UNIVERSITY OF MICHIGAN
UNDERGRADUATE MATH COMPETITION 32
APRIL 11, 2015

Instructions. Write on the front of your blue book your student ID number. Do not write your name anywhere on your blue book. Each question is worth 10 points. For full credit, you must **prove** that your answers are correct even when the question doesn't say "prove". There are lots of problems of widely varying difficulty. It is not expected that anyone will solve them all; look for ones that seem easy and fun. No calculators are allowed.

Problem 1. Suppose that at some time in the future there are N universities in the Big Ten (including Michigan and Ohio State). A single-elimination tournament is to be played as follows: Two teams are selected at random, and play a game. The loser is eliminated and there are $N-1$ undefeated teams. Two of them are selected at random and they play a game. This continues until only one undefeated team is left. in each game, the two teams involved have an equal probability of winning and these probabilities are independent. What is the probability that Michigan and Ohio State play against each other during this tournament?

There are $N-1$ games because each game creates one loser and in the end there are $N-1$ losers. For the i-th game that is played, each of the $\binom{N}{2}$ pairings is equally likely. The probability that the i-th game is a game between Michigan and Ohio state is therefore $\binom{N}{2}^{-1}$. The probability that Michigan and Ohio state meet some time during the tournament is $(N-1)/\binom{N}{2} = \frac{2}{N}$.

Problem 2. Let $a_1 = 1$, $a_2 = 2015$, $a_3 = 2014^{2015}$, $a_4 = 2013^{2014^{2015}}$, and so forth, so that for $1 \leq n < 2015$, $a_{n+1} = (2016-n)^{a_n})$. Find, with proof, the rightmost 2 digits of a_{2015}.

Modulo 100 we have $2^2 \equiv 4$, $2^{10} \equiv 24$, $2^{20} \equiv 76$, $2^{21} \equiv 52$, and $2^{22} \equiv 4$. It follows that behavior thereafter is periodic with period 20, and so $2^{20k+1} \equiv 52 \bmod 100$ for $k \geq 1$. Since $3^4 = 81 \equiv 1 \bmod 20$, it follows that $3^{4h} \equiv 1 \bmod 20$, and so every number of the form $2^{3^{4^s}}$ ends in 52, including a_{2015}.

Problem 3. A house shape is a convex 5-gon $ABCDE$ such that $ABCD$ is a rectangle, and DE and EA have equal length. What is the maximal area of a house shape if the perimeter is 1?

Suppose that $|AB| = |CD| = a$, $|BC| = |AD| = 2b$ and $|AE| = |DE| = c$. The perimeter is $2a + 2b + 2c = 1$ and the area is

$$2ab + b\sqrt{c^2 - b^2} = 2b(\tfrac{1}{2} - b - c) + b\sqrt{c^2 - b^2} = b - 2b^2 + bg(c)$$

where $g(x) = \sqrt{x^2 - b^2} - 2x$. for x with $x \geq b$. Since $g(\sqrt{2}b) = (1 - 2\sqrt{2})b > -2b = g(b)$ and $g(\sqrt{2}b) > -2b > -x \geq g(x)$ for $x > 2b$, the continuous function must have a maximum on the interval $(b, 2b)$. Suppose that g has a maximum at $x = c$. Then we have

$$g'(c) = \frac{c}{\sqrt{c^2 - b^2}} - 2 = 0$$

So we get $c = 2\sqrt{c^2 - b^2}$, $c^2 = 4c^2 - 4b^2$ and $c = \frac{2}{\sqrt{3}}b$. The maximal value of g is $g(c) = \frac{b}{\sqrt{3}} - \frac{4b}{\sqrt{3}} = -\sqrt{3}b$ So $g(x)$ has a maximum at $x = \frac{2}{\sqrt{3}}b$. For fixed b, the maximum area is

$$b - 2b^2 + bg(c) = b - (2 + \sqrt{3})b^2.$$

The maximum is at $b = \frac{1}{2(2+\sqrt{3})} = 1 - \frac{1}{2}\sqrt{3}$ and the maximal area is $\frac{1}{2} - \frac{1}{4}\sqrt{3}$. (We have $c = \frac{2}{\sqrt{3}}b = \frac{2}{3}\sqrt{3} - 1$ and $a = \frac{1}{2} - b - c = \frac{1}{2} - \frac{1}{6}\sqrt{3}$.)

Problem 4. Let α_n be the largest real root of the polynomial

$$C_N(x) = \sum_{i=0}^{n} \frac{(-1)^i x^{2i}}{(2i)!}.$$

Prove that

$$\lim_{n\to\infty} \frac{\alpha_n}{n} = \frac{2}{e}.$$

If $x > 2n$, then the sequence $\frac{x^{2i}}{(2i)!}$ is strictly increasing. By a telescoping argument $C_n(x) > 0$ if n is even and $C_n(x) < 0$ if n is odd. So $0 \leq \alpha_n \leq 2n$. We have

$$C_n(x) = \cos(x) - \sum_{i=n+1}^{\infty} \frac{(-1)^i x^{2i}}{(2i)!}.$$

In particular, we have

$$0 = C_n(\alpha_n) = \cos(\alpha_n) - \sum_{i=n+1}^{\infty} \frac{(-1)^i \alpha_n^{2i}}{(2i)!}$$

So

$$\frac{\alpha_n^{2n+2}}{(2n+4)!} \leq \frac{\alpha_n^{2n+2}}{(2n+2)!} - \frac{(2n)^2\alpha_n^{2n+2}}{(2n+4)!} \leq \frac{\alpha_n^{2n+2}}{(2n+2)!} - \frac{\alpha_n^{2n+4}}{(2n+4)!} \leq \left| \sum_{i=n+1}^{\infty} \frac{(-1)^i \alpha_n^{2i}}{(2i)!} \right| = |\cos(\alpha_n)| \leq 1.$$

It follows that

$$\alpha_n^{2n+2} \leq (2n+4)! \approx \sqrt{2\pi(2n+4)}\Big(\frac{2n+4}{e}\Big)^{2n+4}$$

by Stirling's formula. We get

$$\limsup_{n\to\infty} \frac{\alpha_n}{n} \leq \frac{2}{e}.$$

We can choose β_n such that $\alpha_n - \pi \leq \beta_n \leq \alpha_n$ and $|\cos(\beta_n)| = 1$. We get

$$\frac{\beta_n^{2n+2}}{(2n+2)!} \geq \left| \sum_{i=n+1}^{\infty} \frac{(-1)^i \beta_n^{2i}}{(2i)!} \right| = |\cos(\beta_n)| = 1.$$

A similar calculation shows that

$$\liminf_{n\to\infty} \frac{\alpha_n}{n} = \liminf_{n\to\infty} \frac{\beta_n}{n} = \frac{2}{e}.$$

We conclude that $\lim_{n\to\infty} \frac{\alpha_n}{n} = \frac{2}{e}$.

Problem 5. Find all the solutions to $2^x = y^2 + 15$ where x and y are integers.

Modulo 3 we have $2^x \equiv y^2 \bmod 3$, So y is not divisible by 3 and $2^x \equiv y^2 \equiv 1 \bmod 3$. It follows that x must be even, say $x = 2z$. Then we have $(2^z + y)(2^z - y) = 15$. There are two cases. The first case is where $2^z + y = 15$ and $2^z - y = 1$. Taking the difference yields $2y = 14$ and $y = 7$. It follows that $2^x = 64$ and $x = 6$. In the second case, $2^z + y = 5$ and $2^z - y = 3$. It follows that $2y = 2$ and $y = 1$. So we have $2^x = 16$ and $x = 4$. So the only solutions are $(4, 1)$ and $(6, 7)$.

Problem 6. Suppose that f is a differentiable function from $\mathbb{R}$ to $\mathbb{R}$ such that $f'(x) > 0$ and $f(f(x)) = 2x + 1$ for all $x \in \mathbb{R}$. What is $f(x)$? Prove your answer.

Let $g(x) = f(x-1)+1$. Then we have $g(g(x)) = f(g(x)-1)+1 = f(f(x-1))+1 = (2x-1)+1 = 2x$. Suppose that $g(x) = y$. Then $y = g(x) = g(g(g(x/2))) = 2g(x/2)$. So we get $g(x/2) = y/2$. By induction we get $g(x/2^k) = y/2^k$. By continuity, and taking $k \to \infty$ we get $g(0) = 0$. Moreover we get

$$g'(0) = \lim_{k\to\infty} \frac{g(x/2^k) - g(0)}{x/2^k} = \frac{y/2^k}{x/2^k} = \frac{y}{x}.$$

So $g(x) = y = g'(0)x$ and we have $g(g(x)) = g'(0)^2 x = 2x$. Since $g'(0) > 0$, we get $g'(0) = \sqrt{2}$. It follows that $g(x) = \sqrt{2}x$ and $f(x) = g(x+1) - 1 = \sqrt{2}(x+1) - 1 = \sqrt{2}x + (\sqrt{2} - 1)$.

Problem 7. Numbers 9 and 10 are written on a blackboard. Anne and Pete are playing the following game. Taking turns, the players double one number, subtract 1 from the other number, and substitute the results for the originals. Anne goes first. The player who first reaches or exceeds 1000 wins the game. Both Anne and Pete are very competitive, but only one of them is guaranteed to win by following a proper strategy no matter how the other one plays. Who is it going to be?

We start with the numbers $(9, 10)$. In every turn, a player replaces (x, y) by $(2x, y-1)$ or by $(x-1, 2y)$. Anne wins with the following strategy. First she replaces $(9, 10)$ by $(8, 20)$. After that, she doubles x whenever Pete doubled y, and she doubles y whenever Pete doubles x. This way, after each player has his/her tern, (x, y) gets replaced by

$$(x, y) \to (2x, y-1) \to (2x-1, 2y-2)$$

or

$$(x, y) \to (x-1, 2y) \to (2x-2, 2y-1).$$

After Anne's second turn, we have $(14, 39)$ or $(15, 38)$. The value of (x, y) changes in each round according to the following table:

turn of Anne	x	y
1	$x = 8$	$y = 20$
2	$14 \leq x \leq 15$	$38 \leq y \leq 39$
3	$26 \leq x \leq 29$	$74 \leq y \leq 77$
4	$50 \leq x \leq 57$	$146 \leq y \leq 153$
5	$98 \leq x \leq 113$	$290 \leq y \leq 305$

After Anne's 5th turn, Pete must double x because if he doubles y then Anne will double y as well and $y \geq 4 \cdot 290 > 1000$. So Pete doubles x and we get

$$196 \leq x \leq 226, 289 \leq y \leq 304$$

Now Anne doubles x as well and we get

$$392 \leq x \leq 452, 288 \leq y \leq 303.$$

Now if Pete doubles x, then Anne doubles x and she wins. If Pete doubles y then Anne doubles y and she wins as well.

Problem 8. The coefficients of a quadratic equation $ax^2 + bx + c = 0$ are randomly selected integers between $-n$ and n inclusive, with all choices of an integer equally likely. Let P_n be the probability that the roots of the equation are real. Find $\lim_{n\to\infty} P_n$.

Consider the set $S = \{(a, b, c) \in [-1, 1]^3 \mid b^2 - 4ac \geq 0\}$. Then the roots of $ax^2 + bx + c = 0$ are real if $(\frac{a}{n}, \frac{b}{n}, \frac{c}{n}) \in S$. By taking the limit $n \to \infty$, the limit $\lim_{n\to\infty} P_n$ becomes $\frac{1}{8}\mathrm{vol}(S)$. We compute the volume of S as:

$$\int_{-1}^{1} u(\tfrac{1}{4}b^2)\, db$$

where $u(t)$ is the area of $\{(a, c) \in [-1, 1]^2 \mid ac \leq t\}$, $0 \leq t \leq 1$. This area is

$$4t + 2\int_t^1 1 + \frac{t}{x}\, dx = 2t + 2 - 2t\log(t).$$

So

$$u(\tfrac{1}{4}b^2) = \tfrac{1}{2}b^2 + 2 - \tfrac{1}{2}b^2\log(\tfrac{1}{4}b^2) = (\tfrac{1}{2} + \log(2))b^2 + 2 - b^2\log(|b|).$$

We have

$$\int_{-1}^{1} b^2\log(|b|)\, db = 2\int_0^1 b^2\log(b)\, db = 2\left[\frac{1}{3}b^3\log(b)\right]_0^1 - \frac{2}{3}\int_0^1 b^2\, db = -\frac{2}{9}.$$

It follows that

$$\int_{-1}^{1} u(\frac{1}{4}b^2)\,db = \tfrac{2}{3}(\tfrac{1}{2}+\log(2)) + 4 + \tfrac{2}{9} = \tfrac{41}{9} + \tfrac{2}{3}\log(2)$$

The probability of real roots is $\frac{41}{72} + \frac{1}{12}\log(2)$.

Problem 9. Consider the triangles with sides $0 < a < b < c$ and area $A > 0$ are such that a, b, c, A is an arithmetic progression (i.e., $b - a = c - b = A - c$). Find greatest lower bound of the possible values for b.

We may write a, b, c, A as $b-d, b, b+d, b+2d$, where $0 < d < b$. The semi-perimeter is $3b/2$, and so $(*)\quad (b+2d)^2 = (3b/2)(b/2)(b/2+d)(b/2-d)$ and $16(b+2d)^2 = 3b^2(b^2 - 4d^2)$, i.e. $16b^2+64bd+64d^2 = 3b^4-12b^2d^2$. Therefore $(12b^2+64)d^2+64bd+16b^2-3b^4 = 0$. For $b > 0$, this has a positive root for d if and only if $16b^2 - 3b^4 < 0$, i.e., $b^2 > 16/3$. Hence, the greatest lower bound for values of b is $4/\sqrt{3}$ or $4\sqrt{3}/3$. Note that the equation $(*)$ implies that $b/2 - d$ is positive, and so $b > d$ is automatic.

Problem 10. In the waiting room there are n chairs arranged in a row. A chair is "available" if nobody is sitting on that chair or a chair right next to it. People are entering the room, one person at a time, and sit down in one of the available chairs at random (all the available chairs have the same probability). This continues until there are no more available chairs. Let $f(n)$ be the expected number of people sitting down when there are no more available chairs. For example, if $n = 3$, then with probability $1/3$ the first person sits in the middle and no more people can sit down, and with probability $2/3$ the first person sits at the end and two people get seated, so $f(3) = \frac{1}{3}\cdot 1 + \frac{2}{3}\cdot 2 = \frac{5}{3}$. Show that

$$f(n) = \sum_{i=0}^{n} \frac{(n-i)(-2)^i}{(i+1)!}.$$

Let a_n be the expected value. The first person will sit down on any chair, each chair equally likely. Given that the person sits down in chair i, the expected number of people sitting in a chair below i is a_{i-2} and the expected number of people sitting in a chair above i is a_{n-i-1}. So we have

$$a_n = 1 + \frac{1}{n}\sum_{j=1}^{n}(a_{j-2} + a_{n-j-1}) = 1 + \frac{2}{n}\sum_{j=1}^{n} a_{j-2}$$

If we multiply this equation with x^{n-1} and sum over all positive integers n, we get

$$A'(x) = \frac{1}{(1-x)^2} + \frac{2x}{1-x}A(x).$$

where

$$A(x) = \sum_{n=1}^{\infty} a_n x^n.$$

We have to solve the differential equation

$$A'(x) - \frac{2x}{(1-x)}A(x) = \frac{1}{(1-x)^2}.$$

Let

$$g(x) = \int_0^x \frac{-2y}{1-y}\,dy = \int_0^x 2 - \frac{2}{1-y}\,dy = 2x + 2\log(1-x)$$

We have

$$\left(e^{g(x)}A(x)\right)' = \frac{e^{g(x)}}{(1-x)^2} = e^{2x}.$$

It follows that

$$(1-x)^2 e^{2x} A(x) = e^{g(x)}A(x) = \tfrac{1}{2}e^{2x} - \tfrac{1}{2}$$

and

$$A(x) = \frac{1-e^{-2x}}{2x}\frac{x}{(1-x)^2} = \sum_{i=0}^{\infty} \frac{(-2)^i x^i}{(i+1)!} \sum_{j=0}^{\infty} j x^j$$

Looking at the coefficient of x^n gives the desired formula.