

UNIVERSITY OF MICHIGAN
UNDERGRADUATE MATH COMPETITION 24
APRIL 14, 2007

Instructions. Write on the front of your blue book your student ID number. Do not write your name anywhere on your blue book. Each question is worth 10 points. For full credit, you must **prove** that your answers are correct even when the question doesn't say "prove". There are lots of problems of widely varying difficulty. It is not expected that anyone will solve them all; look for ones that seem easy and fun. No calculators are allowed.

Problem 1. Let S be a set of nonnegative integers that contains 0, is closed under addition, and that contains all positive integers from some point on. Let a be an element of S . Show that the number of integers in S that are not of the form $a + s$ for some $s \in S$ is equal to a .

Proof. For i with $0 \leq i < a$, let $S_i = S \cap i + a\mathbb{Z}$ be the set of all elements of S which are congruent to i modulo a . Since S contains all positive integers from some point on, S_i is nonempty. Let b_i be the smallest element of S_i . Clearly $b_i - a$ does not lie in S and S_i . It follows that $S_i = \{b_i, b_i + a, b_i + 2a, \dots\}$. So S_i has exactly one element that is not of the form $a + s$ with $s \in S$, namely b_i . Since S is the disjoint union of S_1, S_2, \dots, S_{a-1} , the elements with the desired properties are b_0, b_1, \dots, b_{a-1} . So there are exactly a of them. \square

Problem 2. A jar contains 600 jelly beans, 100 red, 200 green, and 300 blue. These are drawn randomly from the jar, one at a time, without replacement. What is the probability that the first color to be exhausted is red?

Proof. There will be a final string of beans of the same color (the color of the last bean drawn), and then a bean preceding it of a different color. The desired probability p is that these two colors are blue and green or green and blue, respectively. The probability that the last bean drawn is blue is $300/600 = 1/2$, and, no matter how long the final string of blue beans is, the probability that the bean of preceding color is green is $200/(100+200) = 2/3$. For green and blue these numbers become $200/600 = 1/3$ and $300/(100 + 300) = 3/4$. Thus, $p = (1/2)(2/3) + (1/3)(3/4) = (1/3) + (1/4) = 7/12$. \square

Problem 3. Let $\langle \rangle$ indicate decimal notation, so that when A, B, C, D are integers between 0 and 9, $\langle ABCD \rangle$ denotes $1000A + 100B + 10C + D$ and $\langle A7 \rangle$ denotes $10A + 7$, etc. Find all integers $N = \langle ABCD \rangle$ with $A, D \neq 0$ such that

$$\langle DCBA \rangle = \langle A1 \rangle^2 + (A + 1)^{D+1}.$$

Proof. Mod 10, we have $A \equiv 1 + (A + 1)^{D+1}$, i.e. $(A + 1)^{D+1} \equiv A - 1$. This is clearly impossible if $A = 1, 4, 5, 6, 8$, or 9 . If $A = 2$, $D + 1$ must be a multiple of 4, and so $D = 3$ or $D = 7$. We have $21^2 + 3^4 < 1000$, but $21^2 + 3^8 = 441 + 6561 = 7002$, so that $N = 2007$ is one solution. If $A = 7$ we must have $8^{D+1} \equiv 2$. This implies $D + 1$ is 3 or 7. But $71^2 + 8^3 = 5041 + 512 = 5553$, and $71^2 + 8^7 > 2^{21} > 1,000,000$. Thus, $N = 2007$ is the only solution. \square

Problem 4. A UFO lands at a random spot on a large square, which is paved with square tiles of size 1ft \times 1ft. The UFO leaves a disc-shaped burn mark of radius 10ft. What is the expected number of tiles that have to be replaced because they are damaged by the UFO?

Proof. Assume that the center (p, q) of the disc is uniform distributed over $[0, 1] \times [0, 1]$. The tile $T_{x,y} := [x, x + 1] \times [y, y + 1]$ ($x, y \in \mathbb{Z}$) intersects the disc if and only if the point $(p - x, q - y)$ has at most distance 10 to $[0, 1] \times [0, 1]$. So the probability that $T_{x,y}$ intersects the disc is the area of $T_{-x,-y} \cap S$ where S is the set of all points with distance ≤ 10 to $[0, 1] \times [0, 1]$. Summing over all $x, y \in \mathbb{Z}$ shows that the expected number of tiles intersecting with the disc is the area of S . The set S is the union of a cross-shaped region

$$\{(x, y) \mid 0 \leq x \leq 1, -10 \leq y \leq 11\} \cup \{(x, y) \mid 0 \leq y \leq 1, -10 \leq x \leq 11\}$$

which has area 41 and its complement, consisting of four quarter discs of radius 10, whose area together is 100π . So the expected number of tiles that have to be replaced is $100\pi + 41$. \square

Problem 5. Let $a_0 = 0$ and, recursively, let $a_n = \sqrt{a_{n-1} + n}$ for every integer $n \geq 1$. Thus, $a_1 = 1$ and $a_2 = \sqrt{3}$. Determine, with proof, $\lim_{n \rightarrow \infty} (a_n - \sqrt{n})$ or show it does not exist.

Proof. We show the limit is $1/2$. It is clear that $a_n \geq 0$ for all n , and so $a_n \geq \sqrt{n}$. By induction, we see that $a_n \leq \sqrt{n} + 1$ for all n : this is clear when $n = 0$, and if $n \geq 1$ and $a_{n-1} \leq \sqrt{n-1} + 1$ we have $a_n^2 = a_{n-1} + n \leq n + \sqrt{n-1} + 1 \leq n + 2\sqrt{n} + 1 = (\sqrt{n} + 1)^2$.

Let $b_n = a_n - \sqrt{n} - 1/2$. Then $b_n \in [-1/2, 1/2]$ for all n . It will suffice to show that $b_n \rightarrow 0$. For $n \geq 1$, $(b_n + \sqrt{n} + 1/2)^2 = a_n^2 = a_{n-1} + n = b_{n-1} + \sqrt{n-1} + 1/2 + n$, and so $b_n^2 + (2\sqrt{n} + 1)b_n + n + \sqrt{n} + 1/4 = b_{n-1} + \sqrt{n-1} + 1/2 + n$. Note that $\sqrt{n} - \sqrt{n-1} = 1/(\sqrt{n} + \sqrt{n-1})$. This yields

$$b_n = \frac{b_{n-1} + 1/4 - 1/(\sqrt{n} + \sqrt{n-1})}{(2\sqrt{n} + 1 + b_n)}.$$

The three terms in the numerator are bounded in absolute value by $1/2$, $1/4$, and 1 , respectively and the denominator is at least $2\sqrt{n}$. It follows that $b_n \rightarrow 0$. \square

Problem 6. Show that it is not possible to partition the set $\{1, 3, 5, \dots, 2007\}$ into two sets $\{x_1, x_2, \dots, x_{502}\}$ and $\{y_1, y_2, \dots, y_{502}\}$ such that

$$\sum_{i=1}^{502} x_i^2 = \sum_{j=1}^{502} y_j^2.$$

Proof. We compute $1^2 + 3^2 + \dots + 2007^2$ modulo 16. □

Problem 7. Prove that

$$e^{e^x} \geq e^{2x+1} - xe^{x+1}$$

for all $x \in \mathbb{R}$.

Proof. Let $f(x) = e^x - x - 1$, then $f'(x) = e^x - 1$. Note that $f'(x) < 0$ for $x < 0$ and $f'(x) > 0$ for $x > 0$ so $f(x)$ is decreasing for $x < 0$ and increasing for $x > 0$. Since $f(x) = 0$ we get $f(x) \geq 0$ for all $x \in \mathbb{R}$. This implies

$$f(f(x)) = e^{e^x - x - 1} - (e^x - x - 1) - 1 = e^{e^x - x - 1} - e^x + x \geq 0$$

Multiplying with e^{x+1} gives us

$$e^{e^x} - e^{2x+1} + xe^{x+1} \geq 0$$

□

Problem 8. A malfunctioning calculator has 12 keys: ten numeric keys labeled **0** through **9**, a key labeled **+**, and a key labeled **=**. A monkey presses keys at random until the **=** key is hit, at which point the calculator evaluates the input expression. All keys are equally likely to be hit. The calculator behaves as follows: if only **+** and **=** have been hit, the value is 0. An initial or final string of consecutive **+** entries is ignored. A string of consecutive **+** signs between two numeric strings is interpreted as a single **+**. Whenever a new numeric string begins, the **calculator enters a decimal point at the beginning of the numeric string**. Thus, if the monkey hits

+, +, +, 0, 4, 5, 6, +, +, 2, 7, 5, +, 0, 0, 9, 1, +, +, =

the number returned is $.0456 + .275 + .0091 = .3297$. What is the expected value of the number obtained by evaluation of the string the monkey enters?

Proof. Let F be the expected value of the numeric string (which may be empty) that precedes the first **+** or **=** in what the monkey types. We condition on the first key struck. If it is **+** or **=**, then the expected value is 0. If it is the integer d , the expected value is $(d+F)/10$, for the expected value of what the monkey types in the first numeric string after d is $F/10$. Hence, F is the average of these, and $12F = 0 + 0 + 4.5 + F$ so that $F = 4.5/11$. Let E be the expected value of the whole expression. We condition on the first key struck after the first numeric string. If it is **=** then expected value is F . If it is **+**, then the expected value is $E + F$. These are equally likely, so that $2E = F + (E + F)$. Hence, $E = 2F = 9/11$. □

Problem 9. Let a_n be the fractional part of $\ln(n)$, $n \geq 1$. Let b_n be the average of the numbers a_1, a_2, \dots, a_n . Find a continuous real-valued function f on $[0, 1]$ such that

$$\lim_{n \rightarrow \infty} (b_n - f(a_n)) = 0.$$

Proof. $nb_n = \sum_{i=1}^n \ln(n) - \sum_{i=1}^n [\ln(n)]$, where $[\]$ indicates integer part. The number of integers between e^{j+1} and e^j is $[e^{j+1}] - [e^j]$. If $e^k \leq n < e^{k+1}$, so that $\ln(n) = k + a_k$, the sum of the integer parts is $k(n - e^k) + (k-1)(e^k - e^{k-1}) + \dots + 2(e^3 - e^2) + (e^2 - e)$ with an error that is bounded by $k + 2(k-1 + k-2 + \dots + 2 + 1) = k^2$ (which arises because each of the exponential terms should have been enclosed in $[\]$). This gives $kn - e^k - e^{k-1} - \dots - e = kn - e(e^k - 1)/(e-1) = (\ln(n) - a_n)n - e(e^{\ln(n) - a_n})/(e-1) = \ln(n)n - a_n n - \frac{e}{e-1} n e^{-a_n}$ with an error bounded by $E_n = k^2 + e/(e-1)$, and $E_n/n \rightarrow 0$, since $k \leq \ln(n)$. Now $\int_2^{n+1} \ln(x) dx \leq \sum_{i=2}^n \ln(n) \leq \int_1^n \ln(x) dx$. the antiderivative of $\ln(x)$ is $x \ln(x) - x$, so that $n \ln(n) - n + 1 \leq \sum_{i=2}^n \ln(i) \leq (n+1) \ln(n+1) - (n+1) - 2 \ln(2) + 2$. Therefore, we may estimate $\sum_{i=1}^n \ln(n)$ as $n \ln(n) - n$ with an error bounded by $C \ln(n)$ for C some constant, and $C \ln(n)/n \rightarrow 0$. The difference of the estimates is $na_n - n + n \frac{e}{e-1} / e^{a_n}$. Dividing by n yields $a_n - 1 + e^{1-a_n}/(e-1)$. Thus, we may take $f(x) = (e-1)^{-1} e^{1-x} - (1-x)$. \square

Problem 10. Let p be a odd prime number.

- Show that among $p+2$ distinct points in the plane with integer coordinates, one can choose 3 distinct points A, B, C such that $2 \text{ area}(\triangle ABC)$ is divisible by p . (Here $\text{area}(\triangle ABC)$ denotes the area of the triangle $\triangle ABC$.)
- Show that it is possible to choose a set of $p+1$ distinct points with integral coordinates, such that p does not divide $2 \text{ area}(\triangle ABC)$ whenever A, B, C are distinct elements of the set.

Proof. (a). Suppose that $P_i = (x_i, y_i)$, $i = 0, 1, 2, \dots, p+1$ are points, no 3 on a line such that $2 \text{ area}(\triangle P_i P_j P_k)$ is not divisible by p for all i, j, k with $1 \leq i < j < k \leq p+2$. After translation, we may assume that $P_0 = (0, 0)$. We have

$$2 \text{ Area}(\triangle P_i P_j P_k) = \pm \det \begin{pmatrix} 1 & 1 & 1 \\ x_i & x_j & x_k \\ y_i & y_j & y_k \end{pmatrix}.$$

Let $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ be the integers modulo p and define $\bar{P}_i = (\bar{x}_i, \bar{y}_i) \in \mathbb{F}_p^2$ and $\bar{x}_i, \bar{y}_i \in \mathbb{F}_p$ are the residue classes modulo p of x_i and y_i respectively for all i . Modulo p we have

$$(1) \quad 0 \neq 2 \text{ Area}(\triangle P_i P_j P_k) \pmod{p} = \pm \det \begin{pmatrix} 1 & 1 & 1 \\ \bar{x}_i & \bar{x}_j & \bar{x}_k \\ \bar{y}_i & \bar{y}_j & \bar{y}_k \end{pmatrix}$$

for all i, j, k . Let $T = \{\bar{P}_0, \dots, \bar{P}_{p+1}\} \subseteq \mathbb{F}_p^2$. The nonvanishing of (1) means that no 3 points of T are colinear. In \mathbb{F}_p^2 , there are $p+1$ rays through the origin $\bar{P}_0 = (\bar{0}, \bar{0})$. At most 1 of the points in $T \setminus \{\bar{P}_0\}$ can lie on each of these rays. Therefore, exactly

1 of these points lies on each ray through the origin. So each ray through $\overline{P_0}$ contains two elements of T . Similarly, every line through $\overline{P_i}$ contains 2 points from T for $i = 1, 2, \dots, p + 1$. This shows that every line in \mathbb{F}_p^2 contains 0 or 2 points from T . Consider the lines $x = \overline{0}, \overline{1}, \dots, \overline{p-1}$. each of these lines contains an even number of elements of T . Since the plane is a disjoint union of these lines, T has an even number of elements. But T has $p + 2$ elements and p is odd. Contradiction!

(b). Let \mathbb{F}_{p^2} be the field with p^2 elements. Then \mathbb{F}_{p^2} has a primitive $(p + 1)$ -th root of unity α . We can identify \mathbb{F}_{p^2} with \mathbb{F}_p^2 as an \mathbb{F}_p -vector space. We claim that among $1, \alpha, \alpha^2, \dots, \alpha^p$ there are no three lying on a line. Note that β, γ, δ lie on a line if and only if

$$\frac{\beta - \delta}{\delta - \delta} \in \mathbb{F}_p$$

Suppose that $\alpha^i, \alpha^j, \alpha^k$ lie on a line. Then

$$(2) \quad \frac{\alpha^k - \alpha^i}{\alpha^j - \alpha^i} = \beta \in \mathbb{F}_p$$

Applying Frobenius and noting that $\alpha^p = \alpha^{-1}$ and $\beta^p = \beta$ we get

$$(3) \quad \frac{\alpha^{-k} - \alpha^{-i}}{\alpha^{-j} - \alpha^{-i}} = \beta$$

Dividing (2) by (3) yields

$$\alpha^{k-j} = 1.$$

This is a contradiction because $1 \leq k - j < p + 1$. We can lift $\alpha_1, \dots, \alpha_{p+1}$ to points in \mathbb{Z}^2 with the desired property. \square