UNIVERSITY OF MICHIGAN
UNDERGRADUATE MATH COMPETITION 24
APRIL 14, 2007

Instructions. Write on the front of your blue book your student ID number. Do not write your name anywhere on your blue book. Each question is worth 10 points. For full credit, you must **prove** that your answers are correct even when the question doesn't say "prove". There are lots of problems of widely varying difficulty. It is not expected that anyone will solve them all; look for ones that seem easy and fun. No calculators are allowed.

Problem 1. Let S be a set of nonnegative integers that contains 0, is closed under addition, and that contains all positive integers from some point on. Let a be an element of S. Show that the number of integers in S that are not of the form $a+s$ for some $s \in S$ is equal to a.

Problem 2. A jar contains 600 jelly beans, 100 red, 200 green, and 300 blue. These are drawn randomly from the jar, one at a time, without replacement. What is the probability that the first color to be exhausted is red?

Problem 3. Let $\langle\ \rangle$ indicate decimal notation, so that when A, B, C, D are integers between 0 and 9, $\langle A\,B\,C\,D\rangle$ denotes $1000A+100B+10C+D$ and $\langle A\,7\rangle$ denotes $10A+7$, etc. Find all integers $N = \langle A\,B\,C\,D\rangle$ with A, $D \neq 0$ such that

$$\langle D\,C\,B\,A\rangle = \langle A\,1\rangle^2 + (A+1)^{D+1}.$$

Problem 4. A UFO lands at a random spot on a large square, which is paved with square tiles of size 1ft × 1ft. The UFO leaves a disc-shaped burn mark of radius 10ft. What is the expected number of tiles that have to be replaced because they are damaged by the UFO?

Problem 5. Let $a_0 = 0$ and, recursively, let $a_n = \sqrt{a_{n-1}+n}$ for every integer $n \geq 1$. Thus, $a_1 = 1$ and $a_2 = \sqrt{3}$. Determine, with proof, $\lim\limits_{n\to\infty}(a_n - \sqrt{n})$ or show it does not exist.

More problems on the other side →

Problem 6. Show that it is not possible to partition the set $\{1, 3, 5, \ldots, 2007\}$ into two sets $\{x_1, x_2, \ldots, x_{502}\}$ and $\{y_1, y_2, \ldots, y_{502}\}$ such that

$$\sum_{i=1}^{502} x_i^2 = \sum_{j=1}^{502} y_i^2.$$

Problem 7. Prove that

$$e^{e^x} \geq e^{2x+1} - xe^{x+1}$$

for all $x \in \mathbb{R}$.

Problem 8. A malfunctioning calculator has 12 keys: ten numeric keys labeled **0** through **9**, a key labeled $+$, and a key labeled $=$. A monkey presses keys at random until the $=$ key is hit, at which point the calculator evaluates the input expression. All keys are equally likely to be hit. The calculator behaves as follows: if only $+$ and $=$ have been hit, the value is 0. An initial or final string of consecutive $+$ entries is ignored. A string of consecutive $+$ signs between two numeric strings is interpreted as a single $+$. Whenever a new numeric string begins, the **calculator enters a decimal point at the beginning of the numeric string**. Thus, if the monkey hits

$$\mathbf{+,\ +,\ +,\ 0,\ 4,\ 5,\ 6,\ +,\ +,\ 2,\ 7,\ 5,\ +,\ 0,\ 0,\ 9,\ 1\ ,\ +,\ +,\ =}$$

the number returned is $.0456 + .275 + .0091 = .3297$. What is the expected value of the number obtained by evaluation of the string the monkey enters?

Problem 9. Let a_n be the fractional part of $\ln(n)$, $n \geq 1$. Let b_n be the average of the numbers $a_1,\ a_2,\ \ldots,\ a_n$. Find a continuous real-valued function f on $[0,\ 1]$ such that

$$\lim_{n\to\infty} \big(b_n - f(a_n)\big) = 0.$$

Problem 10. Let p be a odd prime number.

(a) Show that among $p + 2$ distinct points in the plane with integer coordinates, one can choose 3 distinct points A, B, C such that $2\,\text{area}(\triangle ABC)$ is divisible by p. (Here area($\triangle$ABC) denotes the area of the triangle $\triangle ABC$.)

(b) Show that it is possible to choose a set of $p + 1$ distinct points with integral coordinates, such that p does not divide $2\,\text{area}(\triangle ABC)$ whenever A, B, C are distinct elements of the set.