UNIVERSITY OF MICHIGAN
UNDERGRADUATE MATH COMPETITION 28
APRIL 7, 2011

Instructions. Write on the front of your blue book your student ID number. Do not write your name anywhere on your blue book. Each question is worth 10 points. For full credit, you must **prove** that your answers are correct even when the question doesn't say "prove". There are lots of problems of widely varying difficulty. It is not expected that anyone will solve them all; look for ones that seem easy and fun. No calculators are allowed.

Problem 1. There are some gas stations on a circular roadway. Together the stations contain just enough gas to make it all the way around. Show that it is possible to start at one of the stations with an empty tank (that is large enough to hold all the gas at all these stations) and make it all the way around.

Suppose instead you have a large tank with enough gas to drive around and room for twice as much. Pick a station, and drive around, adding the available gas at each station. There will be a station where the gas you have is minimum. If you start there, even with an empty tank, you can make the trip.

Problem 2. Four distinct points $\{P_1, P_2, P_3, P_4\}$ in the plane have the property that the set of six distances between the pairs of them, P_iP_j for $i \neq j$, consists of precisely 2 real numbers. One example is the set of vertices of a square. Find, up to similarity, all configurations of four points with this property.

Suppose that 3 of the four points, say A, B, C, form an equilateral triangle. Let $x = AB = BC = AC$. If D is the fourth point then either 0, 1 or 2 of the distances AD, BD, CD are equal to x. If none of the distances AD, BD, CD are equal to x, then $AD = BD = CD$ and D is the center of $\triangle ABC$ (case 1). If $AD = x$ and $BD = CD$, then D lies on the perpendicular bisector of B and C, with distance x to A. There are two possibilities: Either AD lie on the same since of the line through B and C (case 2), or they lie on opposite sides (case 3). If $AD = BD = x$, then $\triangle ABC$ and $\triangle ABD$ are two equilateral triangles were A and D lie on opposite sides of BC (case 4).

Suppose none of the 3 points form an equilateral triangle. There are two cases: either $AB = BC = CD = DA$ and $AC = BD$. This is the square (case 5). The other case is where $x = AB = BC = CD$ and $y = BD = DA = AC$ where $x < y$. Then $ABCD$ is a trapezium with base AD. Then A, B, C, D lie on a circle. Choose a point E with $DE = EA = x$ such that A and E are of opposite sides of AD. The triangles $\triangle ABC$ and $\triangle DEA$ are congruent. It follows that E also lies on the circle through $ABCD$.

Since $x = AB = BC = CD = DE = EA$, the points A, B, C, D, E form a regular pentagon. So A, B, C, D consist of 4 of the 5 vertices of a regular pentagon (case 6).

Problem 3. Find all solutions for $x^3 + y^3 = 3^z$ where x, y and z are integers.

From
$$(a+b)(a^2 - ab + b^2) = 3^c$$
and $a^2 - ab + b^2 \geq 0$, follows that both $a+b$ and $a^2 - ab + b^2$ are powers of 3. We can write $a = 3^k p$ and $b = 3^k q$ such that 3 does not divide both p and q. It follows that $p^2 - pq + q^2$ and $p+q$ are powers of 3. If $3 \mid (p+q)$, and $9 \mid p^2 - pq + q^2 = (p+q)^2 - 3pq$, then $9 \mid 3pq$ and $3 \mid pq$. From $3 \mid p+q$ and $3 \mid pq$ follows that $3 \mid p$ and $3 \mid q$. Contradiction. So $p + q = 1$, $p^2 - pq + q^2 = 1$ or $p^2 - pq + q^2 = 3$.

If $p + q = 1$, then $p^2 - pq + q^2 = 3p^2 - 3p + 1 \equiv 1 (\mathrm{mod} 3)$, hence $3p^2 - 3p + 1 = 1$ and $p = 0$ or $p = 1$.

If $p^2 - pq + q^2 = 1$, then $(q - \frac{1}{2}p)^2 + \frac{3}{4}p^2 = 1$. Hence $\frac{3}{4}p^2 \leq 1$.

If $p^2 - pq + q^2 = 3$, then $(q - \frac{1}{2}p)^2 + \frac{3}{4}p^2 = 3$, so $p^2 \leq 4$.

In each of the cases, we have $p^2 \leq 4$ and similarly $q^2 \leq 4$. So we have $p, q \in \{-2, -1, 0, 1, 2\}$. We verify that the only solutions for (p, q) are
$$\{(0,1), (1,0), (1,2), (2,1)\}.$$

Problem 4. What is the largest number of subsets you can choose from $\{1, \ldots, n\}$ with each subset having an odd number of elements and the intersection of any two distinct subsets having an even number of elements?

The largest number possible is n. The n one element sets do work. We shall sow there cannot be $n+1$. Consider the ring F^n where $F = Z/2Z$. Each subset of $\{1, \ldots, n\}$ corresponds to an element, the sum of the corresponding standard basis elements. The elements with an even number of 1s form a vector subspace over F. Given $n+1$ elements they are dependent: one is a sum of others, say $a = b_1 + \cdots + b_n$. Then $a = a \cdot a = ab_1 + \ldots + ab_n$. Each term on the right sums to 0, while a sums to 1, a contradiction.

Problem 5. Suppose that $a_1, a_2, \ldots$ is a sequence of real numbers such that
$$\sum_{i=1}^{n} a_i = \prod_{i=1}^{n} a_i$$
for all positive integers n. For every possible value of a_1, determine $\lim_{n\to\infty} a_n$.

Let
$$b_n = \prod_{i=1}^{n} a_i = \sum_{i=1}^{n} a_i.$$
Then we have $a_{n+1} = b_{n+1} - b_n = b_{n+1}/b_n$ for all $n \geq 1$. From this follows that
$$b_{n+1} = \frac{b_n^2}{b_n - 1}$$

if $a_1 = b_1 \leq 0$ then $b_1 \leq b_2 \leq b_3 \ldots$ and all b_i's are negative. So $\lim_{n\to\infty} b_n$ exists. Suppose that this limit is equal to b. Then we have $b = b^2/(b-1)$ and $b = 0$. From $\lim_{n\to\infty} b_n = b = 0$ follows that $\lim_{n\to\infty} a_n = \lim_{n\to\infty}(b_n - b_{n-1}) = 0$.

Suppose that $0 < a_1 < 1$. Then $a_2 < 0$ and $\lim_{n\to\infty} a_n = 0$ as in the previous case.

Suppose that $a_1 > 1$, then we have $b_1 \leq b_2 \leq \cdots$ and $\lim_{n\to\infty} b_n = \infty$. We have

$$a_n = b_n - b_{n-1} = 1 + \frac{1}{b_{n-1} - 1}$$

So we get

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} 1 + \frac{1}{b_{n-1} - 1} = 1.$$

Problem 6. Let $r > 0$ be a fixed real number. Suppose that a differentiable function $y = f(x)$ can be defined on the interval $(0, B)$, $B > 0$, so that it is positive and satisfies the differential equation $xy' = y + y^{r+1}$. Suppose that $f(1) = a > 0$. Find, as a function of r and a, the largest possible value of B for which there is such a function.

We solve the differential equation:

$$-\frac{\log(y^{-r}+1)}{r} = \int \frac{y^{-r-1}dy}{y^{-r}+1} = \frac{dy}{y + y^{r+1}} = \int \frac{dx}{x} + C = \log(x) + C,$$

so $x(y^{-r}+1)^{1/r}$ is constant. For $x = 1$ we have $y = a$, so

$$x(y^{-r}+1)^{1/r} = (a^{-r}+1)^{1/r}$$

and

$$y = ((a^{-r}+1)x^{-r} - 1)^{-1/r}$$

So $B = (a^{-r}+1)^{1/r}$.

Problem 7. Suppose that G is a finite group containing elements x, y, z such that $yx = x^2y$, $zy = y^2z$ and $xz = z^2x$. Prove that $x = y = z = 1$.

We prove the statement by induction on the group order $|G|$. Let d_x, d_y, d_z be the orders of x, y and z. By cyclically permuting x, y and z, we may assume that $d_y \geq d_x$. The group $\langle y\rangle \cong \mathbb{Z}/d_y\mathbb{Z}$ generated by y acts on $\langle x\rangle \cong \mathbb{Z}/d_x\mathbb{Z}$ by conjugation. Define a group homomorphism $\psi : \langle y\rangle \to \mathrm{Aut}(\langle x\rangle)$ where $\mathrm{Aut}(\langle x\rangle)$ is the automorphism group of $\langle x\rangle$ and $\psi(y)$ is conjugation by y. Then ψ has a nontrivial kernel, because $|\mathrm{Aut}(\langle x\rangle)| = \phi(d_x) < d_x \leq d_y$ where ϕ is Euler's phi function. So there exists a power y^r of y with $y^r \neq 1$ such that y^r commutes with x. It follows that $\langle y\rangle$ is a normal subgroup of G. By induction, we may assume that the images of x, y, z are trivial in the group $G/\langle y\rangle$. So x, y, z all lie in $\langle y\rangle$ and x, y, z commute. From $x^2y = yx = xy$ follows that $x = 1$ and similarly $y = z = 1$.

Problem 8. Show that there is a 4×4 matrix M over the real numbers such that all the entries off the diagonal are nonzero, but all of the entries on the diagonal of M^k are zero when $k \geq 1$ is an integer.

Let A be the matrix of a 45° rotation. All of the entries of A and A^{-1} have absolute value equal to $\sqrt{2}/2$. Note that A^2 is the matrix of a 90° rotation and has diagonal entries 0 and nonzero entries off the diagonal. Also note that $A^4 = -I$. The matrix whose block form is

$$\begin{pmatrix} -A^2 & A \\ A^{-1} & , -A^2 \end{pmatrix}$$

has all entries off the diagonal nonzero, all entries on the diagonal equal to 0, and its square and, hence, all higher powers, are 0.

Problem 9. How many of the binomial coefficients $\binom{2011}{r}$, $r = 0, 1, \ldots, 2011$ are even?

We have $2011 = (1111101101)_2$ in base 2. Computing modulo 2, we have $(1+x)^2 = 1 + x^2$, and $(1+x)^{2^n} = 1 + x^{2^n}$. So we have

$$\sum_{r=0}^{2011} \binom{2011}{r} x^r = (1+x)^{2011} =$$
$$= (1+x)^{1024}(1+x)^{512}(1+x)^{256}(1+x)^{128}(1+x)^{64}(1+x)^{16}(1+x)^{8}(1+x)^{2}(1+x) =$$
$$= (1+x^{1024})(1+x^{512})(1+x^{256})(1+x^{128})(1+x^{64})(1+x^{16})(1+x^{8})(1+x^{2})(1+x)$$

So $(1+x)^{2011}$ has $2^9 = 512$ odd coefficients, and $2012 - 512 = 1500$ even coefficients.

Problem 10. One chooses 5 random points in the unit disc. Assume that we have a uniform distribution on the disc, and that the points are chosen independently. What is the probability that the convex hull of the 5 points is a triangle?

If $V = (x, y) = (r\cos(\varphi), r\sin(\varphi))$ is a random vector in the unit disc, then the expected value of r^2 is (using change to polar coordinates polar coordinates)

$$E(r^2) = \frac{1}{\pi}\int\int_{x^2+y^2\leq 1} x^2 + y^2 dx\, dy = \frac{1}{\pi}\int_0^1\int_0^{2\pi} r^2 \cdot r\, d\varphi\, dr = 2\int_0^1 r^3\, dr = \frac{1}{2}.$$

Let $V_i = (r_i\cos(\varphi_i), r_i\sin(\varphi_i))$ be random vectors in the unit disc for $i = 1, 2, 3, 4, 5$. The area of $\triangle V_1V_2V_3$ is

$$\pm\tfrac{1}{2}(r_1r_2\cos(\beta_3) + r_2r_3\cos(\beta_1) + r_3r_1\cos(\beta_2))$$

where $\beta_3 = \varphi_2 - \varphi_1$,$\beta_1 = \varphi_3 - \varphi_2$ and $\beta_2 = \alpha_1 - \alpha_3$. Then $r_1, r_2, r_3, \beta_1, \beta_2, \beta_3$ are independent random variables. The angles $\beta_1, \beta_2, \beta_3$ have a uniform distribution on $[0, 2\pi]$, and

$$E(\cos^2(\beta_i)) = \frac{1}{2\pi}\int_0^{2\pi}\cos^2(\theta)\, d\theta = \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{2} + \frac{1}{2}\cos(2\theta)\, d\theta = \frac{1}{2}$$

for all i. We have

$$E(r_1^2r_2^2\cos^2(\beta_3)) = E(r_1^2)E(r_2^2)E(\cos^2(\beta_3)) = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{8}.$$

and

$$E(r_1 r_2^2 r_3 \cos(\beta_1)\cos(\beta_3) = E(r_1 r_2^2 r_3 \cos(\beta_1) E(\cos(\beta_3) = 0$$

because $E(\cos(\beta_3)) = 0$. Let A be the area of $\triangle V_1V_2V_3$. Then we have

$$E(A^2) = E(\tfrac{1}{4}(a_1^2a_2^2\cos(\beta_3) + a_2^2a_3^2\cos^2(\beta_1) + a_3^2a_1^2\cos^2(\beta_2)+$$
$$+ 2a_1a_2^2a_3\cos(\beta_1)\cos(\beta_3) + 2a_2a_3^2a_1\cos(\beta_2)\cos(\beta_1) + 2a_3a_1^2a_2\cos(\beta_3)\cos(\beta_2))) =$$
$$\frac{1}{4}(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + 0 + 0 + 0) = \frac{3}{32}.$$

The probability that V_4 and V_5 lie in the convex hull of V_1, V_2, V_3 is A^2/π^2. So for random V_1, V_2, V_3, this probability is

$$E(A^2/\pi^2) = \frac{3}{32\pi^2}.$$

The probability that 2 of the vectors $V_1, V_2, \ldots, V_5$ lie in the convex hull of the other 3 is

$$\binom{5}{2} \cdot \frac{3}{32\pi^2} = \frac{15}{16\pi^2}.$$