UNIVERSITY OF MICHIGAN
UNDERGRADUATE MATH COMPETITION 25
MARCH 29, 2008

Instructions. Write on the front of your blue book your student ID number. Do not write your name anywhere on your blue book. Each question is worth 10 points. For full credit, you must **prove** that your answers are correct even when the question doesn't say "prove". There are lots of problems of widely varying difficulty. It is not expected that anyone will solve them all; look for ones that seem easy and fun. No calculators are allowed.

Problem 1. In a convex tetrahedron $PABC$ edges PA, PB, and PC have length 1. The angle at vertex P in both of the triangles APB and APC is α, the angle at vertex P in triangle BPC is β, while the angle between the half-planes of these triangles (which meet at the line through AP) is θ. Express $\cos\theta$ in terms of α and β.

Problem 2. Let $n \geq 2$ be an integer and let $S(n)$ denote the number of solutions in the positive integers of the equation $x_1 + \cdots + x_n = x_1 \cdots x_n$ such that $x_1 \leq \cdots \leq x_n$. Show that $S(n)$ is nonzero and finite for all n, but that it is not bounded as $n \to \infty$.

Problem 3. Let f be a differentiable real-valued function on $[0, 1]$ such that $f(0) = 0$. Show that for every real $r > 0$ there exists $a \in (0, 1)$ such that $f'(a) = rf(a)/(1 - a)$.

Problem 4. In the game Random Bocce, a ball called the Pallino is placed at random on a large playing field. Each of the n players has k balls of a distinctive color. The players place their balls on the playing field at random. The winning player is the one with a ball closest to the Pallino, and gets a score which is the number of balls of that player's color that are closer to the Pallino than any ball of another color. You may assume that no two balls are placed at the same distance from the Pallino. What is the expected score of the winner, as a function of n and k?

Problem 5. Prove that $\displaystyle\lim_{n\to\infty} n\left(\left(1^1 + 2^2 + 3^3 + \cdots + n^n\right)^{1/n} - n\right) = \frac{1}{e}$.

Problem 6. Suppose that $x_0, x_1, \ldots, x_{2008}$ are real numbers with $x_0 = 0$ and $x_{2008} = 1$. What is the minimal value of

$$(x_1 - 2x_0)^2 + (x_2 - 2x_1)^2 + \cdots + (x_{2008} - 2x_{2007})^2?$$

Problem 7. A mathwhiz has $2n > 0$ students, where n is an integer. She tests them as a group by shuffling a deck of $2n$ cards labeled from 1 to $2n$ and placing them in envelopes also labeled from 1 to $2n$. All permutations of the cards are equally likely. Each student will separately look at the cards in n of the envelopes and then replace them, leaving no traces of what was done, and not communicating with the others. If every student sees the card with her or his number, they will all be promoted to mathwhiz. The students hit on the following strategy: each looks first in the envelope with her or his number, then in the envelope with the number found on the card in the first envelope, etc. Each time, the student looks in the envelope whose number is the number of the card just previously seen, quitting when the student's number is seen, or when the n tries are over. Let p_n be the probability that this strategy succeeds, i.e., that all are promoted. Find $\lim_{n\to\infty} p_n$.

Problem 8. For a set $A = \{S_1, S_2, \ldots, S_k\}$ where $S_1, \ldots, S_k$ are distinct subsets of $\{1, 2, \ldots, n\}$ we define another set $F(A)$ of subsets of $\{1, 2, \ldots, n\}$ as follows: A subset $S \subseteq \{1, 2, \ldots, n\}$ is a member of $F(A)$ if and only if there is an odd number of i for which S_i is a subset of S. Prove that $F(F(A)) = A$ for every set A of subsets of $\{1, 2, \ldots, n\}$.

Problem 9. Suppose that $f : [0, 1] \to \mathbb{R}$ is a real-valued function (not necessarily continuous) for which every $x \in [0, 1]$ is a local maximum, i.e., for every $x \in [0, 1]$ there exists an $\epsilon > 0$ such that for every $y \in [0, 1]$ with $|x-y| < \varepsilon$ we have $f(y) \leq f(x)$. Prove that f has a global maximum, i.e., there exists an $x \in [0, 1]$ such that $f(y) \leq f(x)$ for all $y \in [0, 1]$.

Problem 10. Prove that every positive integer can be multiplied by a power of 2, such that the product has at least as many 8's as it has 4's in its decimal expansion.