to V_{n-2} and the road from V_{n-1} to V_0 belong to different clans. Note that these two roads do not intersect any other road. If we forget about the village V_{n-1} for a moment, then we can apply the induction hypothesis to the villages $V_0, V_1, \ldots, V_{n-2}$. So we can pave certain roads belonging to only one clan C so that we can go from village V_i to village V_j using paved roads if $1 \leq i, j \leq n-2$, and so that no two paved roads cross. Either the road from V_n to V_0 or the road from V_{n-1} to V_1 belongs to the clan C. Pave this road as well, and the desired conditions are satisfied.

Problem 8. Call a nonnegative integer *powerful* if its decimal representation can be partitioned into strings of consecutive digits each of which is either 0 or the decimal representation of a power of 2. For example, 0, 1, 2014, and 32116512 are powerful: the last can be broken up as 32 1 16 512. Let S_n denote the number of powerful nonnegative integers with at most n digits. Show that the limit $L = \lim_{n\to\infty} \sqrt[n]{S_n}$ exists, and that for all n, $\sqrt[n]{S_n} \leq L \leq 6$.

Solution. We estimate the number S_n of powerful integers with at most n digits. Each powerful integer has 0 or a power of two with at most n digits occurring leftmost, and the remaining digits must also form a special integer. It will be convenient to make the convention that $S_0 = 1$. Hence, $S_n \leq \sum_{k=1}^{n} P_k S_{n-k}$ where P_k is the number of elements in $\{0, 1, 2, 4, 8, \ldots, 2^n, \ldots\}$ with exactly k digits. One only has an inequality because some numbers have more than one allowable partition of digits, e.g., 128 and 164. Note that $P_1 = 5$ and for $k \geq 2$, P_k is 3 or 4 (the least power of two A greater than 10^{k-1} has first digit 1, or half of it is still greater than 10^{k-1}, and then $A, 2A, 4A$ will be in $[10^{k-1}, 10^k)$ while $A, 2A, 4A, 8A, 16A$ are not all in that interval. It follows that the integers T_n satisfying $T_n = \sum_{k=1}^{n} 5S_{n-k}$ bound S_n, and a straightforward induction yields that $T_n = 5(6^n)$, and so $\sqrt[n]{S_n} \leq 6\sqrt[n]{5}$ for all n. Since $\sqrt[n]{5} \to 1$ as $n \to \infty$ we have that $\sqrt[n]{S_n} : n \geq 1\}$ is bounded. We next show that $(*)$ for all $n \gg t$ and all $\epsilon > 0$, $\sqrt[n]{S_n} \geq \sqrt[t]{S_t} - \epsilon$. It then follows easily that the limit of $\sqrt[n]{S_n}$ is the least upper bound L of the set $\{\sqrt[n]{S_n} : n \geq 1\}$: if S_t is within ϵ of L, $\sqrt[t]{S_n}$ is within 2ϵ of L for all $n \gg 0$. To prove $(*)$, write $n = qt + r$, $0 \leq r \leq t - 1$. We can obtain a powerful integer with n digits by concatenating q elements of S_t with one powerful integer with r digits. Hence, $S_n \geq S_t^q$ and $\sqrt[n]{S_n} \geq S_t^{q/n} = (\sqrt[t]{S_t})S_t^{q/n-1/t}$. For fixed t, $q/n - 1/t = q/(qt+r) - 1/t = 1/(t + \frac{r}{q}) - 1/t \to 0$ as $n \to \infty$ ($r \leq t - 1$ and $q \geq \frac{n}{t} - 1$).

Problem 9. On a river there are 6 islands connected by a system of bridges (see figure A). During the summer flood a part of the bridges has been destroyed. Each bridge is destroyed with probability $\frac{1}{2}$, independently of the other bridges. What is the probability that after the described flood it is possible to cross the river using the remaining bridges? (In the case shown in figure B it is possible to cross the river; in the case shown in figure C it is not possible to cross the river.)



Solution. Suppose that there are two persons: a walker who tries to cross the river and a boatman who tries to go down the river. Assume that the boatman cannot go beneath a bridge, so he can pass only through the destroyed bridges. The possible ways of the boatman lie along the dashed lines in figure D. It is clear that

$$p = P(\text{the walker can cross the river}) = P(\text{the boatman can go down the river}) =: q$$

by symmetry (switching horizontal/vertical and walker/boatman). It is easy to see that the boatman can go down the river if and only if the walker cannot cross the river. Thus, $p + q = 1$. As a result, $p = \frac{1}{2}$.

Problem 10. Suppose that $p = x^2 + y^2$ is an odd number, where x and y are relatively prime integers. Show that the set

$$\{(a, b) \in \mathbb{Z}^2 \mid 0 \le b \le a, \quad \sqrt{a+b} + \sqrt{a-b} \le \sqrt{2p}\}$$

has $(2p^2 + 9p + 13)/12$ elements.

Solution. We rewrite the inequality (assuming $0 \le b \le a$)

$$\sqrt{a+b} + \sqrt{a-b} \le \sqrt{2p}.$$

Squaring and dividing by 2 yields

$$a + \sqrt{a^2 - b^2} \le p.$$

Moving a to the other side and squaring yields

$$a^2 - b^2 \le p^2 - 2ap + a^2.$$

It follows that

$$b \le a \le \tfrac{1}{2}p + \frac{b^2}{2p} = \tfrac{1}{2}(p-1) + \frac{b^2 + p}{2p}$$

Note here that p is even. For a fixed value of b, there are

$$\tfrac{1}{2}(p-1) + \left\lfloor \frac{b^2+p}{2p} \right\rfloor - b + 1 = \tfrac{1}{2}(p+1) - b + \frac{b^2+p}{2p} - \left\{ \frac{b^2+p}{2p} \right\} = \tfrac{1}{2}p - b + 1 + \frac{b^2}{2p} - \left\{ \frac{b^2+p}{2p} \right\}$$

where $\lfloor a \rfloor = a - \lfloor a \rfloor$ is the fractional part of a. solutions. So the total number of solutions is obtained by summing this for $b = 0, 1, \dots, p$:

$$\tfrac{1}{2}p(p+1) - \tfrac{1}{2}p(p+1) + (p+1) + \frac{1}{2p} \cdot \frac{p(p+1)(2p+1)}{6} - \sum_{b=0}^{p} \left\{ \frac{b^2+p}{2p} \right\} =$$

$$= \frac{2p^2 + 15p + 13}{6} - \sum_{b=0}^{p} \left\{ \frac{b^2+p}{2p} \right\}.$$

Note that

$$\left\{ \frac{p+c}{2p} \right\} + \left\{ \frac{p-c}{2p} \right\} = 1$$

unless $c \equiv p \bmod 2p$.

We have that $(x+y)^2+(y-x)^2 \equiv 2(x^2+y^2) \equiv 0 \bmod 2p$. Note that $x+y$ is relatively prime to $2p$, so we can choose $z \in \mathbb{Z}$ such that $z(x+y) \equiv (x-y) \bmod 2p$. Then we have $z^2(x+y)^2 \equiv (x-y)^2 \equiv -(x+y)^2 \bmod 2p$. It follows that $z^2 \equiv -1 \bmod 2p$.

We have

$$2\sum_{b=0}^{p}\left\{\frac{b^2+p}{2p}\right\} = \sum_{b=0}^{p}\left\{\frac{b^2+p}{2p}\right\}+\sum_{b=0}^{p}\left\{\frac{(zb)^2+p}{2p}\right\} = \sum_{b=0}^{p}\left\{\frac{p+b^2}{2p}\right\}+\sum_{b=0}^{p}\left\{\frac{p-b^2}{2p}\right\} = p$$

So the total number of lattice points in the region is

$$\frac{2p^2+15p+13}{12} - \sum_{b=0}^{p}\left\{\frac{b^2+p}{2p}\right\} = \frac{2p^2+15p+13}{12} - \frac{p}{2} = \frac{2p^2+9p+13}{12}.$$