MORE ON THE MAGNUS REPRESENTATION OF THE TORELLI GROUP

MIRA WATTAL

Abstract. The Magnus representation of the Torelli group $\mathcal{I}(\Sigma_g^1)$ of a surface Σ_g^1 is a group homomorphism $r_1 : \mathcal{I}(\Sigma_g^1) \to \mathrm{GL}_{2g}(\mathbb{Z}[H_1(\Sigma_g^1; \mathbb{Z})])$. This paper characterizes the kernel of r_1 for commutators of simply intersecting pairs, commutators of Dehn twists whose associated curves have trivial algebraic intersection. Specifically, we show that under certain conditions, this family of commutators is not in the kernel. To prove our result, we motivate and harness a geometric interpretation of r_1 that is due to Suzuki [13].

Contents

Introduction

Given a surface Σ_g^1 of genus g with one boundary component, we can study its group of symmetries. We can study the **mapping class group** of Σ_g^1. We define the mapping class group $\mathrm{MCG}(\Sigma_g^1)$ to be the group of isotopy classes of orientation preserving homeomorphisms of Σ_g^1 that fix the boundary pointwise.

One natural representation of this group is the **symplectic representation**, the action induced by $\mathrm{MCG}(\Sigma_g^1)$ on the first homology group $H = H_1(\Sigma_g^1;\mathbb{Z})$. In this paper, we study the kernel of the symplectic representation, called the **Torelli group** of Σ_g^1. The Torelli group $\mathcal{I}(\Sigma_g^1)$ is the subgroup of the mapping class group that acts by the identity on H.

A different representation of the mapping class group is the **Magnus representation**, an action of $\mathrm{MCG}(\Sigma_g^1)$ on the fundamental group $\Gamma = \pi_1(\Sigma_g^1, *)$. Here, $*$ is a basepoint on the boundary of Σ_g^1. Unlike the symplectic representation, the Magnus representation of $\mathrm{MCG}(\Sigma_g^1)$ is *not* a genuine representation but a function

$$r : \mathrm{MCG}(\Sigma_g^1) \to \mathrm{GL}_{2g}(\mathbb{Z}[\Gamma]).$$

However, r can be modified to be genuine on one distinguished subgroup: the Torelli group of Σ_g^1.

We define the **Magnus representation of the Torelli group** of Σ_g^1 by restricting the mapping r to $\mathcal{I}(\Sigma_g^1)$ and abelianizing the coefficients linearly. We denote the modified representation by

$$r_1 : \mathcal{I}(\Sigma_g^1) \to \mathrm{GL}_{2g}(\mathbb{Z}[H]).$$

This paper characterizes the kernel of r_1 for **commutators of simply intersecting pairs**, commutators of Dehn twists whose associated curves have trivial algebraic intersection [10]. Specifically, Margalit[1] asked us the following question:

Are there families of commutators of simply intersecting pairs that are not in the kernel of r_1?

and we answered in the affirmative:

Let $[T_\alpha, T_\beta] : \Sigma_g^1 \to \Sigma_g^1$ be a commutator of a simply intersecting pair such that $g \geq 3$, $i(\alpha, \beta) > 0$, and $\alpha \cup \beta$ does not separate Σ_g^1. Then, $[T_\alpha, T_\beta] \notin \ker r_1$.

The kernel of r_1 has been characterized before. In 2001, Suzuki [12] showed that r_1 was not faithful when $g \geq 2$. Four years later, he [13] proved that commutators of simply intersecting pairs whose associated curves are separating and have a geometric intersection of two are in the kernel of r_1 when $g \geq 3$. This paper considers a case not characterized by Suzuki. We prove that commutators of simply intersecting pairs whose associated curves are non-disjoint and form a non-separating pair are *not* in the kernel when $g \geq 3$.[2]

To motivate our main result, we split the paper into two halves. In the first half, we give a survey of surfaces, curves, and mapping class groups. We outline the technical setting required for our theorem, including Dehn twists and algebraic and geometric intersection. Our exposition builds to a computational interpretation of r_1, where we introduce the Fox derivative [4]. Then, the climax occurs in Section 8. Applying the Fox derivative, we prove a specific case of our theorem.

[1] D. Margalit (personal communication, June 2019)

[2] In either ours or Suzuki's case, the specified family of commutators does not occur in $\mathrm{MCG}(\Sigma_g^1)$ when $g < 3$.

We strengthen this specific case to our main result in the second half of the paper. At this point, our exposition becomes markedly more nuanced. We express the first homology of a surface as a symplectic vector space. We define the universal abelian covering space. Finally, we construct and study a cyclic covering space. This machinery from algebraic topology and covering space theory allows us to motivate and harness a geometric interpretation of r_1 that is due to Suzuki [13]. With Suzuki's interpretation, we can finally prove our strongest result.

Note to the Reader

This paper is largely exposition. It is intended for the reader whose mathematical maturity mirrors mine at the start of this research—uninitiated into the world of surface topology. Naturally, most of the paper is devoted to developing the theory needed to prove my result. Only Sections 8, 11, and 12 contain original research. For the initiated, if this format proves too tedious, then you could consider skipping Sections 1-6. However, it was important to me that this paper prioritized accessibility over efficiency.

1. Surfaces

This story opens with our first main objects of study: **compact, oriented surfaces**. Compact, oriented surfaces are compact 2-manifolds that admit an orientation. Some familiar examples include spheres and tori. We can generate more complicated ones by gluing handles (increasing genus) or removing open disks (adding boundary components).

Up to homeomorphism, there are only countably many such manifolds. This brings us to the first classification theorem of the paper:

Theorem 1.1 (Classification Theorem for Surfaces)**.** *Any compact, oriented surface is homeomorphic to one of the following "model" surfaces, classified by its genus and boundary components:*

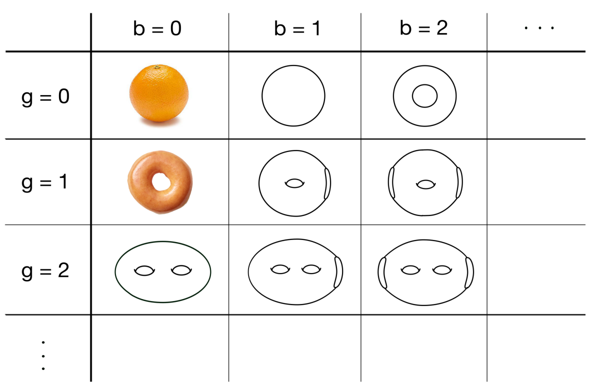

Figure 1. Model surfaces

Let Σ_g^b be a compact, oriented surface of genus g with b boundary components.

The Classification Theorem for Surfaces should *not* be taken lightly. It is deeply rooted in a result from algebraic topology, which states:

Theorem 1.2. *Any compact, oriented surface admits a finite triangulation.*

A **finite triangulation** of a surface Σ_g^b is a finite graph consisting of vertices, edges, and faces of triangles. When embedded into Euclidean space, this finite triangulation is homeomorphic to the starting surface. It roughly describes a decomposition of Σ_g^b into finitely many triangles. In this vein, Theorem 1.1 is the statement:

Any finite triangulation of a compact, oriented surface is homeomorphic to one of the model surfaces.

Not unexpectedly, there is a slick, algebraic formula connecting 1.1 to 1.2.

Corollary 1.3. *Endow Σ_g^b with a finite triangulation consisting of V vertices, E edges, and F faces. Then, the Euler characteristic $\chi(\Sigma_g^b)$ satisfies the equality:*

$$\chi(\Sigma_g^b) = V - E + F = 2 - 2g - b.$$

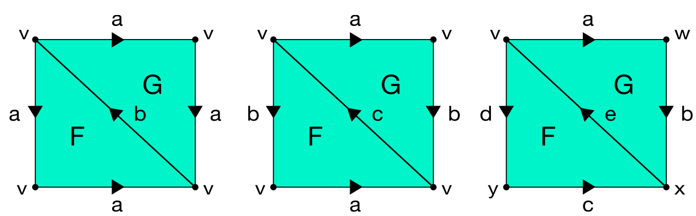

FIGURE 2. Finite triangulations of surfaces pictured from left to right: sphere, torus, disk

2. CURVES

In studying surfaces, we are just as equally interested in studying their **simple closed curves**. In a surface Σ_g^b, a simple closed curve is an embedding $\gamma : S^1 \to \Sigma_g^b$. That is, γ is a non-intersecting, closed loop.

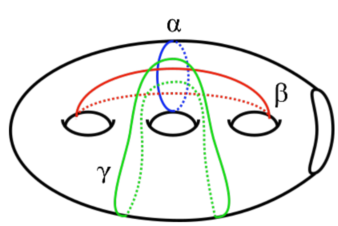

FIGURE 3. Examples of simple closed curves

Despite the name, simple closed curves can be anything but simple. The curves in the above figure are technically simple. (Simple just means that the maps are injective.) However, simple closed curves can range in their properties. To account for the diversity of curves that can occur in a surface, we will need to define some terms. The following definition describes an equivalence relation that we can place on any embedding, including curves.

Definition 2.1. Two embeddings $f : X \to \Sigma_g^b$ and $g : X \to \Sigma_g^b$ are **isotopic** iff f and g related by a homotopy $H : X \times I \to \Sigma_g^b$ so that for each $t \in I$, $H_t : X \times \{t\} \to \Sigma_g^b$ is an embedding. We denote isotopy equivalence by $\simeq$.

Isotopy classes of curves and later, homeomorphisms are key to the theory of mapping class groups. Keep Definition 2.1 in your back-pocket! If this definition is confusing to you, then use the following heuristic description instead:

Two embeddings are isotopic iff one can be "wiggled" into the other.

In what follows, we designate isotopy (wiggle) classes of curves with standard script.

Definition 2.2. For a pair of isotopy classes of simple closed curves $\{a, b\}$, their **geometric intersection** $i(a, b)$ is the minimum number of intersections among all pairs of representatives for the given classes. Simply put,

$$i(a, b) = \min\{|\alpha \cap \beta| : \alpha \in a \text{ and } \beta \in b\}.$$

In Figure 3, the geometric intersection between α and β is 2. Conveniently, the pictured curves already realize their minimum intersection: $i(\alpha, \beta) = |\alpha \cap \beta|$. When this occurs, we say that α and β are in **minimal position**.

There is an easy way to check that two curves are in minimal position. They cannot share tangential intersections or form **bigons**. The neighborhood of a bigon has the following picture:

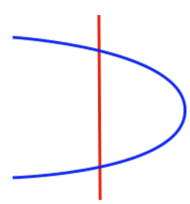

FIGURE 4. Neighborhood of a bigon

Before we move on, we make one final note on curve intersections.

Definition 2.3. For a pair of simple closed curves $\{\alpha, \beta\}$, their **algebraic intersection** $\hat{i}(\alpha, \beta)$ is the sum of the *signed*, transversal intersections between α and β.

The specified orientation on a surface induces signs on all of the intersections between curves. In the case of a surface with a counterclockwise orientation, the neighborhoods of the positive and negative intersections have the following pictures:

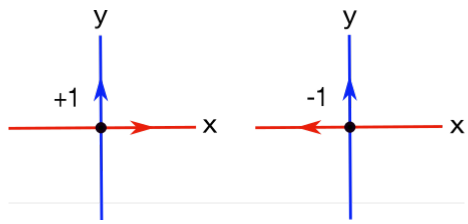

FIGURE 5. Signs induced by a counterclockwise orientation

3. Change of Coordinates Principle

The next definition gives rise to a classification theorem for curves.

Definition 3.1. Let $\beta : S^1 \to \Sigma_g^b$ be a simple closed curve. If we cut Σ_g^b along β, then we obtain a compact, **cut surface** Σ_g^b/β. The cut surface is equipped with a homeomorphism h between two of its boundary components so that

(1) the gluing $\Sigma_g^b/\beta/_{h(x)\sim x}$ is homeomorphic to Σ_g^b, and
(2) the images of these components after gluing is β [3, pg. 36].[3]

Cut surfaces can be connected or disconnected. In Figure 3, the cut surface Σ_g^b/γ has two connected components while Σ_g^b/α and Σ_g^b/β each have only one. We will call the curves that separate surfaces **separating** and the curves that do not separate surfaces, **non-separating**. In fact,

Proposition 3.2. *All non-separating simple closed curves in Σ_g^b are related by a homeomorphism of the surface.*

Proof. Endow Σ_g^b with a finite triangulation. Let $\alpha : S^1 \to \Sigma_g^b$ be any non-separating curve. Up to isotopy, we can assume that α does not bisect any of the faces in the triangulation.

If we cut along α in Σ_g^b, then the number of faces in the finite triangulation does not change. Cutting only doubles the number of edges and vertices along α and adds two boundary components to the surface. Because $\chi(\Sigma_g^b/\alpha) = \chi(\Sigma_g^b)$, the cut surface Σ_g^b/α is homeomorphic to Σ_{g-1}^{b+2} by Corollary 1.3.

Since α was arbitrary, for any other non-separating curve $\beta : S^1 \to \Sigma_g^b$, the cut surfaces Σ_g^b/α and Σ_g^b/β are homeomorphic. Choose a homeomorphism between Σ_g^b/α and Σ_g^b/β that respects the equivalence relations given by Definition 3.1. Such a homeomorphism descends and restricts to a homeomorphism between α to β.[4] □

This result is not just an incarnation of the Classification Theorem for Surfaces [1.1] but of a trick in surface topology known as the Change of Coordinates Principle. In the same way that the theorem classifies surfaces, the trick classifies collections of simple closed curves that share the same "intersection pattern" in a

[3]We slightly abuse notation by using two meanings of the forward slash in the gluing quotient $\Sigma_g^b/\beta/_{h(x)\sim x}$. The first instance describes the cut surface, while the second describes the quotient of the cut surface by the equivalence relation.

[4]Exercise: By a similar argument, we can show that two separating curves in Σ_g^b are related by a homeomorphism of the surface iff their cut surfaces are homeomorphic. Use this argument to classify all separating curves in Σ_g^b.

surface [3, pg. 38]. This classification is up to a homeomorphism of the surface. Thus, up to some homeomorphism, Change of Coordinates can transform any specific collection of curves into a general picture. We end this chapter with one more of the trick's (and the theorem's) incarnations.

Proposition 3.3. *All pairs of simple closed curves* $\{\alpha, \beta\} \subset \Sigma_g^b$ *satisfying* $i(\alpha, \beta) = 1$ *are related by a homeomorphism of the surface.*

Proof. Let $\{\alpha, \beta\}$ be constructed as above. First, we cut along α. (We could have just as easily cut along β. By Change of Coordinates, the order of cutting does not matter.)

The cut surface Σ_g^b/α has two more boundary components and one less handle than Σ_g^b. In addition, β is broken into a single arc connecting the boundary components created by cutting along α.

Cutting along the remaining arc of β does not remove any handles but connects the two α-boundary components. Thus, $\Sigma_g^b/(\alpha \cup \beta)$ is a surface with one more boundary component and one less handle than the original. It is homeomorphic to Σ_{g-1}^{b+1}.

We conclude this argument by the concluding one in Proposition 3.2. Take any other cut surface associated to a pair of curves $\{\alpha', \beta'\}$ constructed as in Proposition 3.3. Then, find a homeomorphism between both cut surfaces that descends and restricts to a homeomorphism between $\{\alpha, \beta\}$ and $\{\alpha', \beta'\}$. □

4. Symmetries

We can improve the previous propositions by requiring that the desired homeomorphisms between curves preserve the specified orientations and restrict to the identity on the boundary components of the starting surfaces. In the associated proofs, we just needed to choose similarly orientation preserving homeomorphisms between the cut surfaces that fix the uncut boundary components pointwise. (These "uncut" boundary components are fixed by the quotients that equip the cut surfaces.) Such homeomorphisms induce homeomorphisms with the same properties between curves.

We can think of the orientation preserving homeomorphisms of a surface that fix the boundary pointwise as the symmetries of the surface. For surfaces without boundary, their symmetries always include rotations.

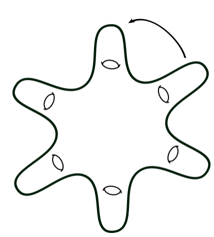

Figure 6. Rotation by $\pi/3$

A more complicated family of symmetries arises for every surface: **Dehn twists** about simple closed curves. We can define a Dehn twist for any simple closed curve in a surface.

Definition 4.1. Let $\alpha : S^1 \to \Sigma_g^b$ be a simple closed curve. Choose a closed regular neighborhood N of α and an orientation preserving homeomorphism $\phi : N \to A$, where $A = S^1 \times [0,1]$ is an annulus. Define the twist homeomorphism $T : A \to A$ by the formula:

$$T(\theta, t) \coloneqq (\theta + 2\pi t, t).$$

Then, the **Dehn twist** about α is defined piecewise by

$$T_\alpha(x) \coloneqq \begin{cases} \phi^{-1} T \phi(x) & x \in N \\ x & \text{otherwise.} \end{cases}$$

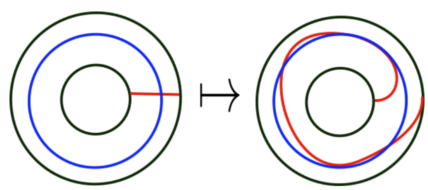

FIGURE 7. The Dehn twist about the boundary curve in an annulus

Our definition for Dehn twists is well-defined up to isotopy. Though we made two choices: a closed regular neighborhood N and an orientation preserving homeomorphism ϕ, both of these choices are unique up to isotopy. Note that regular neighborhoods of isotopic curves are isotopic [7].

In our context, isotopy-equivalence is sufficient for equivalence. The next propositions are based on this notion of equivalence. In Section 6, isotopy-equivalence becomes ubiquitous.

Proposition 4.2. *Let $\alpha : S^1 \to \Sigma_g^b$ and $\alpha' : S^1 \to \Sigma_g^b$ be isotopic simple closed curves. Then, $T_\alpha : \Sigma_g^b \to \Sigma_g^b$ and $T_{\alpha'} : \Sigma_g^b \to \Sigma_g^b$ are isotopic Dehn twists.*

Proof. Once again, regular neighborhoods of isotopic curves are isotopic. □

Proposition 4.3. *Let $\alpha : S^1 \to \Sigma_g^b$ and $\beta : S^1 \to \Sigma_g^b$ be simple closed curves. Let $f : \Sigma_g^b \to \Sigma_g^b$ be a homeomorphism such that $f(\alpha) = \beta$. Then, we have the equivalence*

$$f T_\alpha f^{-1} \simeq T_{f(\alpha)} = T_\beta.$$

Proof. Even better, we have equality

$$f T_\alpha f^{-1} = T_{f(\alpha)}.$$

The homeomorphism f^{-1} maps a closed regular neighborhood of β to a closed regular neighborhood of α. Then, the Dehn twist rotates the α-annulus by 2π before f maps the rotated α-annulus to a rotated β-annulus. Taken together, these steps define the Dehn twist about β. □

5. Curve Surgery

To study the action of Dehn twists on simple closed curves, we define **curve surgery**—a non-medical procedure that depicts this action on curve intersections. Let α and β be two simple closed curves in minimal position in a surface. Then, the Dehn twist of β about α has the following recipe via curve surgery.

Cut α and β at points in their respective images such that the cut or **surgered** arcs consist of one arc from β and $i(\alpha, \beta)$ arcs from α. In addition, the β-arc should intersect the α-arcs in $i(\alpha, \beta)$ points. Up to isotopy, the picture for these surgered arcs is the leftmost diagram in Figure 8.

At each point of intersection between α and β, we operate on the surgered arcs as in the rightmost diagram. That is, we trace a parallel copy of β in the leftmost diagram until we hit a point of intersection. Then, we turn left and follow α. Once we are back at the intersection point, we turn right. If we perform this surgery for all intersections, then from the parallel copy, we obtain $T_\alpha(\beta)$.

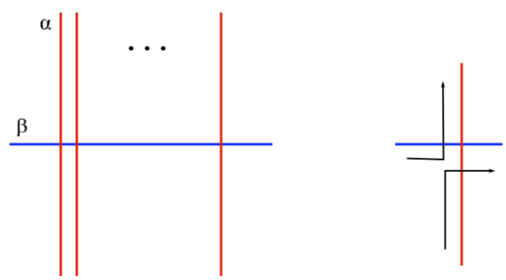

Figure 8. Pictured left to right: the surgered arcs of α and β and the recipe for surgery

From Figure 8, it appears that if $i(\alpha, \beta) = 1$, then $i(\beta, T_\alpha(\beta)) = 1$ as well. However, this supposes that $T_\alpha(\beta)$ and β are in minimal position in a diagram that does *not* depict genus or boundary components. Are these curves in minimal position?

Proposition 5.1. *Let a and b be two isotopy classes of simple closed curves in a surface Σ_g^b. Let n be an integer. Then,*

$$i(T_a^n(b), b) = |n| i(a, b)^2$$

Proof. Choose two representatives $\alpha \in a$ and $\beta \in b$ so that α and β are in minimal position. As in Figure 8, we compute $T_\alpha^n(\beta)$ with curve surgery, starting with the first Dehn twist. Trace a parallel copy of β in the leftmost diagram, performing surgery as in the rightmost whenever β intersects α. Since we perform this surgery $i(a, b)$ times, the resulting curve $T_\alpha(\beta)$ must cross β in $i(a, b)^2$ points in the diagram. After twisting $n-1$ more times, the curves must cross each other in $|n| i(a, b)^2$ points.

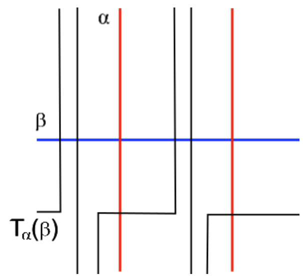

FIGURE 9. The Dehn twist of β about α when $i(a, b) = 2$ and $n = 1$

Though we have assumed that α and β are in minimal position, we should check that $T_\alpha^n(\beta)$ and β are in minimal position in the diagram. Cut β and $T_\alpha^n(\beta)$ at their points of intersection. This produces two cut curves: two collections of arcs originating from β and $T_\alpha^n(\beta)$.

If β and $T_\alpha^n(\beta)$ formed a bigon, then this bigon is the union of an arc taken from the β-collection and an arc taken from the $T_\alpha^n(\beta)$-collection. The arcs must share a trivial algebraic intersection. Thus, the candidate bigon has just one picture up to isotopy. When $i(a, b) = 2$ and $n = 1$, this picture is Figure 10.

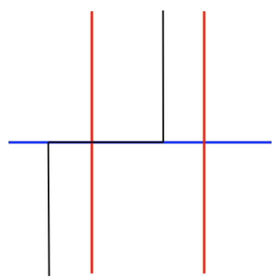

FIGURE 10. The candidate bigon when $i(a, b) = 2$ and $n = 1$

In general, the arc originating from $T_\alpha^n(\beta)$ runs *parallel* to α. If this arc formed a bigon with β, then α and β must also form one—a contradiction. Therefore, no pictures can depict bigons. The curve surgery diagrams must depict $T_\alpha^n(\beta)$ and β in minimal position. □

The next proposition kicks the previous one up a notch. To prove it, we need to assume that for a triple of simple closed curves in a surface, each curve can be put into minimal position with respect to the other two. For a proof of this result, see [3], Lemma 3.3. Technicalities aside, we move onto the main event of this section.

Proposition 5.2. *Let a, b, and c be three isotopy classes of simple closed curves in a surface Σ_g^b. Let n be an integer. Then,*

$$\left| i(T_a^n(b), c) - |n| i(a, b) i(a, c) \right| \leq i(b, c)$$

Proof. With Proposition 5.1 and Lemma 3.3 [3], we fix the following triple of curves. Let $\beta \in b$, $\beta' \in T_a(b)$, and $\gamma \in c$ be representatives such that they are in minimal position with respect to each other. We also assume that γ does not intersect $\beta \cap \beta'$ by applying an isotopy.

There is a continuous map from the disjoint union of $|n|i(a,b)$ copies of S^1 into Σ_g^b with image $\beta \cup \beta'$. The image of each copy of S^1 is isotopic to a representative $\alpha_i \in a$. In addition, any α_i will intersect γ in at least $i(a,c)$ points. Thus, we derive the inequality

$$|n|i(a,b)i(a,c) \leq |(\beta \cup \beta') \cap \gamma| = i(T_a^n(b), c) + i(b,c).$$

We now find representatives for $T_a^n(b)$ and c that intersect in $|n|i(a,b)i(a,c)+i(b,c)$ points. If they exist, then we can prove the reverse inequality of the absolute value:

$$i(T_a^n(b), c) \leq |n|i(a,b)i(a,c) + i(b,c).$$

Choose β' so that it lies in the union of β and small, regular neighborhoods of the images of the disjoint copies of S^1. Choose γ so that it transversally intersects each of these neighborhoods in $i(a,c)$ points and β in $i(b,c)$ points *disjoint* from these neighborhoods. Finally, count the intersections to prove that β' and γ are the desired representatives. □

Unravelling these propositions,

Corollary 5.3. *Dehn twists can have infinite order.*

6. The Group of Symmetries

With the next definition, we change perspectives from topology to group theory.

Definition 6.1. Let $\mathrm{Homeo}^+(\Sigma_g^b, \partial\Sigma_g^b)$ be the set of orientation preserving homeomorphisms of Σ_g^b that fix the boundary pointwise. Let $\simeq$ denote the isotopy equivalence relation from Definition 2.1. Then, the quotient

$$\mathrm{Homeo}^+(\Sigma_g^b, \partial\Sigma_g^b)/_{\simeq}$$

forms a group under functional composition. We take this group to be the **mapping class group** $\mathrm{MCG}(\Sigma_g^b)$ of a surface Σ_g^b and call its elements **mapping classes**.

As group theorists, we might ask: Which mapping classes generate the mapping class group? Take Corollary 5.3 as a clue.

Theorem 6.2 (Dehn-Lickorish Twist Theorem [2])**.** *The mapping class group of a surface is finitely generated by Dehn twists about simple closed curves.*

Dehn twists are the natural candidate. Mapping class groups can be big and Dehn twists can have infinite order as mapping classes. The many proofs of Theorem 6.2 are also big but in a different sense. For brevity, we provide the bases cases of an inductive proof and direct you to [11] for the inductive step.

Proposition 6.3. *The mapping class group of a disk is trivial.*

Proof. Let D^2 be the closed unit disk. Let $f : D^2 \to D^2$ be an orientation-preserving homeomorphism that fixes the boundary pointwise. Then, f is isotopic to the identity via **Alexander's trick**:

$$F(x,t) := \begin{cases} (1-t)f(\frac{x}{1-t}) & 0 \leq |x| < 1-t \\ x & 1-t \leq |x| \leq 1. \end{cases}$$

□

Proposition 6.4. *The mapping class group of a sphere is also trivial.*

Proof. Identify the sphere S^2 with the image of the natural projection $\pi : D^2 \to D^2/_{\partial D^2}$. Fix a point $x \in D^2/_{\partial D^2}$. Let $f : D^2/_{\partial D^2} \to D^2/_{\partial D^2}$ be an orientation preserving homeomorphism (that fixes the boundary vacuously). Up to isotopy, we can assume that $f(x) = \pi(\partial D^2) = x$.

There is a second orientation preserving function $\hat{f} : D^2 \to D^2$ that makes the following diagram commute.

$$\begin{tikzcd} D^2 \arrow[r, "\hat{f}"] \arrow[d, "\pi"] & D^2 \arrow[d, "\pi"] \\ D^2/_{\partial D^2} \arrow[r, "f"] & D^2/_{\partial D^2} \end{tikzcd}$$

The function $\hat{f}$ is defined by $\hat{f} \restriction_{D^2 - \partial D^2} := f$ and $\hat{f} \restriction_{\partial D^2} := \mathrm{id}$. It is not difficult to check that $\hat{f}$ is a homeomorphism or that it is an element of $\mathrm{MCG}(D^2) \cong 0$. (A proof of continuity with open sets should suffice.) Thus, if $F : D^2 \times [0, 1] \to D^2$ is an isotopy between $\hat{f}$ and the identity, then F descends uniquely to an isotopy between f and the identity. $\square$

7. Representations

We have defined surfaces, curves in surfaces, symmetries specified by curves, and a group of symmetries. With all of this theory, we can finally define our main objects of study: **representations** of the mapping class group.

Definition 7.1. A **representation** of the mapping class group is a group homomorphism from this group to a group of matrices.

Representations greatly simplify our analysis of mapping class groups. Much of linear algebra has already been discovered! In this research, we are concerned with two such representations: the **symplectic representation** and the **Magnus representation**.

7.1. The Symplectic Representation.

The symplectic representation of $\mathrm{MCG}(\Sigma_g^b)$ is the action of mapping classes on a symplectic vector space over $\mathbb{Z}$. This vector space is maximally generated by the following curves in Σ_g^b.

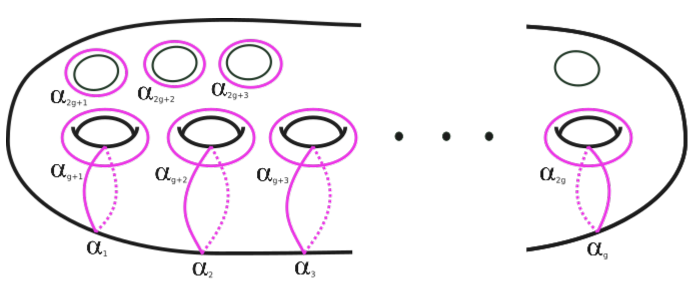

FIGURE 11. $2g + b - 1$ simple closed curves in Σ_g^b

The basis in Figure 11 is a basis for first homology. Thus, we re-express our initial description in terms of this new vocabulary.

Definition 7.2. Let $f : \Sigma_g^b \to \Sigma_g^b$ be a mapping class and $f_\star : H_1(\Sigma_g^b;\mathbb{Z}) \to H_1(\Sigma_g^b;\mathbb{Z})$ be the induced map on first homology. Choose the basis $\{\alpha_i\}$ from Figure 11. Then, the **symplectic representation** of $\mathrm{MCG}(\Sigma_g^b)$ is a group homomorphism

$$s : \mathrm{MCG}(\Sigma_g^b) \to \mathrm{Sp}_{|\alpha_i|}(\mathbb{Z})$$

that sends a mapping class $f : \Sigma_g^1 \to \Sigma_g^1$ to a matrix whose i^{th} column corresponds to $f_\star(\alpha_i)$.

The mapping class group of a torus is isomorphic to $\mathrm{Sp}_2(\mathbb{Z})$. Naturally, its symplectic representation is a group *iso*morphism. Symplectic representations with more interesting kernels correspond to surfaces of higher genus. Nevertheless, whether they are interesting or empty, the kernels of these representations are objects of significant mathematical study.

Definition 7.3. The **Torelli group** $\mathcal{I}(\Sigma_g^b)$ of a surface Σ_g^b is the kernel of the symplectic representation of $\mathrm{MCG}(\Sigma_g^b)$. We call its elements **Torelli classes**.

In the 1970's, Birman and Powell [9] found a generating set for $\mathcal{I}(\Sigma_g^b)$. This set consists of the following kinds of homeomorphisms.

(1) **Separating twists.** This notation is shorthand for Dehn twists about separating curves.
(2) **Bounding pair maps.** Bounding pair maps are composites of Dehn twists $T_\alpha T_\beta^{-1}$ whose associated curves form a **bounding pair**. In other words, $\{\alpha, \beta\}$ is a disjoint, homologous, and non-isotopic pair of simple closed curves.

Three decades later, Putman [10] constructed an infinite presentation, adding one more family of homeomorphisms to Birman and Powell's set.

(3) **Commutators of simply intersecting pairs.** Commutators of simply intersecting pairs are commutators of Dehn twists $[T_\alpha, T_\beta]$ whose associated curves **simply intersect**. In this case, $\{\alpha, \beta\}$ share a trivial algebraic intersection.

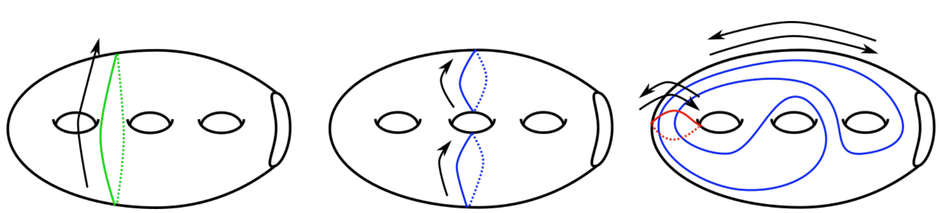

FIGURE 12. Pictured from left to right: separating twist, bounding pair map, and commutator of a simply intersecting pair

7.2. **The Magnus Representation.** The Magnus representation of the mapping class group is defined for surfaces with one boundary component. It is an action of mapping classes on a free group that is maximally generated by the following *based* curves in Σ_g^1.

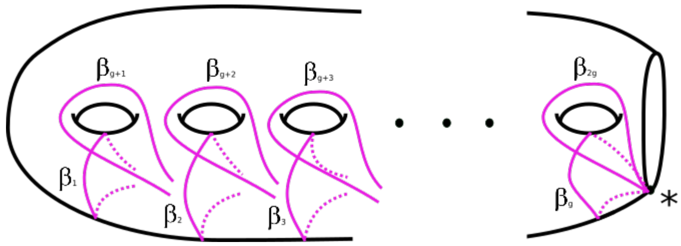

FIGURE 13. $2g$ simple closed curves based at $*$

We can define the Magnus representation with the following formula from Fox calculus [4].

Definition 7.4. Given a free group G with generators $\{\beta_i\}$, the **Fox derivative** with respect to β_i is a function

$$\frac{\partial}{\partial \beta_i} : G \to \mathbb{Z}[G].$$

Here, $\mathbb{Z}[G]$ is the free module over $\mathbb{Z}$ with basis G.[5] We first define this derivative on the generators of G:

$$\frac{\partial \beta_i}{\partial \beta_j} := \begin{cases} 1 & i = j \\ 0 & \text{otherwise.} \end{cases}$$

After setting the group multiplication to concatenation, we extend to all of G with two more rules.

(1) **Product Rule.** $\forall a, b \in G$:

$$\frac{\partial (ab)}{\partial \beta_i} := \frac{\partial a}{\partial \beta_i} + a \frac{\partial b}{\partial \beta_i}$$

(2) **Reciprocal Rule.**

$$\frac{\partial \beta_i^{-1}}{\partial \beta_j} := \begin{cases} -\beta_i^{-1} & i = j \\ 0 & \text{otherwise} \end{cases}$$

Like the analytic derivative, we can define the Fox derivative on the automorphisms of G. If G is $\pi_1(\Sigma_g^1, -)$, the fundamental group of Σ_g^1, then we can even define this derivative on the automorphisms induced by mapping classes. Note that Figure 13 depicts a possible basepoint and generators for $\pi_1(\Sigma_g^1, -)$.

Definition 7.5. Let $f : \Sigma_g^1 \to \Sigma_g^1$ be a mapping class. Choose the basepoint $*$ on the boundary and the generators $\{\beta_i\}$ in Figure 13. Let $f_* : \pi_1(\Sigma_g^1, *) \to \pi_1(\Sigma_g^1, *)$ be the induced map on fundamental groups. Then, the Fox derivative of f is a matrix with entries:

$$\left[\frac{\partial f_*(\beta_i)}{\partial \beta_j}\right]_{ij}.$$

Out of ease, we set $\Gamma = \pi_1(\Sigma_g^1, *)$ and $H = H_1(\Sigma_g^1; \mathbb{Z})$. Putting definitions and notation together,

[5]That is, $\mathbb{Z}[G]$ consists of polynomials of elements in G with integer coefficients.

Definition 7.6. The **Magnus representation** of $\mathrm{MCG}(\Sigma_g^1)$ is a function

$$r : \mathrm{MCG}(\Sigma_g^1) \to \mathrm{GL}_{2g}(\mathbb{Z}[\Gamma])$$

that sends a mapping class $f : \Sigma_g^1 \to \Sigma_g^1$ to a matrix with entries:

$$\left[\overline{\frac{\partial f_*(\beta_j)}{\partial \beta_i}}\right]_{ij}.$$

Here, $\overline{(\ )} : \mathbb{Z}[\Gamma] \to \mathbb{Z}[\Gamma]$ denotes the anti-automorphism, sending β_i to β_i^{-1}.

The Magnus representation of the mapping class group is *not* a genuine representation. This representation satisfies a "crossed" formula that was initially described by Morita [8].

Proposition 7.7. *Let $f : \Sigma_g^1 \to \Sigma_g^1$ and $g : \Sigma_g^1 \to \Sigma_g^1$ be mapping classes. Then,*

$$r(fg) = r(f)^{f_*}r(g),$$

where ${}^{f_*}r(g)$ is the matrix obtained from $r(g)$ by applying f_* to each entry. We extend the induced map linearly if necessary.[6]

However, r can be modified to be a genuine representation on one distinguished subgroup: the Torelli group of Σ_g^1.

Definition 7.8. The **Magnus representation of the Torelli group** of Σ_g^1 is a group homomorphism

$$r_1 : \mathcal{I}(\Sigma_g^1) \to \mathrm{GL}_{2g}(\mathbb{Z}[H])$$

that sends a Torelli class $f : \Sigma_g^1 \to \Sigma_g^1$ to a matrix

$$r_1(f) \coloneqq^{\mathfrak{a}} r(f).$$

Here, $\mathfrak{a} : \Gamma \to H$ is the abelianization of Γ. Thus, ${}^{\mathfrak{a}}r(f)$ is the matrix obtained from $r(f)$ by applying $\mathfrak{a}$ to each entry.

8. The Specific Case

Armed with a computational interpretation of r_1, we can prove a preliminary result. This result is a specific case of our main theorem. We choose α and β to be the curves embedded into the subsurface of Σ_g^1 that has three handles and one boundary component. This surface and these curves are depicted in Figure 14. With α and β fixed as in the figure, we prove that $[T_\alpha, T_\beta]$ is not in the kernel of r_1.

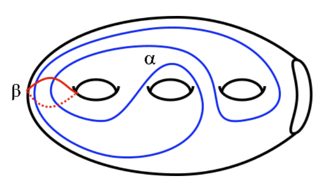

FIGURE 14. The fixed simple closed curves α and β in a subsurface of Σ_g^1

[6]Remember that we defined f_* on the free non-abelian group Γ, not the free module $\mathbb{Z}[\Gamma]$. To fix this, we just extend f_* linearly: Given $a, b \in \Gamma$ and $n \in \mathbb{Z}$, set $f_*(a + nb) \coloneqq f_*(a) + nf_*(b)$.

Theorem 8.1. *Let $\{\alpha, \beta\}$ be constructed as above. Then, $[T_\alpha, T_\beta] : \Sigma_g^1 \to \Sigma_g^1$ is a commutator of a simply intersecting pair that is not in the kernel of r_1.*

Proof. Specifically, we prove that $r_1([T_\alpha, T_\beta])_{12} \neq 0$. We compute $r_1([T_\alpha, T_\beta])_{12}$ with the recipe outlined in Subsection 7.2. To start, we twist β_2.

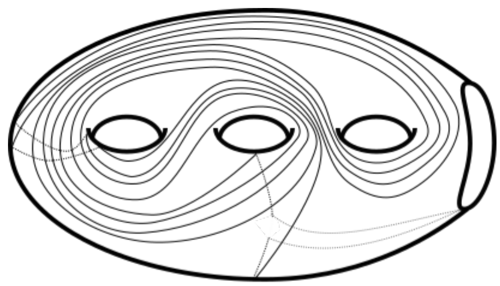

FIGURE 15. The image of β_2 under $[T_\alpha, T_\beta]$

The image of β_2 under $[T_\alpha, T_\beta]$ is the following seventeen letter word in $\pi_1(\Sigma_g^1, *)$:

$$\beta_2\beta_5^{-1}\beta_1\beta_6\beta_4^{-1}\beta_5\beta_4\beta_6^{-1}\beta_4^{-1}\beta_5^{-1}\beta_4\beta_6^{-1}\beta_1^{-1}\beta_5\beta_4\beta_6\beta_4^{-1}.$$

We evaluate the Fox derivative of $[T_\alpha, T_\beta]$ with respect to β_1 at this word. Set $T = [T_\alpha, T_\beta]$. Denote the i through j letters of $T_*(\beta_2)$ by B_i^j. Then, we write

$$\begin{aligned}\frac{\partial T_*(\beta_2)}{\partial \beta_1} &= \frac{\partial \beta_2}{\partial \beta_1} + \beta_2 \frac{\partial B_2^{17}}{\partial \beta_1} \\ &= \beta_2 \frac{\partial B_2^{17}}{\partial \beta_1}.\end{aligned}$$

More applications of Fox derivatives yield

$$= \beta_2(\beta_5^{-1}(1 + B_3^{12}(-\beta_1^{-1} + B_{13}^{16}\frac{\partial \beta_4^{-1}}{\partial \beta_1}))),$$

which simplifies to a non-zero element in $\mathbb{Z}[H]$:

$$= \beta_2\beta_5^{-1} - \beta_2\beta_5^{-1}\beta_6^{-1}.$$

Applying the anti-automorphism, we obtain

$$\begin{aligned}\overline{\beta_2\beta_5^{-1} - \beta_2\beta_5^{-1}\beta_6^{-1}} &= \beta_2^{-1}\beta_5 - \beta_2^{-1}\beta_5\beta_6 \\ &\neq 0.\end{aligned}$$

□

Second Note to the Reader

Depending on your mathematical background, you could consider treating this last section as the conclusion of the paper. The penultimate three sections are more nuanced and thus, less accessible than the previous. For brevity, we assume that the reader is familiar with intermediate constructions of first homology, fundamental groups, and covering spaces.

9. The Torelli Group, Revisited

With some additional facts from algebraic topology, we can prove that separating twists, bounding pair maps, and commutators of simply intersecting pairs are all elements in $\mathcal{I}(\Sigma_g^b)$. In what follows, we designate homology classes of simple closed curves with square brackets.

Fact 9.1. *The algebraic intersection*

$$\hat{i}(-,-) : H_1(\Sigma_g^b;\mathbb{Z}) \times H_1(\Sigma_g^b;\mathbb{Z}) \to \mathbb{Z}$$

is a symplectic bilinear form on first homology.

Remember that we called the first homology of a surface a *symplectic* vector space. Indeed, it has a symplectic bilinear form: the algebraic intersection. With this form, we can express the second fact.

Fact 9.2. *Let $\alpha : S^1 \to \Sigma_g^b$ be a simple closed curve and $T_{\alpha\star} : H_1(\Sigma_g^b;\mathbb{Z}) \to H_1(\Sigma_g^b;\mathbb{Z})$ be the induced twist on first homology. Then, $T_{\alpha\star}$ is defined by*

$$T_{\alpha\star}(-) = [-] + \hat{i}(\alpha,-)[\alpha].$$

The third and final fact follows from the definition of first homology. Boundaries are null-homologous.

Fact 9.3. *A simple closed curve $\alpha : S^1 \to \Sigma_g^b$ is separating iff α is null-homologous.*

The curves that separate surfaces are inherently boundaries! They bound embedded submanifolds in their respective surfaces. Putting these facts together,

Proposition 9.4. *The following are elements in $\mathcal{I}(\Sigma_g^1)$: separating twists, bounding pairs maps, and commutators of simply intersecting pairs.*

Proof. The proof of each case in the proposition is the application of at least two of the three preceding facts. For brevity, we only check the case of commutators of simply intersecting pairs.

Let $[T_\alpha, T_\beta] : \Sigma_g^1 \to \Sigma_g^1$ be a commutator of a simply intersecting pair. By Proposition 4.3,

$$T_\beta T_\alpha^{-1} T_\beta^{-1} = T_{T_\beta(\alpha)}^{-1},$$

and by Fact 9.2,

$$T_{\beta_\star}(\alpha) = [\alpha] + \hat{i}(\alpha,\beta)[\beta] = [\alpha].$$

A corollary to the second fact is that Dehn twists about homologous curves induce the same group homomorphisms on first homology. So after choosing α as our homology representative for $T_{\beta_\star}(\alpha)$, we obtain

$$[T_\alpha, T_\beta]_\star = (T_\alpha T_{T_\beta(\alpha)}^{-1})_\star = T_{\alpha\star} {T_\alpha^{-1}}_\star = \mathrm{id}_\star.$$

□

10. The Magnus Representation, Revisited

In 2003, Suzuki [13] gave an original interpretation of the Magnus representation of the Torelli group. His interpretation avoids the computational hairiness of taking the Fox derivative, but it is deep into the weeds of abstract mathematics. It uses the **universal abelian covering space** of Σ_g^1.

Definition 10.1. A path-connected covering of Σ_g^1 is **abelian** if it is normal and has an abelian deck transformation group.

By the classification theorem for covering spaces, subgroups of the fundamental group $\pi_1(\Sigma_g^1, -)$ determine path-connected coverings of Σ_g^1. In particular, normal subgroups $N \subset \pi_1(\Sigma_g^1, -)$ determine normal covering spaces whose deck transformation groups are precisely the quotients $\pi_1(\Sigma_g^1, -)/_N$. In this way,

Definition 10.2. The **universal abelian covering space** of Σ_g^1 is the abelian path-connected covering $\overline{p} : \overline{\Sigma} \to \Sigma_g^1$ that corresponds to the commutator subgroup of $\pi_1(\Sigma_g^1, -)$. Necessarily, its deck transformation group is the abelianization of $\pi_1(\Sigma_g^1, -)$: the first homology group $H_1(\Sigma_g^1; \mathbb{Z})$.

Suzuki re-expressed r_1 as an action of Torelli classes on a vector space that can be associated to the universal abelian covering space. This vector space is the **relative homology group** $H_1(\overline{\Sigma}, \overline{p}^{-1}(*); \mathbb{Z})$, where $*$ is a basepoint on the boundary of Σ_g^1. Recall that elements in $H_1(\overline{\Sigma}, \overline{p}^{-1}(*); \mathbb{Z})$ are represented by 1-chains or edges whose endpoints lie in $\overline{p}^{-1}(*)$. In addition, we say that an edge $x \in H_1(\overline{\Sigma}, \overline{p}^{-1}(*); \mathbb{Z})$ is trivial iff it forms a boundary: $x = \partial y$ for some $y \in H_2(\overline{\Sigma}; \mathbb{Z})$.

To realize this action on $H_1(\overline{\Sigma}, \overline{p}^{-1}(*); \mathbb{Z})$ as the representation

$$r_1 : \mathcal{I}(\Sigma_g^1) \to \mathrm{GL}_{2g}(\mathbb{Z}[H]),$$

we need a correspondence between automorphisms of the relative homology group and automorphisms of the free $\mathbb{Z}[H]$-module of rank $2g$: $\mathbb{Z}[H]^{2g}$.[7] We claim that a bijective correspondence exists. This correspondence is rigorously detailed by Church and Pixton in [1]. We briefly outline their exposition, before proceeding with Suzuki's interpretation.

Loosely speaking, the bijection maps any automorphism of $H_1(\overline{\Sigma}, \overline{p}^{-1}(*); \mathbb{Z})$ to its action on an unnatural $\mathbb{Z}[H]$-module basis for the relative homology group. This basis consists of the lifts of an unnatural choice of generators for the fundamental group of Σ_g^1. Thus, the action is expressed by a matrix whose $2g$ columns correspond to the images of these lifts under the automorphism.

More precisely: Let $\{\beta_i\}$ be the generating set for $\pi_1(\Sigma_g^1, *)$ that is depicted in Figure 13. Choose a lift $\overline{*} \in \overline{p}^{-1}(*)$ of the basepoint. Note that each β_i lifts uniquely to an edge $\overline{\beta}_i \subset \overline{\Sigma}$ that begins at the point $\overline{*}$ and ends at a different point in $\overline{p}^{-1}(*)$. Thus, each $\overline{\beta}_i$ defines a nontrival cycle in $H_1(\overline{\Sigma}, \overline{p}^{-1}(*); \mathbb{Z})$. In particular, these lifts constitute the unnatural $\mathbb{Z}[H]$-module basis described above. Every automorphism $F : H_1(\overline{\Sigma}, \overline{p}^{-1}(*); \mathbb{Z}) \to H_1(\overline{\Sigma}, \overline{p}^{-1}(*); \mathbb{Z})$ determines an invertible matrix whose i^{th} column corresponds to $F(\overline{\beta}_i)$.

10.1. Suzuki's Interpretation.

Let $f : \Sigma_g^1 \to \Sigma_g^1$ be a Torelli class. Choose a basepoint $* \in \partial\Sigma_g^1$ and a corresponding lift $\overline{*} \in \overline{p}^{-1}(*)$. Recall that f **lifts** to a homeomorphism $\overline{f} : \overline{\Sigma} \to \overline{\Sigma}$ iff $f\overline{p} = \overline{p}\overline{f}$. This lift is unique if we require that it fixes $\overline{*}$.

For Suzuki's interpretation, we use an equivalent lifting criterion for homeomorphisms. The homeomorphism f lifts to a homeomorphism $\overline{f} : \overline{\Sigma} \to \overline{\Sigma}$ iff $f_*\overline{p}_*(\pi_1(\overline{\Sigma}, -)) = \overline{p}_*(\pi_1(\overline{\Sigma}, -))$. Conveniently, this lifting criterion is automatic.

[7] Note that this treatment of $\mathbb{Z}[H]$ is different from that in previous sections. In Section 7, we introduced it as the free $\mathbb{Z}$-module with basis H. In this section, $\mathbb{Z}[H]$ is a module over itself.

Since the commutator subgroup of $\pi_1(\Sigma_g^1, -)$ is characteristic, every homeomorphism f lifts. As in the first lifting criterion, we define $\overline{f}$ to be the unique lift of f that fixes $\overline{*}$.

The lifted homeomorphism induces an automorphism of $H_1(\overline{\Sigma}, \overline{p}^{-1}(*); \mathbb{Z})$:

$$\overline{f}_\sharp : H_1(\overline{\Sigma}, \overline{p}^{-1}(*); \mathbb{Z}) \to H_1(\overline{\Sigma}, \overline{p}^{-1}(*); \mathbb{Z}),$$

which we take to be an automorphism of $\mathbb{Z}[H]^{2g}$ [1, pg. 180]. Thus, we can identify $\overline{f}_\sharp$ with a matrix F in $\mathrm{GL}_{2g}(\mathbb{Z}[H])$. According to Suzuki [13], an automorphism of this matrix is the representation $r_1(f)$. The homomorphism r_1 maps f to the matrix $\overline{F^t}$, where t is the transpose and $\overline{(\ )}$ is the anti-automorphism from Definition 7.6.

In this paper, we use a slightly edited version of r_1 (as did Church and Pixton in [1]). We define $r_1(f)$ to simply be F. This simplification does not affect the validity of our final result, but it allows us to treat r_1 as *the* action of Torelli classes on $H_1(\overline{\Sigma}, \overline{p}^{-1}(*); \mathbb{Z})$.

10.2. **An Intermediate Abelian Shortcut.** The universal abelian covering space is "universal" in the following sense. It covers every abelian covering of Σ_g^1. Put precisely,

Theorem 10.3. *Let $\widetilde{p} : \widetilde{\Sigma} \to \Sigma_g^1$ be an abelian covering space. Fix three basepoints $* \in \Sigma_g^1$, $\widetilde{*} \in \widetilde{p}^{-1}(*)$, and $\overline{*} \in \overline{p}^{-1}(*)$. Then, there is a unique abelian covering $q : \overline{\Sigma} \to \widetilde{\Sigma}$ such that $\overline{p} = \widetilde{p}q$ and $q(\overline{*}) = \widetilde{*}$.*

With this universal property, we have an intermediate abelian shortcut to prove our main result. To prove Theorem 12.5, it is enough to prove that the distinguished family of commutators does not act by the identity on the first homology of an intermediate abelian covering space.

11. Covering Modulo Two

Our goal for this section is to construct and study the intermediate abelian covering space that will be key to the proof of Theorem 12.5.

11.1. **Construction.** Let $\widetilde{\Sigma}$ be a compact, oriented surface of genus $2g-1$ with two boundary components. When $g = 3$, we depict it as in Figure 16. However, Figure 16 can be realized as a subsurface of $\widetilde{\Sigma}$ for higher g.

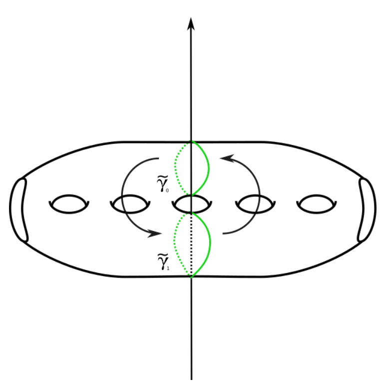

Figure 16. $\widetilde{\Sigma}$ when $g = 3$

The surface $\widetilde{\Sigma}$ has order two rotational symmetry. The symmetry is generated by a rotational angle of π about the axis through the central hole that does not intersect the surface. Thus, the cyclic group $\mathbb{Z}_2$ acts on $\widetilde{\Sigma}$ by the aforementioned rotation. This action is **free** and **properly discontinuous**.

Definition 11.1. An action of a group G on a topological space X is **free** and **properly discontinuous** iff each $x \in X$ has a neighborhood U such that the images $g(U)$ for varying $g \in G$ are disjoint.

Free and properly discontinuous group actions on topological spaces give rise to normal covering spaces. These covering spaces are the quotients of the actions. Thus, the group action of $\mathbb{Z}_2$ on $\widetilde{\Sigma}$ gives rise to a normal covering space $\widetilde{p} : \widetilde{\Sigma} \to \widetilde{\Sigma}/\mathbb{Z}_2$. Note that $\widetilde{p}$ is abelian because its deck transformation group is precisely $\mathbb{Z}_2$. It remains to prove that the quotient space $\widetilde{\Sigma}/\mathbb{Z}_2$ is homeomorphic to Σ_g^1.

We form the quotient by first cutting along the bounding pair in $\widetilde{\Sigma}$ whose associated curves intersect the central hole and are related by the pertinent π-rotation. When $g = 3$, this pair is $\{\widetilde{\gamma}_0, \widetilde{\gamma}_1\}$ in Figure 16. Cutting produces two disconnected subsurfaces, each of genus $g-1$ with three boundary components. These subsurfaces are the closures of the sheets of the covering map $\widetilde{p}$. Not only are these subsurfaces (or the closures of the sheets) related by the deck transformation, but they each have two boundary components that are related in this way as well. Thus, if we glue together the distinguished boundary components in one of the subsurfaces, then from this subsurface, we obtain the quotient space $\widetilde{\Sigma}/\mathbb{Z}_2$. In particular, we recover a surface of genus g with one boundary component.

11.2. **Lifting Modulo Two.** The following proposition characterizes any simple closed curve that lifts to the intermediate abelian covering space.

Proposition 11.2. *Let $\alpha : S^1 \to \Sigma_g^1$ be a simple closed curve. Let $\{\widetilde{\gamma}_0, \widetilde{\gamma}_1\}$ be the bounding pair in $\widetilde{\Sigma}$ defined in the previous section. Then, α lifts to a simple closed curve $\widetilde{\alpha} : S^1 \to \widetilde{\Sigma}$ iff $\hat{i}(\alpha, \gamma) \equiv 0 \bmod 2$, where $\gamma = \widetilde{p}(\widetilde{\gamma}_i)$.*

We have modified the lifting criterion in Proposition 11.2 from Ghaswala and Winarski's lifting criterion in [5], Lemma 3.4. The following results are even motivated by their exposition.

To prove the first direction of our lifting criterion, we use a combinatorial construction. We have illustrated this construction on the subsurface of $\widetilde{\Sigma}$ that is depicted in Figure 16.

Lemma 11.3. *Consider the following, weighted digraph G that is drawn on the subsurface of $\widetilde{\Sigma}$ described above.*

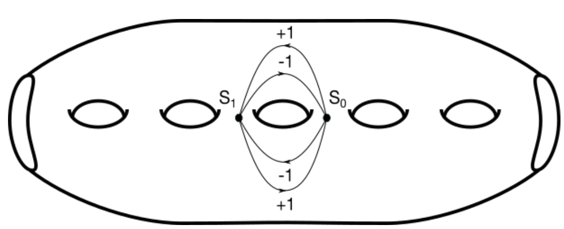

FIGURE 17. Weighted digraph on $\widetilde{\Sigma}$

Let F be a finite walk in G. Let $w(F)$ be the sum of the weights of the edges traversed in the walk. Then, F begins and ends at the same vertex iff $w(F) \equiv 0 \bmod 2$.

Proof. Count. ☐

For the backwards direction, we need a special group homomorphism. This mapping's existence and uniqueness is ensured by the following result.

Lemma 11.4. *Let $\mathfrak{a} : \pi_1(\Sigma_g^1, -) \to H_1(\Sigma_g^1; \mathbb{Z})$ be the abelianization of $\pi_1(\Sigma_g^1, -)$. Let $\varphi : \pi_1(\Sigma_g^1, -) \to \mathbb{Z}_2$ be the quotient by the subgroup $\widetilde{p}_*(\pi_1(\widetilde{\Sigma}, -))$. Up to a change in basepoint, there is a unique group homomorphism $\varphi_{\mathfrak{a}} : H_1(\Sigma_g^1; \mathbb{Z}) \to \mathbb{Z}_2$ satisfying $\varphi = \varphi_{\mathfrak{a}}\mathfrak{a}$.*

Proof. In the background of the above lemma is the universal property of quotient maps. We have applied this property before in Propositions 3.2, 3.3, and 6.4. We choose to name it now only because its present application is more subtle.

For the universal property to guarantee the existence and uniqueness of $\varphi_{\mathfrak{a}}$, the kernel of $\mathfrak{a}$ must be a subgroup of the kernel of φ. However, this is not too difficult to prove (provided we recall some facts from algebraic topology). In particular, let $\overline{p} : \overline{\Sigma} \to \Sigma_g^1$ be the universal abelian covering space from Section 10, which corresponds to the commutator subgroup of $\pi_1(\Sigma_g^1, -)$.

By definition, the image of $\overline{p}$ is equal to the kernel of $\mathfrak{a}$. Since covering maps induce inclusions on fundamental groups and since $\overline{p}$ is universal [10.3], we obtain:

$$\ker \mathfrak{a} = \overline{p}_*(\pi_1(\overline{\Sigma}, -)) \subset \widetilde{p}_*(\pi_1(\widetilde{\Sigma}, -)) = \ker \varphi.$$

Thus, $\varphi_{\mathfrak{a}}$ exists and is unique. ☐

We can give $\varphi_{\mathfrak{a}}$ a hands-on description. It maps all homology classes of curves that lift to $\widetilde{\Sigma}$ to *zero*. The kernel of $\varphi_{\mathfrak{a}}$ is a preliminary lifting criterion for curves.

Lemma 11.5. *Let $\alpha : S^1 \to \Sigma_g^1$ be a simple closed curve. Then, α lifts to $\widetilde{\Sigma}$ iff $[\alpha] \in \ker \varphi_{\mathfrak{a}}$.*

Proof. Fix a basepoint $* \in \Sigma_g^1$ and a corresponding lift $\widetilde{*} \in \widetilde{p}^{-1}(*)$ in Lemma 11.4. Let $\alpha : S^1 \to \Sigma_g^1$ be a simple closed curve based at $*$. Then, the homomorphism $\varphi_{\mathfrak{a}}$ is uniquely defined by $\varphi_{\mathfrak{a}}([\alpha]) \coloneqq \varphi(\alpha)$. It follows that $[\alpha] \in \ker \varphi_{\mathfrak{a}}$ iff $\alpha \in \widetilde{p}_*(\pi_1(\widetilde{\Sigma}, \widetilde{*}))$.

The preceding biconditional is secretly Lemma 11.5. By a well-known lifting criterion, α lifts to $\widetilde{\Sigma}$ iff $\alpha \in \widetilde{p}_*(\pi_1(\widetilde{\Sigma}, \widetilde{*}))$. ☐

Equipped with the previous lemmas, homomorphisms, and pictures, we can finally prove Proposition 11.2.

Proof of Proposition 11.2. ($\implies$) Fix a basepoint $* \in \Sigma_g^1$ and a corresponding lift $\widetilde{*} \in \widetilde{p}^{-1}(*)$. Let $\alpha : S^1 \to \Sigma_g^1$ be a simple closed curve based at $*$ that lifts.

Because $\widetilde{p}$ is a two-sheeted covering space, $\widetilde{p}^{-1}(\alpha)$ consists of two, disjoint simple closed curves: $\widetilde{\alpha}_0 : S^1 \to \widetilde{\Sigma}$ and $\widetilde{\alpha}_1 : S^1 \to \widetilde{\Sigma}$. Let $\widetilde{\alpha}_0$ be the curve beginning at $\widetilde{*}$. Up to isotopy, we can assume that the multicurves $\widetilde{\alpha} = \widetilde{\alpha}_0 \cup \widetilde{\alpha}_1$ and $\widetilde{\gamma} = \widetilde{\gamma}_0 \cup \widetilde{\gamma}_1$ are transverse.

We consider the action of $\mathbb{Z}_2$ on the intersection of these multicurves: $\widetilde{\alpha} \cap \widetilde{\gamma}$. This action is transitive. It transposes the points in $\widetilde{\alpha}_0 \cap \widetilde{\gamma}$ with the points in $\widetilde{\alpha}_1 \cap \widetilde{\gamma}$, even preserving their signs of intersection. Thus, all components of $\widetilde{\alpha}$ have the same algebraic and geometric intersections with $\widetilde{\gamma}$. What is more: $\hat{i}(\widetilde{\alpha}_i, \widetilde{\gamma}) = \hat{i}(\alpha, \gamma)$ for any $i \in \{0, 1\}$.

Let $\widetilde{S}_0$ and $\widetilde{S}_1$ be the two sheets of $\widetilde{p}$ such that $\widetilde{S}_0$ contains the basepoint $\widetilde{*}$. If G is the weighted diagraph from Lemma 11.3, then we let its vertices S_i correspond to the sheets $\widetilde{S}_i$.

In this suggestively decorated diagraph, we construct a finite walk F_i. The walk corresponds to the path traversed by $\widetilde{\alpha}_i$ in $\widetilde{\Sigma}$. Naturally, it begins at the point S_i. If $\widetilde{\alpha}_i \cap \widetilde{\gamma}$ is empty, then the walk is simply S_i.

If $\widetilde{\alpha}_i \cap \widetilde{\gamma} \neq \varnothing$, then the recipe for F_i is as follows. Parametrize $\widetilde{\alpha}_i : [0, 1] \to \widetilde{\Sigma}$. Because $|\widetilde{\alpha} \cap \widetilde{\gamma}| < \infty$ by compactness, there is a finite set $\{t_j\} \subset [0, 1]$ satisfying $\widetilde{\alpha}_i(t_j) \in \widetilde{\alpha}_i \cap \widetilde{\gamma}$ and $t_j < t_{j+1}$. Choose a value $\epsilon > 0$ such that $\widetilde{\alpha}_i(t_j - \epsilon, t_j + \epsilon) \cap \widetilde{\gamma} = \widetilde{\alpha}_i(t_j)$ for all j. Then, for each t_j in the order of increasing j,

(1) add the vertex that corresponds to the subsurface containing $\widetilde{\alpha}_i(t_j + \epsilon)$ to F_i; and
(2) add the edge that connects the vertex corresponding to the subsurface containing $\widetilde{\alpha}_i(t_j - \epsilon)$ to the vertex corresponding to the subsurface containing $\widetilde{\alpha}_i(t_j + \epsilon)$, whose weight equals the sign of intersection assigned to $\widetilde{\alpha}_i(t_j)$.

This process yields a finite walk beginning and ending at S_i.

By Lemma 11.3, $w(F_i) \equiv 0 \bmod 2$. Note that $w(F_i)$ sums the signed intersections between $\widetilde{\alpha}_i$ and $\widetilde{\gamma}$. So by the previous discussion,

$$\hat{i}(\alpha, \gamma) \equiv \hat{i}(\widetilde{\alpha}_i, \widetilde{\gamma}) \equiv 0 \bmod 2.$$

($\Longleftarrow$) Let $\pi_2 : \mathbb{Z} \to \mathbb{Z}_2$ be a surjective group homomorphism. Define the homomorphism $\phi = \pi_2 \circ \hat{i}(-, \gamma) : H_1(\Sigma_g^1; \mathbb{Z}) \to \mathbb{Z}_2$. Note that ϕ is also surjective. There is a curve γ' such that $i(\gamma', \gamma) = 1$.

The kernel of ϕ has the following description:

$$\ker \phi \coloneqq \{[\alpha] \in H_1(\Sigma_g^1; \mathbb{Z}) : \hat{i}(\alpha, \gamma) \equiv 0 \bmod 2\}.$$

The forward direction implies that this kernel contains the kernel of $\varphi_{\mathfrak{a}}$, which has a prima facie stronger description:

$$\ker \varphi_{\mathfrak{a}} \coloneqq \{[\alpha] \in H_1(\Sigma_g^1; \mathbb{Z}) : \alpha \text{ lifts}\}.$$

However, both of these kernels are index two subgroups! Thus, their descriptions are equivalent:

$$\{[\alpha] \in H_1(\Sigma_g^1; \mathbb{Z}) : \alpha \text{ lifts}\} = \{[\alpha] \in H_1(\Sigma_g^1; \mathbb{Z}) : \hat{i}(\alpha, \gamma) \equiv 0 \bmod 2\}.$$

□

11.3. **Twisting Modulo Two.** In addition to the lifting criterion for curves, we have a lifting criterion for Dehn twists. Mercifully, its proof is succinct.

Corollary 11.6. *Let $\{\alpha, \gamma\}$ be constructed as above. If $\hat{i}(\alpha, \gamma) \equiv 0 \mod 2$, then $T_\alpha : \Sigma_g^1 \to \Sigma_g^1$ lifts uniquely to a homeomorphism of $\widetilde{\Sigma}$ that fixes $\widetilde{p}^{-1}(\alpha)$.*

Proof. Our suspect for the lift of T_α is $\widetilde{T}_\alpha = T_{\widetilde{\alpha}_1} T_{\widetilde{\alpha}_0} : \widetilde{\Sigma} \to \widetilde{\Sigma}$, where $\widetilde{\alpha}_0$ and $\widetilde{\alpha}_1$ form the disjoint components of $\widetilde{p}^{-1}(\alpha)$. Since these components are simple closed curves by Proposition 11.2, $\widetilde{T}_\alpha$ is well-defined. It fixes $\widetilde{p}^{-1}(\alpha)$ by definition [4.1].

While T_α rotates a closed regular neighborhood of α, its conjectured lift rotates closed regular neighborhoods of $\widetilde{\alpha}_0$ and $\widetilde{\alpha}_1$. Since $\widetilde{p}$ maps the (resp. rotated) $\widetilde{\alpha}_0$-

and $\widetilde{\alpha}_1$-annuli to the (resp. rotated) α-annulus, we have that $\widetilde{p}\widetilde{T}_\alpha = T_\alpha \widetilde{p}$. In sum: T_α lifts. $\square$

12. The General Case

The final section is split into five claims: four lemmas and one theorem. The first three lemmas build to the fourth, which is the intermediate abelian shortcut alluded to in Subsection 10.2. Once we have shown this intermediate case, we can prove the fifth claim and only theorem of this chapter: Theorem 12.5.

Lemma 12.1. *There is a simple closed curve $\gamma : S^1 \to \Sigma_g^b$ so that $i(\alpha, \gamma) = 1$ and $i(\beta, \gamma) = 0$.*

Proof. Let $\Sigma_g^1/(\alpha \cup \beta)$ be the path-connected cut surface equipped with a quotient $g : \Sigma_g^1/(\alpha \cup \beta) \to \Sigma_g^1$ [3.1]. Choose a point $x \in \alpha - \beta$ and two distinct preimages $x_0 \neq x_1 \in g^{-1}(x)$. By path-connectedness, there is an embedding $\hat{\gamma} : [0, 1] \to \Sigma_g^1/(\alpha \cup \beta)$ such that $\hat{\gamma}(0) = x_0$ and $\hat{\gamma}(1) = x_1$. We can also assume that $\hat{\gamma} \cap g^{-1}(\alpha \cup \beta) = \{x_0, x_1\}$ by applying an isotopy. Thus, $g(\hat{\gamma}) : [0, 1] \to \Sigma_g^1$ is a closed curve, which intersects α only in x but is disjoint from β. Since we can identify S^1 with the quotient $[0, 1]/_{0\sim 1}$, there is a simple closed curve $\gamma : S^1 \to \Sigma_g^1$ with the same image as $g(\hat{\gamma})$. $\square$

In what follows, we choose a representative for $\{\alpha, \gamma\}$. We depict this choice in Figure 18. Just like Figure 14, we can realize 18 as a subsurface of Σ_g^1.

This choice of representative is not really a "choice" in the sense that it is restrictive. In fact, if the theorem holds for one representative of $\{\alpha, \gamma\}$, then it holds for *all* representatives. Though these statements are non-obvious (and likely sound imprecise), we will unpack them in greater detail only at the end.

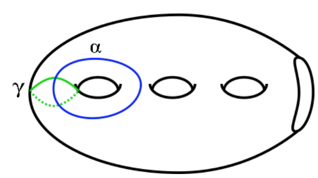

Figure 18. Representative for $\{\alpha, \gamma\}$

Lemma 12.2. *The union of $T_\alpha(\beta)$ and γ does not separate Σ_g^1.*

Proof. It is enough to prove that the cut surface $\Sigma_g^1/(\beta \cup T_\alpha^{-1}(\gamma))$ is path-connected. Note that the pairs $\{T_\alpha(\beta), \gamma\}$ and $\{\beta, T_\alpha^{-1}(\gamma)\}$ are related by a homeomorphism of Σ_g^1.

Consider the second cut surface $\Sigma_g^1/(\alpha \cup \beta \cup \gamma)$. This surface is equipped with two quotients. The first is given by Definition 3.1. It is the quotient $h : \Sigma_g^1/(\alpha \cup \beta \cup \gamma) \to \Sigma_g^1$ that undoes the cuts along α, β, and γ.

The second quotient is less canonical. It arises from the observation that we can cut $\Sigma_g^1/(\beta \cup T_\alpha^{-1}(\gamma))$ into a surface that is homeomorphic to $\Sigma_g^1/(\alpha \cup \beta \cup \gamma)$. Thus, there is a second quotient $k : \Sigma_g^1/(\alpha \cup \beta \cup \gamma) \to \Sigma_g^1/(\beta \cup T_\alpha^{-1}(\gamma))$ that glues together two disconnected subsets taken from possibly distinct boundary components of $\Sigma_g^1 - (\alpha \cup \beta \cup \gamma)$. We depict k up to a homeomorphism of the domain or codomain in Figure 19.

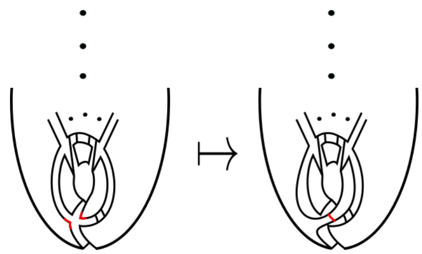

FIGURE 19. Gluing $\Sigma_g^1/(\alpha \cup \beta \cup \gamma)$ into $\Sigma_g^1/(\beta \cup T_\alpha^{-1}(\gamma))$

Let y_0 and y_1 be distinct points in $\Sigma_g^1/(\beta \cup T_\alpha^{-1}(\gamma))$. With the second quotient, we show that a path exists between y_0 and y_1. Choose representatives $y_0' \in k^{-1}(y_0)$ and $y_1' \in k^{-1}(y_1)$. Let $p : [0,1] \to \Sigma_g^1/(\alpha \cup \beta \cup \gamma)$ be a function with minimal discontinuities such that $p(0) = y_0'$ and $p(1) = y_1'$ and whose discontinuities (if they exist) occur on the boundary of $\Sigma_g^1/(\alpha \cup \beta \cup \gamma)$. In fact, we assume that these discontinuities exist. If p were continuous, then kp is already a path between y_0 and y_1.

Let D_p be the set of discontinuous points of p. By minimality, D_p consists of a finite number of jump discontinuities. In addition, $p(D_p)$ is a subset of $h^{-1}(\gamma)$. Since $\Sigma_g^1/(\alpha\cup\beta)$ is path-connected, the discontinuities only occur where $\Sigma_g^1/(\alpha\cup\beta)$ was cut along γ.

Because k is continuous, the set of discontinuities D_{kp} of kp are a subset of D_p. By the previous discussion, $kp(D_{kp})$ is a subset of $k(h^{-1}(\gamma))$. However, $k(h^{-1}(\gamma))$ is path-connected. Up to a homeomorphism of $\Sigma_g^1/(\beta \cup T_\alpha^{-1}(\gamma))$, it is the red arc in Figure 20. Thus, there is a way to remove each discontinuity of kp (if it exists) via concatenation of paths. There is a path between y_0 and y_1. $\square$

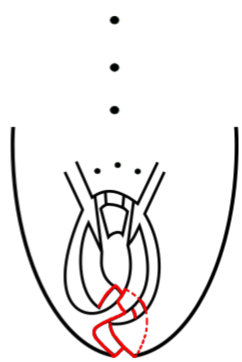

FIGURE 20. The set $k(h^{-1}(\gamma))$ up to a homeomorphism of $\Sigma_g^1/(\beta\cup T_\alpha^{-1}(\gamma))$

Our choice of γ is suspect. It is isotopic to $\widetilde{p}(\widetilde{\gamma}_i)$, where $\widetilde{\gamma}_i$ is a curve in the bounding pair depicted in Figure 16. Moreover, $\hat{i}(\beta,\gamma) = 0$. Thus, β lifts to $\widetilde{\Sigma}$ by our lifting criterion in Section 11 [11.2].

Even $T_\alpha(\beta)$ lifts. Because α and β have trivial algebraic intersection, $T_{\alpha\star}$ acts by the identity on $[\beta]$. Accordingly, it acts trivially in the β-coordinate of algebraic intersection: $\hat{i}(\beta,\gamma) = \hat{i}(T_\alpha(\beta),\gamma) = 0$ [9.1].

To this end, we consider the lifts of β and $T_\alpha(\beta)$. Choose a basepoint $*$ on the boundary of Σ_g^1 and a corresponding lift $\widetilde{*} \in \widetilde{p}^{-1}(*)$. Up to isotopy, we can assume that β and $T_\alpha(\beta)$ are based at $*$. In this way, we define the lifts $\widetilde{p}^{-1}(\beta) = \{\widetilde{\beta}_0, \widetilde{\beta}_1\}$ and $\widetilde{p}^{-1}(T_\alpha(\beta)) = \{\widetilde{\eta}_0, \widetilde{\eta}_1\}$ such that $\widetilde{\beta}_0$ and $\widetilde{\eta}_0$ are based at $\widetilde{*}$.

Lemma 12.3. *Let $\{\widetilde{\beta}_0, \widetilde{\beta}_1, \widetilde{\eta}_0, \widetilde{\eta}_1\}$ be constructed as above. Then, $\forall i, j : [\widetilde{\beta}_i] \neq [\widetilde{\eta}_j]$.*

Proof. Fix any $i, j \in \{0, 1\}$. To prove that $[\widetilde{\beta}_i] \neq [\widetilde{\eta}_j]$, it is sufficient to find a curve that shares a nontrivial algebraic intersection with $\widetilde{\eta}_j$ and a trivial intersection with $\widetilde{\beta}_i$. Note that the algebraic intersection is a well-defined mapping on the product of first homology groups [9.1].

To find this curve, we consider a crucial difference between β and $T_\alpha(\beta)$. Though β is disjoint from γ, the opposite is true for $T_\alpha(\beta)$. Via curve surgery, we compute that $T_\alpha(\beta) \geq i(\alpha, \beta)$ [5.2]. From this crucial difference, we can make a germane and crucial observation about the lifts of β and $T_\alpha(\beta)$. While $\widetilde{\beta}_i$ is properly contained in the sheet of $\widetilde{p}$ that contains its basepoint, $\widetilde{\eta}_j$ is not properly contained in *any* sheet.

We can say more about $\widetilde{\eta}_j$. Let $\widetilde{S}_0$ and $\widetilde{S}_1$ be the two sheets of $\widetilde{p}$ such that $\widetilde{*} \in \widetilde{S}_0$. Then, the subsurface $\widetilde{S}_k - \widetilde{\eta}_j$ is connected for any $k \in \{0, 1\}$. This is because $\widetilde{S}_k - \widetilde{\eta}_j$ is homeomorphic to its image $\widetilde{p}(\widetilde{S}_k - \widetilde{\eta}_j)$ and the image is connected by Lemma 12.2.

Now fix $k \in \{0, 1\}$ such that $k \neq i$. Since $\widetilde{S}_k - \widetilde{\eta}_j$ is connected, there is a curve that is properly contained in $\widetilde{S}_k$ and intersects $\widetilde{\eta}_j$ in one point. (See the proof of Lemma 12.1 for more details.) Because $\widetilde{\beta}_i$ is disjoint from $\widetilde{S}_k$, the proof of the lemma is complete. $\square$

Since β and $T_\alpha(\beta)$ lift to $\widetilde{\Sigma}$, $T_{T_\alpha(\beta)} T_\beta^{-1}$ lifts to a homeomorphism of $\widetilde{\Sigma}$. Let

$$\widetilde{T} = T_{\widetilde{\eta}_1} T_{\widetilde{\eta}_0} T_{\widetilde{\beta}_1}^{-1} T_{\widetilde{\beta}_0}^{-1} : \widetilde{\Sigma} \to \widetilde{\Sigma}.$$

be the unique lift of $T_{T_\alpha(\beta)} T_\beta^{-1}$ that fixes $\widetilde{*}$ [11.6]. This homeomorphism twists regular neighborhoods of the lifts of β, before twisting regular neighborhoods of the lifts of $T_\alpha(\beta)$.

Secretly, $\widetilde{T}$ is also the lift of $[T_\alpha, T_\beta]$. Note that the commutator and $T_{T_\alpha(\beta)} T_\beta^{-1}$ define the same homeomorphism on Σ_g^1 [4.3].

Lemma 12.4. *The induced isomorphism $\widetilde{T}_\star : H_1(\widetilde{\Sigma}; \mathbb{Z}) \to H_1(\widetilde{\Sigma}; \mathbb{Z})$ is not the identity.*

Proof. By construction, the pair $\{\beta, \gamma\}$ is disjoint and non-homologous. So not only is $\beta \cup \gamma$ a representative for the class $[\beta] + [\gamma]$, but it is also *not* null-homologous. It is also non-separating [9.3].

In the same vein as Lemma 12.1, there is a simple closed curve ζ based at $*$ such that $i(\beta, \zeta) = \hat{i}(\beta, \zeta) = 1$ and $i(\gamma, \zeta) = 0$. Hence, ζ lifts to $\widetilde{\Sigma}$ [11.2]. As before, we define the components in the preimage $\widetilde{p}^{-1}(\zeta) = \{\widetilde{\zeta}_0, \widetilde{\zeta}_1\}$ such that $\widetilde{\zeta}_0$ is based $\widetilde{*}$.

Now set $\widetilde{p}^{-1}(\zeta) = \widetilde{\zeta}$ and $\widetilde{p}^{-1}(T_\alpha(\beta)) = \widetilde{\eta}$. By the action of the deck group, these preimages satisfy $i(\widetilde{\beta}_i, \widetilde{\eta}) = 0$, $i(\widetilde{\beta}_i, \widetilde{\zeta}) = 1$, and $i(\widetilde{\zeta}_i, \widetilde{\eta}) = 1$ for any $i \in \{0, 1\}$. We can also assume that $\hat{i}(\widetilde{\beta}_0, \widetilde{\zeta}_0) = 1$. (If not, then we switch $\widetilde{\zeta}_0$ with $\widetilde{\zeta}_1$ in the following discussion.) Under this construction, we claim that the induced homomorphism $\widetilde{T}_\star$ does not act by the identity on the class $[\widetilde{\zeta}_0]$.

Consider the action of $\widetilde{T}_\star$ on $[\widetilde{\zeta}_0]$, which can be expressed as the action of $T_{\widetilde{\eta}_1 \star}$ on $(T_{\widetilde{\eta}_0} T^{-1}_{\widetilde{\beta}_0})_\star(\widetilde{\zeta}_0)$.

$$\widetilde{T}_\star(\widetilde{\zeta}_0) = (T_{\widetilde{\eta}_0} T^{-1}_{\widetilde{\beta}_0})_\star(\widetilde{\zeta}_0) + \hat{i}(\widetilde{\eta}_1, T_{\widetilde{\eta}_0} T^{-1}_{\widetilde{\beta}_0}(\widetilde{\zeta}_0))[\widetilde{\eta}_1]. \quad [9.2]$$

Because $\hat{i}$ is bilinear, we can simplify further:

$$= [\widetilde{\zeta}_0] - [\widetilde{\beta}_0] + (\hat{i}(\widetilde{\eta}_0, \widetilde{\zeta}_0) - \hat{i}(\widetilde{\eta}_0, \widetilde{\beta}_0))[\widetilde{\eta}_0] + (\hat{i}(\widetilde{\eta}_1, \widetilde{\zeta}_0) - \hat{i}(\widetilde{\eta}_1, \widetilde{\beta}_0))[\widetilde{\eta}_1]. \quad [9.1]$$

Finally, we set $\hat{i}(\widetilde{\eta}_0, \widetilde{\beta}_0) = m$ and $\hat{i}(\widetilde{\eta}_0, \widetilde{\zeta}_0) = n$. This gives the more readable expression

$$\widetilde{T}_\star(\widetilde{\zeta}_0) = [\widetilde{\zeta}_0] - [\widetilde{\beta}_0] + (n-m)[\widetilde{\eta}_0] + (1-n+m)[\widetilde{\eta}_1].$$

It remains to prove that $[\widetilde{\beta}_0] \neq (n-m)[\widetilde{\eta}_0] + (1-n+m)[\widetilde{\eta}_1]$, which we do by contradiction.

If $[\widetilde{\beta}_0] = (n-m)[\widetilde{\eta}_0] + (1-n+m)[\widetilde{\eta}_1]$, then

$$\begin{aligned} \hat{i}(\widetilde{\beta}_0, \widetilde{\zeta}_0) &= \hat{i}((n-m)[\widetilde{\eta}_0] + (1-n+m)[\widetilde{\eta}_1], \widetilde{\zeta}_0) \quad [9.1] \\ &= n(n-m) + (1-n)(1-n+m) \quad [9.1] \\ &= 1. \end{aligned}$$

The above expression has integer solutions iff $m = 0$ and $n = 0$ or 1, that is, iff $[\widetilde{\beta}_0] = [\widetilde{\eta}_0]$ or $[\widetilde{\eta}_1]$. However, this contradicts Lemma 12.3. So our original assumption is false, and the fourth lemma is proved. $\square$

For the fifth and final claim, we apply the universal property of abelian covering spaces. After fixing the basepoint $\overline{*} \in \overline{p}^{-1}(*)$, we assert that there is a unique abelian covering $q : \overline{\Sigma} \to \widetilde{\Sigma}$ satisfying $\overline{p} = \widetilde{p}q$ and $q(\overline{*}) = \widetilde{*}$ [10.3].

The universal property gives rise to the following proposition. If $\overline{T} : \overline{\Sigma} \to \overline{\Sigma}$ is the unique lift of $[T_\alpha, T_\beta]$ that fixes $\overline{*} \in \overline{p}^{-1}(*)$, then it is also the unique lift of $\widetilde{T}$ that fixes $\overline{*} \in q^{-1}(\widetilde{*})$. (See Subsection 10.1 for more details.) In other words, we have a commutative diagram on covering spaces that expresses the preceding proposition. However, we consider the diagram that the first diagram induces on relative homology groups.

$$\begin{tikzcd}
H_1(\overline{\Sigma}, \overline{p}^{-1}(*); \mathbb{Z}) \arrow[r, "\overline{T}_\sharp"] \arrow[d, "q_\sharp"] & H_1(\overline{\Sigma}, \overline{p}^{-1}(*); \mathbb{Z}) \arrow[d, "q_\sharp"] \\
H_1(\widetilde{\Sigma}, \widetilde{p}^{-1}(*); \mathbb{Z}) \arrow[r, "\widetilde{T}_\sharp"] \arrow[d, "\widetilde{p}_\sharp"] & H_1(\widetilde{\Sigma}, \widetilde{p}^{-1}(*); \mathbb{Z}) \arrow[d, "\widetilde{p}_\sharp"] \\
H_1(\Sigma^1_g, *; \mathbb{Z}) \arrow[r, "{[T_\alpha, T_\beta]_\sharp}"] & H_1(\Sigma^1_g, *; \mathbb{Z})
\end{tikzcd}$$

In the depicted diagram, the subscript $\sharp$ denotes the natural action on relative homology. We will use this notation for all of the subsequent mappings, including homology cycles.

Theorem 12.5. *Let $[T_\alpha, T_\beta] : \Sigma_g^1 \to \Sigma_g^1$ be a commutator of a simply intersecting pair such that $g \geq 3$, $i(\alpha, \beta) > 0$, and $\alpha \cup \beta$ does not separate Σ_g^1. Then, $[T_\alpha, T_\beta] \notin \ker r_1$.*

Proof. Let $\overline{\zeta}_0 \subset \overline{\Sigma}$ be the unique lift of ζ beginning at the basepoint $\overline{*}$. Then by uniqueness, $q_\sharp(\overline{\zeta}_0) = [\widetilde{\zeta}_0]_\sharp$. By commutativity, $q_\sharp \overline{T}_\sharp(\overline{\zeta}_0) = \widetilde{T}_\sharp(\widetilde{\zeta}_0)$. We claim that these images are not equal: $[\widetilde{\zeta}_0]_\sharp \neq \widetilde{T}_\sharp(\widetilde{\zeta}_0)$.

As we stated in Section 10, an edge $x \in H_1(\widetilde{\Sigma}, \widetilde{p}^{-1}(*); \mathbb{Z})$ is trivial iff there exists a $y \in H_2(\widetilde{\Sigma}; \mathbb{Z})$ such that $x = \partial y$. In this way, the difference $[\widetilde{\zeta}_0]_\sharp - \widetilde{T}_\sharp(\widetilde{\zeta}_0) \in H_1(\widetilde{\Sigma}, \widetilde{p}^{-1}(*); \mathbb{Z})$ must be *non*trivial. By Lemma 12.4, there is no $y \in H_2(\widetilde{\Sigma}; \mathbb{Z})$ such that $[\widetilde{\zeta}_0]_\sharp - \widetilde{T}_\sharp(\widetilde{\zeta}_0) = \partial y$.

Since $\widetilde{T}_\sharp$ does not act by the identity on $[\widetilde{\zeta}_0]_\sharp$, the analogous statement is true for $\overline{T}_\sharp$ and $[\overline{\zeta}_0]_\sharp$. Because $q_\sharp$ is well defined, the preimages $[\overline{\zeta}_0]_\sharp$ and $\overline{T}_\sharp(\overline{\zeta}_0)$ are not equal. In fact, $r_1([T_\alpha, T_\beta])$ is not the identity either! By our definition of r_1 [10.1], $\overline{T}_\sharp$ is the identity iff $r_1([T_\alpha, T_\beta])$ is too. ☐

Remark 12.6. Though we chose $\{\alpha, \gamma\}$ to be the pair of curves pictured in Figure 18, this choice does not invalidate the previous lemmas. By Change of Coordinates, any pair of curves $\{\alpha, \gamma\}$ with $i(\alpha, \gamma) = 1$ is unique up to a homeomorphism of Σ_g^1 [3.3]. Thus, if we fixed a different pair $\{\alpha', \gamma'\}$ with the same intersection pattern, then there is a homeomorphism $f : \Sigma_g^1 \to \Sigma_g^1$ satisfying $\{\alpha', \gamma'\} = f(\{\alpha, \gamma\})$ and $[T_{\alpha'}, T_\beta] = f[T_\alpha, T_\beta]f^{-1}$ [4.3]. Moreover:

$$\begin{aligned}
\overline{[T_{\alpha'}, T_\beta]}_\sharp &= (\overline{f[T_\alpha, T_\beta]f^{-1}})_\sharp \\
&= (\overline{f}\,\overline{[T_\alpha, T_\beta]}\,\overline{f}^{-1})_\sharp \\
&= \overline{f}_\sharp \overline{[T_\alpha, T_\beta]}_\sharp \overline{f_\sharp}^{-1} \\
&\neq \mathrm{id}.
\end{aligned}$$

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