UNIVERSITY OF MICHIGAN
UNDERGRADUATE MATH COMPETITION 27
MARCH 28, 2010

Instructions. Write on the front of your blue book your student ID number. Do not write your name anywhere on your blue book. Each question is worth 10 points. For full credit, you must **prove** that your answers are correct even when the question doesn't say "prove". There are lots of problems of widely varying difficulty. It is not expected that anyone will solve them all; look for ones that seem easy and fun. No calculators are allowed.

Problem 1. Suppose that n is a positive integer and that $a_{i,j}$, where $1 \leq i \leq n$, $1 \leq j \leq 2^n + 1$ are $n(2^n + 1)$ real numbers in the closed interval $[0, 2]$. Show that there are distinct integers $1 \leq j < k \leq 2^n + 1$ such that $\sum_{i=1}^{n}(a_{i,j} - a_{i,k})^2 \leq n$.

Solution. Think of the hypercube $[0, 2]^n$ in n-space as the union of 2^n hypercubes of side one, each the product of unit intervals each of which is $[0, 1]$ or $[1, 2]$. One has, in effect, $2^n + 1$ points in the union. By the pigeonhole principle, two of them lie in the same cube of side 1, and the squared distance between them is at most n.

Problem 2. Determine the number of integers n with $0 \leq n < 2010$ with the property there exists a positive integer m such that $n^{2^m} - 1$ is divisible by 2010.

Solution. By the Chinese Remainder Theorem, one may consider the problem mod 2, 3, 5, and 67, respectively. The respective numbers of solutions are 1, 2, 4, and 2, so there are $1 \cdot 2 \cdot 4 \cdot 2 = 16$ in all.

Problem 3. A sequence of real numbers is defined recursively by the rules $a_1 = 1$ and $a_{n+1} = \sqrt{2010 + 37a_n}$ for $n \geq 1$. Determine whether $\lim\limits_{n\to\infty} a_n$ exists and, if so, find its value.

Solution. If $1 \leq a \leq 67$, then $b = \sqrt{(2010+37a)}$ is such that $a < b < 67$: to see this, it suffices to check $a^2 < b^2 < 67^2$ or that $a^2 < 2010 + 37a < 67^2$. The left inequality is equivalent to $a^2 - 37a - 2010 < 0$ or $(a - 67)(a + 30) < 0$, which is clear. The right hand inequality is equivalent to $37a < 67^3 - 2010 = 67^2 - 67 \cdot 30 = 67(67 - 30) = 67 \cdot 37$ which is also clear. Hence, the sequence is increasing and bounded above by 67, and so approaches a positive limit $L \leq 67$. Since both a_{n+1} and a_n approach L, we must have that $L = \sqrt{(2010 + 37L)}$. If we square we see that L satisfies the equation $(L - 67)(L + 30) = 0$. Hence, $L = 67$.

Problem 4. There is a blackboard on which the following four triples of integers are written: $(0,0,0),(1,0,0),(0,1,0),(0,0,1)$. You are allowed to perform the following operation as often as you choose. If the triple (a,b,c) is on the blackboard, you may erase it and add the triples $(a+1,b,c),(a,b+1,c),(a,b,c+1)$. If at any time the same triple appears on the board more than once, you will be fed to the Jabberwock. (That's bad.) Your goal is for the blackboard not to contain any of the triples $(0,0,0),(1,0,0),(0,1,0),(0,0,1)$. Prove that it is impossible to achieve your goal without being fed to the Jabberwock.

Solution. For a set S of triples, define $f(S)=\sum_{(a,b,c)\in S}3^{-a-b-c}$. If S is the set of triples on the blackboard, then $f(S)$ is invariant under the operations. In the beginning, we have $f(S)=1+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}=2$. Suppose that S does not contain any of the 4 triples. Then we have

$$4=f(S)+1+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}\leq\sum_{(a,b,c)\in\mathbb{N}^3}3^{-a-b-c}=\big(\sum_{a=0}^{\infty}3^{-a}\big)^3-=\big(\frac{3}{2}\big)^3=\frac{27}{8}.$$

Contradiction.

Problem 5. Suppose that X is a set of points in the plane such that the distance between every two elements of X is an integer. Prove that X is finite, or all points of X lie on a line.

Solution. Suppose that A,B,C are points of X, no two on a line. If P is a point, not on the line through A and B, then by the triangle inequality we have

$$|d(P,A)-d(P,B)|\leq d(A,B)$$

So there are only finitely many possible values for $d(P,A)-d(P,B)$ and $d(P,B)-d(P,C)$. It suffices to show that the equations

$$d(P,A)-d(P,B)=a, d(P,B)-d(P,C)=b$$

have only finitely many solutions for fixed a and b. have only finitely many common solutions. Let C_1 be the curve defined by $d(P,A)-d(P,B)=a$ and C_2 be the curve defined by $d(P,A)-d(P,B)=b$. Either, C_1 is the perpendicular bisector of A and B ($a=0$), C_1 is contained in the line through A and B ($a=\pm d(A,B)$) or C_1 is a hyperbolic for which the line through A and B is the only symmetry axis.

if $p\neq 0$, then C_1 is (a connected component) of a hyperbolic, for which the line through A and B is the symmetry axis. If $q\neq 0$, then C_2 is a hyperbolic with a different symmetry axis, so $C_1\neq C_2$ and C_1 and C_2 intersect in finitely many points. If $p=0$ or $q=0$, then C_1 or C_2 is a straight line, and we still have that C_1 and C_2 intersect in finitely many points.

Problem 6. Let n be an integer greater than 1. How many ways are there to label the squares of an $n\times n$ chessboard with the symbols $\heartsuit,\diamondsuit,\spadesuit,\clubsuit$ (one symbol per square) so that, in each of the $(n-1)^2$ subsquares of size 2×2, each symbol appears exactly once? (We do consider two labelings distinct, even if they differ only by rotation or reflection; for example, when $n = 2$ there are 24 labelings.)

Solution. Consider a 3×3 subsquare. If the first row contains 3 distinct symbols, then the other two rows are completely determined by the first row, and each column contains exactly two distinct symbols. One of the following must be true:

(1) Every row contains only two distinct symbols, and these symbols alternate.
(2) Every column contains only two distinct symbols, and these symbols alternate.

In case (1), there are $\binom{4}{2} = 6$ ways to choose the two symbols for the first row. Then all odd rows have these two symbols, and all even rows have the other two symbols. Every row can start with each of the two symbols, so there are $2^n \cdot 6$ possiblities. In case (2), there are also $2^n \cdot 6$ possibilities. If both (1) and (2) hold, then the square is determined by the 2×2 in the upper left corner, so there are 24 possibilities. So the total number is $6 \cdot 2^n + 6 \cdot 2^n - 24 = 12 \cdot 2^n - 24$.

Problem 7. A person repeatedly rolls a die. Suppose that the outcomes are

$$a_1, a_2, a_3, \cdots \in \{1, 2, 3, 4, 5, 6\}.$$

For a positive integer n, let p_n be the probability that one of the partial sums

$$a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots$$

is equal to n. Determine $\lim_{n\to\infty} p_n$.

Solution. We have a recurrence relation $p_n = \frac{1}{6}(p_{n-1} + p_{n-2} + \cdots + p_{n-6})$ with the initial conditions $p_0 = 1$, $p_{-1} = p_{-2} = \cdots = p_{-5} = 0$. We have

$$P(z) := \sum_{n=0}^{\infty} p_n z^n = \frac{1}{1 - \frac{1}{6}(z + z^2 + \cdots + z^6)}.$$

Let $\alpha_1 = 1, \alpha_2, \ldots, \alpha_6$ be the roots of the denominator of $P(z)$. We have $|\alpha_i| > 1$ for $i \geq 2$. We can write $P(z)$ are a sum of partial fractions:

$$P(z) = \sum_{i=1}^{6} \frac{\beta_i}{1 - \frac{z}{\alpha_i}}$$

It follows that

$$p_n = \sum_{i=1}^{6} \frac{\beta_i}{\alpha_i^n}$$

Hence $\lim_{n\to\infty} p_n = \beta_1$. To find β_1, we compute

$$\beta_1 = \lim_{z\to 1}(1 - z)P(z) = -\frac{1}{(1 - \frac{1}{6}(z + z^2 + \cdots z^6))'} |_{z=1}= \frac{1}{\frac{1}{6}(1 + 2 + \cdots + 6)} = \frac{6}{21} = \frac{2}{7}.$$

Problem 8. Hoodwink the magician likes to shuffle a deck with an even number of cards as follows. Place the top card on the bottom card and put these two on the table, starting a new stack. Now place the new top card from the original stack on top of the new bottom card from the original stack, and place these two on top of the new stack. Continue in this way until all of the cards have been used. For example, if the cards are originally 1,2,3,4,5,6,7,8 (top to bottom) they will wind up in the order 4,5,3,6,2,7,1,8. Suppose that one has a deck containing 2^n cards, where $n \geq 2$ is a positive integer. Prove that $n+1$ Hoodwink shuffles restore the cards to their original order.

Solution. Number the cards $2^n - 1$ to 0 with the largest number for the top card. Consider the effect of the map corresponding to the shuffle on the binary representations of these numbers. Think of the binary representation as a string of precisely n elements, each of which is 0 or 1. If we think of these strings as elements in the vector space F^n, where $F = \{0, 1\}$ is the field with two elements, one can check that the map T takes $a_1, \dots a_n$ to $a_2, \dots, a_n, 0$ if $a_1 = 0$ (this doubles the number) and to $1-a_2, \dots, 1-a_{n-1}, 1$ if $a_1 = 1$. Since $a = -a$ in F, this map sends $a_1, \dots, a_n$ to $a_1 + a_2, \dots, a_1 + a_{n-1}, a_1$ in both cases. Thus T is F-linear. Let e_j denote the j th standard basis vector in F^n. The orbit of e_n is $e_n, e_{n-1}, \cdots, e_1, e_1 + \cdots + e_n$, where the last vector maps to e_n. Hence, for all of the vectors v in this orbit, $T^{n+1}(v) = v$. Since the vectors in the orbit span F^n, T^{n+1} is the identity map.

Problem 9. Prove or disprove each of the following statements:
(a) One can fit a dodecahedron inside a cube such that each vertex of the dodecahedron lies on a face of the cube.
(b) One can fit an icosahedron inside a cube such that each vertex of the icosahedron lies on a face of the cube.

Solution. **(a)** The dodecahedron has 20 vertices. By the pigeonhole principle, one face of the cube must at least contain 3 vertex. So one face A of the dodecahedron (with 5 vertices) lies entirely in in a face F of the cube. Consider the 10 vertices $P_1, \dots, P_{10}$ of the dodecahedron that do not lie in A or the face opposite of A. These vertices do not lie in the face F of the cube or the face F' opposite to F. Consider the orthogonal projection onto F. The shadow of the cube is a square, and the shadow of the dodecahedron is a regular 10-gon, whose vertices are the projections of $P_1, \dots, P_{10}$. Since the points $P_1, \dots, P_{10}$ do not lie in F or F', they lie in the other 4 faces of the cube. And the projections of $P_1, \dots, P_{10}$ lie on the sides of the square. Now we have a regular 10-gon whose vertices lie on the sides of a square. By the pigeonhole principal, at least one side contains 3 vertices of the 10-gon. Contradiction!

(b) Consider the cube $[-1, 1] \times [-1, 1] \times [-1, 1]$. Let S be the set of 12 points $(\pm 1, \pm a, 0), (0, \pm 1, \pm a), (\pm a, 0, \pm 1)$. Consider the point $P = (1, a, 0)$, and its 5 neighbors $\{(1, -a, 0), (0, 1, \pm a), (a, 0, \pm 1)\}$. The neighbors have the same distance to P if $4a^2 = (1-a)^2 + a^2 + 1$. The latter equation is equivalent to $a^2 + a - 1 = 0$. Which has a solution $a = (-1+\sqrt{5})/2$. For this value of a, (by symmetry) every element of S has 5 neighbors with distance $2a$. The elements of S form the vertices of an icosahedron.

Problem 10. Show that

$$(1! \cdot 2! \cdot 3! \cdots (p-1)!)^4 - 1$$

is divisible by p for every prime p.

Solution. Modulo p we have

$$1! \cdot 2! \cdots (p-1)! = 1^{p-1} 2^{p-2} \cdots (p-1)^1 = \prod_{i=1}^{p-i} i^{p-i} = \prod_{i=1}^{p-i} (p-i)^i \equiv \prod_{i=1}^{p-i} (-i)^i.$$

so

$$(1! \cdot 2! \cdots (p-1)!)^2 \equiv \prod_{i=1}^{p-1} i^{p-i}(-i)^i = \pm \prod_{i=1}^{p-1} i^p \equiv \prod_{i=1}^{p-1} i = \pm(p-1)! \equiv \pm 1.$$

Here we used, Fermat's theorem: $i^p \equiv i$, and Wilson's theorem: $(p-1)! \equiv -1$. We conclude:

$$(1! \cdot 2! \cdots (p-1)!)^4 \equiv 1.$$