UNIVERSITY OF MICHIGAN UNDERGRADUATE MATHEMATICS COMPETITION 19

1. Given a set S with n elements (n is a positive integer), what is the number of subsets of subsets of S? (More precisely, we want to count the number of pairs of subsets (X, Y) with $X \subseteq Y \subseteq S$.)
2. What is the smallest positive integer k such that the sum of the decimal digits of $k(10^{2002} - 1)$ is not equal to 18018?
3. Suppose that $f \in C^\infty(-1, 1)$, and suppose that there exist nonzero points $x_1, x_2, x_3, \dots$ in $(-1, 1)$ with $\lim_{n\to\infty} x_n = 0$ and $f(x_n) = 0$ for all n. Prove that every derivative of f vanishes at $x = 0$. (By $f \in C^\infty(-1, 1)$ we mean that f is a real valued function on the interval $(-1, 1)$ whose k-th derivative exists for any k.)
4. Start with the sequence
$$1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \dots$$
(each number n appears n times) and form its partial sums
$$1, 3, 5, 8, 11, 14, 18, 22, 26, 30, \dots$$
Identify all the prime numbers in the latter sequence.
5. A permutation of the set $X = \{1, 2, \dots, 2n\}$ is called *complementing* if there exists an n-element subset $Y \subset X$ such that $\pi(Y)$ is the complement of Y. Show that the number of complementing permutations is a square.
6. Show that if s is a real number, $s > 1$, then
$$\log \frac{s}{s-1} = \int_1^\infty \frac{1 - 1/x}{\log x} x^{-s}\, dx.$$
(In the formula, log stands for the natural logarithm.)
7. Is there a binary operation $*$ on a set S that consists of three distinct elements that is commutative, i.e., $x * y = y * x$ for all $x, y \in S$, and also satisfies $x * (x * y) = y$ for all $x, y \in S$?
8. An interior point P in an equilateral triangle ABC is connected to the vertices, and perpendiculars are dropped to the sides, hitting AB, BC, and CA at X, Y, and Z respectively. Is it necessarily true that the sum of the lengths of AX, BY, and CZ is equal to half the perimeter of the triangle?

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9. Let $n \geq 2$ be a positive integer. Show that every complex number c with $|c| \leq n$ can be written as $c = a_1 + a_2 + \cdots + a_n$ where $|a_j| = 1$ for every j.
10. Consider the sequence of first digits in the successive powers of 2:

$$2, 4, 8, 1, 3, 6, 1, \ldots$$

Does one of the digits 7 and 8 appear more often in the sequence than the other one? (We say for example that 5 appears more often than 6 in the sequence if there exists a positive integer N such that for all $n \geq N$, 5 appears more often than 6 among the first n terms of the sequence.)