UNIVERSITY OF MICHIGAN
UNDERGRADUATE MATH COMPETITION 23
APRIL 2, 2006

Instructions. Write on the front of your blue book your student ID number. Do not write your name anywhere on your blue book. Each question is worth 10 points. For full credit, you must **prove** that your answers are correct even when the question doesn't say "prove". There are lots of problems of widely varying difficulty. It is not expected that anyone will solve them all; look for ones that seem easy and fun. No calculators are allowed.

Problem 1. S, T, and U are three finite non-empty mutually disjoint subsets of the plane. Let $A = S \cup T \cup U$. No three points in A are collinear. Show that there must exist a triangle with one vertex in each of the three sets and no point of A in its interior.

Solution. Choose $P \in S$, $Q \in T$, $R \in U$ such that the area of $\triangle PQR$ is minimal. If $X \in S$ and X lies inside $\triangle PQR$, then the area of $\triangle XQR$ is smaller than the area of $\triangle PQR$. Contradiction, so X lies outside $\triangle PQR$. Similarly, elements from T and U lie outside $\triangle PQR$.

Problem 2. Suppose that S is a sphere in 3-dimensional space which is tangent to each of the 6 sides of a tetrahedron $ABCD$. Show that

$$|AB| + |CD| = |AC| + |BD| = |AD| + |BC|,$$

where $|PQ|$ denotes the distance between two points P, Q. (A sphere is tangent to a line segment AB if the line through AB intersects the sphere in a unique point, and this point lies on the line segment AB.)

Solution. Let P be a point on AB, AC or AD which lies on the sphere. Set $a = |AP|$. Then $a^2 + r^2 = |AQ|^2$ where Q is the center and r the radius of the sphere. So a does not depend on whether P lies on AB, on AC or on AD. Similarly let b be the distance of B to the tangent points on AB, BC, BD, c the distance of C to the tangent points on AC, BC, CD and d be the distance of D to the tangent points on AD, BD, CD. If P is a tangent point on AB, then $|AB| = |AP| + |PB| = a + b$. So we have

$$|AB|+|CD| = (a+b)+(c+d) = (a+c)+(b+d) = |AC|+|BD| = (a+d)+(b+c) = |AD|+|BC|.$$

Problem 3. The sequence $\{a_n\}_n$ is defined recursively such that $a_1 = 1$ and $a_{n+1} = (n+1)^{a_n}$, $n \geq 1$. For example, $a_2 = 2^1 = 2$, $a_3 = 3^2 = 9$, and $a_4 = 4^9$. Find the last two digits of a_{2006}.

Solution. The term a_{2005} is a power of 2005 and so has the form $5k$ for some positive integer k. Modulo 100, $a_{2006} = 2006^{5k} \equiv 6^{5k} \equiv (6^5)^k \equiv 76^k \equiv 76$, since $76^2 - 76 = 76 \cdot 75 = 19 \cdot 4 \cdot 25 \cdot 3 \equiv 0$. The required digits are 7, 6.

Problem 4. One morning, tiny bits of cheese start falling from the sky over a broad area including the home of Mervyn the mouse, who just loves cheese storms. The rate of fall is constant for several hours. At 11 a.m., Mervyn starts eating his way in a straight line towards the home of his friend Millie. His rate of progress is inversely proportional to the height of the accumulated cheese, and hence to the amount of time since the cheese storm started. He covers six yards by noon, and three more yards by 1 p.m., when he arrives at Millie's. At what time did the cheese fall start?

Solution. Say the fall starts a hours before 11 a.m. Let $y = y(t)$ be the distance covered by time t, where $t = 0$ corresponds to 11 a.m. Then $dy/dt = c/(t+a)$ for some constant c, and so $y = c\ln(t+a)+b$. The values of y at times 0, 1, and 2 are 0, 6, and 9, respectively. From the first, $b = -c\ln(a)$, and $y = c\ln(t+a) - c\ln(a) = c\ln((t+a)/a)$. Then $6 = c\ln((a+1)/a)$, and $9 = c\ln((a+2)/a)$. Dividing the second equation by the first gives $3/2 = \ln((2+a)/a)/\ln((a+1)/a)$ or $\ln(((a+1)/a)^3) = \ln(((a+2)/a)^2)$. Thus, $(a+1)^3/a^3 = (a+2)^2/a^2$, and multiplying by a^3 gives $(a+1)^3 = a(a+2)^2$ or $a^3 + 3a^2 + 3a + 1 = a^3 + 4a^2 + 4a$ or $a^2 + a - 1 = 0$, and so $a = (-1+\sqrt{5})/2$ hours. (This gives approximately 10:23 a.m.)

Problem 5. Evaluate

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{i+j}(i-j)}{(i-j)^2 - \frac{1}{4}} \quad\text{and}\quad \sum_{j=0}^{\infty}\sum_{i=0}^{\infty} \frac{(-1)^{i+j}(i-j)}{(i-j)^2 - \frac{1}{4}}.$$

Solution. By interchanging i and j in one of the sums, it is clear that the second sum is the negative of the first. So it suffices to compute the first sum.

$$\sum_{j=0}^{\infty} \frac{(-1)^{i+j}(i-j)}{(i-j)^2 - \frac{1}{4}} = \tfrac{1}{2}\sum_{j=0}^{\infty}(-1)^{i+j}\left(\frac{1}{i-j+\frac{1}{2}} + \frac{1}{i-j-\frac{1}{2}}\right) =$$

$$= \tfrac{1}{2}(-1)^i\left(\left(\frac{1}{i+\frac{1}{2}} + \frac{1}{i-\frac{1}{2}}\right) - \left(\frac{1}{i-\frac{1}{2}} + \frac{1}{i-\frac{3}{2}}\right) + \left(\frac{1}{i-\frac{3}{2}} + \frac{1}{i-\frac{5}{2}}\right) - \cdots\right)$$

This is clearly a converging telescope sum whose value is

$$\tfrac{1}{2}(-1)^i\frac{1}{i+\frac{1}{2}} = \frac{(-1)^i}{2i+1}.$$

Therefore

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{(-1)^{i+j}(i-j)}{(i-j)^2 - \frac{1}{4}} = \sum_{i=0}^{\infty}\frac{(-1)^i}{2i+1} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.$$

Problem 6. Let $p(z)$ be a polynomial, and put

$$q(z) = p(z+i) - p(z-i).$$

Show that if all zeros of $p(z)$ are real, then all zeros of $q(z)$ are real.

Solution. Write $p(z) = \alpha \prod_{i=1}^{n}(z-\beta_i)$ with $\alpha \in \mathbb{C}$ and $\beta_1, \ldots, \beta_n \in \mathbb{R}$. Suppose that $p(z+i) = p(z-i)$. If $\operatorname{Im} z > 0$ then $q(z) \neq 0$ and $p(z-i) \neq 0$. We have

$$1 = \frac{p(z+i)}{p(z-i)} = \prod \left| \frac{z-\beta_i+i}{z-\beta-i} \right| > 1$$

because

$$\left| \frac{z-\beta+i}{z-\beta-i} \right| > 1$$

for all i. Contradiction. Similarly $\operatorname{Im} z < 0$ leads to a contradiction as well.

Problem 7. A walk starts at the origin O. The n-th leg of the walk is a straight line of length $1/2^n$ miles, for $n \geq 1$. For each new leg an angle is selected at random (for intervals of equal length in $[0, 2\pi]$ it is as likely to be in one as in the other), and the direction of the new leg makes that angle with the positive x-axis. Let P_n be the position after the n-th leg, and let P be the limit of P_n as $n \to \infty$. Find the expected value of D^2, where D is the distance from P to O.

Solution. Suppose the nth angle is θ_n. The coordinates of P may be thought of as $\sum_{n=1}^{\infty} 1/2^n(\cos\theta_n, \sin\theta_n)$ and so $D^2 = (\sum_{n=1}^{\infty} 2^{-n}\cos\theta_n)^2 + (\sum_{n=1}^{\infty} 2^{-n}\sin\theta_n)^2$. The squares that occurs can be written as $\sum_{n=1}^{\infty}(1/2^{2n})(\cos\theta_n^2 + \sin\theta_n^2) = \sum_{n=1}^{\infty}(1/4^n) = (1/4)/(3/4) = 1/3$. The other terms, each of which is a constant multiple of $\cos\theta_m \cos\theta_n$ or $\sin\theta_m \sin\theta_n$, all have expected value 0. (All sums converge absolutely by comparison with the terms of $2(\sum_{n=1}^{\infty} 1/2^n)^2$.) The expected value of D^2 is $1/3$.

Problem 8. Let $\mathcal{S}$ be a set of $2n+1$ nonzero points in $\mathbb{R}^m$. Show that it is possible to choose a subset $\mathcal{A}$ consisting of $n+1$ of the elements of $\mathcal{S}$, say $\mathcal{A} = \{a_1, \ldots, a_{n+1}\}$, in such a way that if $\varepsilon_i = 0$ or 1, then $\sum_{i=1}^{n+1} \varepsilon_i a_i = 0$ only when $\varepsilon_i = 0$ for all i.

Solution. Choose a vector $v \in \mathbb{R}^m$ such that $\langle v, a_i \rangle \neq 0$ for all i. Let

$$\mathcal{A}_1 = \{a \in \mathcal{S} \mid \langle v, a \rangle > 0\}$$

and

$$\mathcal{A}_2 = \{a \in \mathcal{S} \mid \langle v, a \rangle < 0\}.$$

Then $\mathcal{A}_1 \cup \mathcal{A}_2 = \mathcal{S}$, so one of the sets $\mathcal{A}_1, \mathcal{A}_2$ contains at least $n+1$ elements. By possibly replacing v by $-v$, we may assume that $\mathcal{A}_1$ has at least $n+1$ elements. Choose a subset $\mathcal{A} = \{a_1, \ldots, a_{n+1}\} \subseteq \mathcal{A}_1$ of exactly $n+1$ elements. If $\sum_i \varepsilon_i a_i = 0$ with $\varepsilon_i \in \{0, 1\}$ for all i, then

$$0 = \langle \sum_i \varepsilon_i a_i, v \rangle = \sum_i \varepsilon_i \langle a_i, v \rangle$$

Since $\langle a_i, v \rangle > 0$ for all i, this implies that $\varepsilon_1 = \cdots = \varepsilon_{n+1} = 0$.

Problem 9. A continuous function $f : [0,1) \to [0,\infty)$ satisfies

$$f(\tfrac{1}{2}x + \tfrac{1}{2}) = f(x) + 1$$

and

$$f(1-x) = \frac{1}{f(x)}$$

for $x \in (0,1)$. Evaluate

$$\int_0^1 f(x)\,dx.$$

Solution.

$$f(\tfrac{1}{2} - \tfrac{1}{2}x) = f(1 - (\tfrac{1}{2} + \tfrac{1}{2}x)) = \frac{1}{f(\tfrac{1}{2} + \tfrac{1}{2}x)} = \frac{1}{1 + f(x)}$$

$$f(\tfrac{1}{2}x) = f(\tfrac{1}{2} - \tfrac{1}{2}(1-x)) = \frac{1}{1 + f(1-x)} = \frac{1}{1 + \frac{1}{f(x)}} = \frac{f(x)}{1 + f(x)} = 1 - f(\tfrac{1}{2} - \tfrac{1}{2}x).$$

So

$$f(x) + f(\tfrac{1}{2} - x) = 1$$

and

$$\tfrac{1}{4} = \int_0^{\frac{1}{4}} f(x) + f(\tfrac{1}{2} - x)\,dx = \int_0^{\frac{1}{4}} f(x)\,dy + \int_{\frac{1}{4}}^{\frac{1}{2}} f(x)\,dx = \int_0^{\frac{1}{2}} f(x)\,dx$$

Define

$$d_n := \int_{1-(\frac{1}{2})^n}^{1-(\frac{1}{2})^{n+1}} f(x)\,dx.$$

Then we have

$$\int_0^1 f(x)\,dx = \sum_{n=0}^{\infty} d_n.$$

In fact, since $f(x)$ is nonnegative, and $d_0, d_1, \ldots$ is a nonnegative sequence, the left-hand side converges if and only if the right-hand side converges.

We have $d_0 = \frac{1}{4}$ and for $n > 0$, we have

$$d_n = \int_{1-(\frac{1}{2})^n}^{1-(\frac{1}{2})^{n+1}} f(x)\,dx = \tfrac{1}{2} \int_{1-(\frac{1}{2})^{n-1}}^{1-(\frac{1}{2})^n} f(\tfrac{1}{2}y + \tfrac{1}{2})\,dy = \tfrac{1}{2} \int_{1-(\frac{1}{2})^{n-1}}^{1-(\frac{1}{2})^n} (f(y)+1)\,dy = (\tfrac{1}{2})^{n+1} + \tfrac{1}{2} d_{n-1}.$$

We compute $d_0, d_1, d_2, d_3, \ldots$:

$$\frac{1}{4}, \frac{3}{8}, \frac{5}{16}, \frac{7}{32}, \ldots$$

So we guess $d_n = \frac{2n+1}{2^{n+2}}$ and this can easily be proven by induction. We obtain

$$\int_0^1 f(x)\,dx = \sum_{n=0}^{\infty} d_n = \sum_{n=0}^{\infty} \frac{2n+1}{2^{n+2}} = \frac{1}{2} \sum_{n=0}^{\infty} \frac{n+1}{2^n} - \frac{1}{4} \sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{2} \cdot 4 - \frac{1}{4} \cdot 2 = \frac{3}{2}.$$

Here we used the well-known formulas

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

and

$$1 + 2x + 3x^2 + 4x^3 + \cdots = \frac{1}{(1-x)^2}$$

for $x = \frac{1}{2}$.

Problem 10. Suppose that $p^n - 1$ divides

$$a_1 p^{k_1} + a_2 p^{k_2} + \cdots + a_r p^{k_r}$$

where p is a prime, $n, k_1, k_2, \ldots, k_r$ are nonnegative integers and $a_1, a_2, \ldots, a_r$ are integers satisfying

$$\prod_{i=1}^{r} (|a_i| + 1) < p^n.$$

Prove that there exist nonnegative integers $l_1, l_2, \ldots, l_r$ such that

$$a_1 p^{l_1} + a_2 p^{l_2} + \cdots + a_r p^{l_r} = 0.$$

Without loss of generality we may assume that $0 \le k_1 \le k_2 \le \cdots \le k_r < n$ (we can permute the k_i's and subtract multiples of n from $k_1, \ldots, k_r$ without changing the divisibility condition).

The integer

$$a_1 + a_2 p^{k_2 - k_1} + \cdots + a_r p^{k_r - k_1}$$

is divisible by $p^n - 1$ because $p^n - 1$ and p are relatively prime.

Suppose that

$$b_s := \prod_{i=s}^{r} (|a_i| + 1) \le p^{n + k_1 - k_s}. \tag{1}$$

for $s = 2, \ldots, n$ and that

$$b_1 := \prod_{i=1}^{r} (|a_i| + 1) \le p^n - 1. \tag{2}$$

Then we have

$$b_1 \ge b_2 \ge \cdots \ge b_r \ge 1 \tag{3}$$

We have

$$|a_1 + a_2 p^{k_2 - k_1} + \cdots + a_r p^{k_r - k_1}| \le f(b_1, b_2, \ldots, b_r)$$

where

$$f(b_1, b_2, \ldots, b_r) = \Big(\frac{b_1}{b_2} - 1\Big) + \Big(\frac{b_2}{b_3} - 1\Big) p^{k_2 - k_1} + \cdots + \Big(\frac{b_{r-1}}{b_r} - 1\Big) p^{k_{r-1} - k_1} + (b_r - 1) p^{k_r - k_1}.$$

The function $x \mapsto ax + b/x$ with $a, b > 0$ is concave up for $x > 0$. This means that on every interval $[c, d]$ with $c, d > 0$ the maximum is attained at one of the endpoints of the interval $[c, d]$. Let us view f as a function of real variables $b_1, b_2, \ldots, b_r$ with the restraints (1), (2) and (3). If f is maximal, then $b_1 = p^n - 1$, and for all i, $b_i = b_{i+1}$, $b_i = b_{i-1}$ or $b_i = p^{n+k_1-k_i}$. If $b_i = p^{n+k_1-k_i}$ for all $i \geq 2$, then the value of f is $p^n - 2$. If $b_i = b_{i+1}$ for some i then $a_i = 0$ and we can use induction w.r.t. r to see that the value of f in that case is also $p^n - 2$. So

$$|a_1 + a_2 p^{k_2-k_1} + \cdots + a_r p^{k_r-k_1}| < p^n - 1$$

and we are done (take $l_1 = 0, l_2 = k_2 - k_1, \ldots, l_r = k_r - k_1$). If the theorem is wrong, then $b_s > p^{n+k_1-k_s}$ for some s.

Define a_j for $j > r$ inductively by $a_j = a_{j-r}$. Define k_j inductively for $j > r$ by $k_j = k_{j-r} + n$. We have $k_1 \leq k_2 \leq k_3 \leq \cdots$. For every i we have that $p^n - 1$ divides

$$a_{i+1} p^{k_{i+1}} + \cdots + a_{i+r} p^{k_{i+r}}$$

There exists an s such that

$$\prod_{i=s}^{r} (|a_i| + 1) > p^{n+k_{i+1}-k_{s+i}} = p^{k_{r+i+1}-k_{s+i}}.$$

In other words, for every m there exists an m' such that

$$\prod_{i=m'+1}^{m} (|a_i| + 1) > p^{k_{m+1}-k_{m'+1}}$$

with $m - r < m' < m$.

Let m_0 be a very large postive integer (at least r^2). Define $m_1, m_2, \ldots$ by $m_i - r < m_{i+1} < m_i$ and

$$\prod_{j=m_{i+1}+1}^{m_i} (|a_i| + 1) > p^{k_{m_{i+1}+1}-k_{m_{i+1}}}$$

There exist a and b such that $m_a - m_b$ is divisible by r, say $m_a - m_b = rl$. Then we have

$$\left(\prod_{j=1}^{r} (|a_i| + 1)\right)^l = \prod_{i=a}^{b-1} \prod_{j=m_{i+1}+1}^{m_i} (|a_i| + 1) > p^{k_{m_a+1}-k_{m_b+1}} = p^{nl} = (p^n)^l.$$

So

$$\prod_{j=1}^{r} (|a_i| + 1) \geq p^n$$

Contradiction.