UNIVERSITY OF MICHIGAN
UNDERGRADUATE MATH COMPETITION 23
APRIL 2, 2006

Instructions. Write on the front of your blue book your student ID number. Do not write your name anywhere on your blue book. Each question is worth 10 points. For full credit, you must **prove** that your answers are correct even when the question doesn't say "prove". There are lots of problems of widely varying difficulty. It is not expected that anyone will solve them all; look for ones that seem easy and fun. No calculators are allowed.

Problem 1. S, T, and U are three finite non-empty mutually disjoint subsets of the plane. Let $A = S \cup T \cup U$. No three points in A are collinear. Show that there must exist a triangle with one vertex in each of the three sets and no point of A in its interior.

Problem 2. Suppose that S is a sphere in 3-dimensional space which is tangent to each of the 6 sides of a tetrahedron $ABCD$. Show that

$$|AB| + |CD| = |AC| + |BD| = |AD| + |BC|,$$

where $|PQ|$ denotes the distance between two points P, Q. (A sphere is tangent to a line segment AB if the line through AB intersects the sphere in a unique point, and this point lies on the line segment AB.)

Problem 3. The sequence $\{a_n\}_n$ is defined recursively such that $a_1 = 1$ and $a_{n+1} = (n+1)^{a_n}$, $n \geq 1$. For example, $a_2 = 2^1 = 2$, $a_3 = 3^2 = 9$, and $a_4 = 4^9$. Find the last two digits of a_{2006}.

Problem 4. One morning, tiny bits of cheese start falling from the sky over a broad area including the home of Mervyn the mouse, who just loves cheese storms. The rate of fall is constant for several hours. At 11 a.m., Mervyn starts eating his way in a straight line towards the home of his friend Millie. His rate of progress is inversely proportional to the height of the accumulated cheese, and hence to the amount of time since the cheese storm started. He covers six yards by noon, and three more yards by 1 p.m., when he arrives at Millie's. At what time did the cheese fall start?

Problem 5. Evaluate

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{i+j}(i-j)}{(i-j)^2 - \frac{1}{4}} \text{ and } \sum_{j=0}^{\infty} \sum_{i=0}^{\infty} \frac{(-1)^{i+j}(i-j)}{(i-j)^2 - \frac{1}{4}}.$$

(continued on the other side)

Problem 6. Let $p(z)$ be a polynomial, and put

$$q(z) = p(z+i) - p(z-i).$$

Show that if all zeros of $p(z)$ are real, then all zeros of $q(z)$ are real.

Problem 7. A walk starts at the origin O. The n-th leg of the walk is a straight line of length $1/2^n$ miles, for $n \geq 1$. For each new leg an angle is selected at random (for intervals of equal length in $[0, 2\pi]$ it is as likely to be in one as in the other), and the direction of the new leg makes that angle with the positive x-axis. Let P_n be the position after the n-th leg, and let P be the limit of P_n as $n \to \infty$. Find the expected value of D^2, where D is the distance from P to O.

Problem 8. Let $\mathcal{S}$ be a set of $2n+1$ nonzero points in $\mathbb{R}^m$. Show that it is possible to choose a subset $\mathcal{A}$ consisting of $n+1$ of the elements of $\mathcal{S}$, say $\mathcal{A} = \{a_1, \ldots, a_{n+1}\}$, in such a way that if $\varepsilon_i = 0$ or 1, then $\sum_{i=1}^{n+1} \varepsilon_i a_i = 0$ only when $\varepsilon_i = 0$ for all i.

Problem 9. A continuous function $f : [0, 1) \to [0, \infty)$ satisfies

$$f(\tfrac{1}{2}x + \tfrac{1}{2}) = f(x) + 1$$

and

$$f(1-x) = \frac{1}{f(x)}$$

for $x \in (0, 1)$. Evaluate

$$\int_0^1 f(x)\, dx.$$

Problem 10. Suppose that $p^n - 1$ divides

$$a_1 p^{k_1} + a_2 p^{k_2} + \cdots + a_r p^{k_r}$$

where p is a prime, $n, k_1, k_2, \ldots, k_r$ are nonnegative integers and $a_1, a_2, \ldots, a_r$ are integers satisfying

$$\prod_{i=1}^{r} (|a_i| + 1) < p^n.$$

Prove that there exist nonnegative integers $l_1, l_2, \ldots, l_r$ such that

$$a_1 p^{l_1} + a_2 p^{l_2} + \cdots + a_r p^{l_r} = 0.$$