UNIVERSITY OF MICHIGAN
UNDERGRADUATE MATH COMPETITION 31
MARCH 29, 2014

Instructions. Write on the front of your blue book your student ID number. Do not write your name anywhere on your blue book. Each question is worth 10 points. For full credit, you must **prove** that your answers are correct even when the question doesn't say "prove". There are lots of problems of widely varying difficulty. It is not expected that anyone will solve them all; look for ones that seem easy and fun. No calculators are allowed.

Problem 1. An island has a population of chameleons of which, currently, 21 are blue, 34 are maize, and 47 are green. Whenever two chameleons of different colors meet, they both change to the third color. There are no color changes otherwise. Is it possible that at some future time a sequence of meetings will lead to their being all the same color? Prove your answer.

Problem 2. A point traverses one edge AB of a regular tetrahedron $ABCD$ with constant speed, taking one minute to travel from A to B. Another point, starting at the same time, traverses the opposite edge CD at constant speed, taking thirty seconds to travel from C to D, and then remains stationary. How many seconds after starting are the two points closest to one another?

Problem 3. A *stylish partition* of n is a collection of positive integers, not necessarily all different, which sum to n, such that, for each integer $1 \leq m \leq n$, there exists a unique subcollection which sums to m. For example, the stylish partitions of 9 are $5 + 1 + 1 + 1 + 1$, $2 + 2 + 2 + 2 + 1$, and $1 + 1 + 1 + \cdots + 1$. Notice that $1 + 1 + \cdots + 1$ is always a stylish partition for any n, called the trivial stylish partition. Find, with proof, the smallest $n \geq 2014$ such that n has exactly one nontrivial stylish partition.

Problem 4. Suppose that $P_1, P_2, \ldots, P_6$ are points in $\mathbb{R}^3$. Let $D = (d_{i,j})$ be a 6×6 matrix, where $d_{i,j}$ is the square of the distance between P_i and P_j. Show that $\det(D) = 0$.

Problem 5. Define a_n recursively by $a_1 = 2$ and $a_{n+1} = (n + 2)^{a_n}$ for $n \geq 1$. Thus, $a_2 = 3^2$, $a_3 = 4^{3^2}$, and so forth. Let $\ln^{[n]}$ denote the n-fold iterated composition of the natural logarithm function, so $\ln^{[2]}(x) = \ln(\ln(x))$, $\ln^{[3]}(x) = \ln(\ln(\ln(x)))$, etc. Show that $\lim_{n \to \infty} \ln^{[n]}(a_n)$ exists.

Problem 6. Suppose that A and B are two opposite sides of an icosahedron. A frog starts at A. Every second, the frog jumps to one of the 3 adjacent sides. The probability for each direction is $1/3$. The frog stops when it arrives at side B. What is the expected number of jumps it takes for the frog to move from A to B?

Problem 7. On a certain island, $n \geq 3$ villages are arranged in a circle. Between each pair of villages, there is a single straight dirt road. Each dirt road is considered to be the property of one of the two rival clans. Is it always possible to pave a certain set of roads so that the following conditions are all met?

- It is possible to go from any village to any other using paved roads only (passing through other villages as needed along the way).
- The paved roads do not cross (except at villages).
- All the paved roads belong to the same clan.

Problem 8. Call a nonnegative integer *powerful* if its decimal representation can be partitioned into strings of consecutive digits each of which is either 0 or the decimal representation of a power of 2. For example, 0, 1, 2014, and 32116512 are powerful: the last can be broken up as 32 1 16 512. Let S_n denote the number of powerful nonnegative integers with at most n digits. Show that the limit $L = \lim_{n \to \infty} \sqrt[n]{S_n}$ exists, and that for all n, $\sqrt[n]{S_n} \leq L \leq 6$.

Problem 9. On a river there are 6 islands connected by a system of bridges (see figure A). During the summer flood a part of the bridges has been destroyed. Each bridge is destroyed with probability $\frac{1}{2}$, independently of the other bridges. What is the probability that after the described flood it is possible to cross the river using the remaining bridges? (In the case shown in figure B it is possible to cross the river; in the case shown in figure C it is not possible to cross the river.)
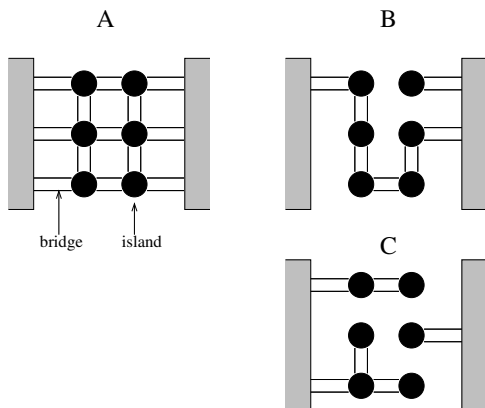
Problem 10. Suppose that $p = x^2 + y^2$ is an odd number, where x and y are relatively prime integers. Show that the set

$$\{(a, b) \in \mathbb{Z}^2 \mid 0 \leq b \leq a, \quad \sqrt{a+b} + \sqrt{a-b} \leq \sqrt{2p}\}$$

has $(2p^2 + 9p + 13)/12$ elements.