RAMSEY THEORY IN MODELS OF SET THEORY WHERE THE AXIOM OF CHOICE FAILS

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Abstract. Blass [1] showed in 1977 that Ramsey's theorem for pairs fails in the basic Cohen model and holds in the basic Fraenkel model. His proofs can be easily extended to the full Ramsey's theorem. Ramsey's theorem in the ordered Mostowski model and open Ramsey theorem in these models, however, had not been investigated. In this paper, we will prove that Ramsey's theorem holds in the ordered Mostowski model; open Ramsey theorem fails in the basic Cohen model, holds in the basic Fraenkel model, and fails in the ordered Mostowski model. Also, the usual proof of open Ramsey theorem on ω given by Galvin and Prikry [2] assumes the Axiom of Dependent Choice, and we will give an alternative proof that requires no choice.

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1. Introduction

Ramsey theory aims to find large homogeneous sets under certain colorings. There are a lot of theorems in Ramsey theory whose proof explicitly or implicitly uses the Axiom of Choice to pick non-constructive elements. In fact, without the Axiom of Choice, it is even not clear whether Ramsey's theorem for pairs implies Ramsey's theorem for triples.

This project focuses on Ramsey's theorem and open Ramsey theorem in models of set theory where the Axiom of Choice fails. We are interested in two kinds of models: symmetric submodels of generic extensions obtained by forcing (the basic Cohen model) and permutation models with atoms (the basic Fraenkel model and the ordered Mostowski model), in both of which the Axiom of Choice fails because "symmetry" is required. The idea is to find symmetric colorings without symmetric homogeneous sets, or to show that every symmetric coloring has a symmetric homogeneous set in these models.

In section 2, we will investigate Ramsey's theorem and open Ramsey theorem. We will look at the usual proofs of the two theorems and identify how much choice is involved in the proofs. We will also give a proof of open Ramsey theorem on well-orderable sets without using any choice.

In section 3, 4, and 5, we will investigate Ramsey's theorem and open Ramsey theorem in the basic Cohen model, the basic Fraenkel model, and the ordered Mostowski model respectively.

2. Two Ramsey-Theoretic Statements

2.1. **Ramsey's Theorem.**

Theorem 2.1 (Ramsey's Theorem)**.** *For any infinite set X, for any $n, r \in \omega - \{0\}$, and for any colouring $\pi : [X]^n \to r$, there is an infinite subset H of X such that H is homogeneous for π, i.e., $[H]^n$ is monochromatic.*

Theorem 2.2 (Ramsey's Theorem on infinite subsets of ω)**.** *For any $S \in [\omega]^\omega$, for any $n, r \in \omega - \{0\}$, and for any colouring $\pi : [S]^n \to r$, there is an infinite subset H of S such that H is homogeneous for π, i.e., $[H]^n$ is monochromatic.*

Proof [3, Theorem 4.1]. Fix $r \in \omega - \{0\}$, we will prove the theorem by induction on n.

Base Case: $n = 1$. This case follows easily from the Infinite Pigeonhole Principle. Fix $S \in [\omega]^\omega$ and fix a coloring $\pi : [S]^1 \to r$. Note that $[S]^1 = \{\{x\} : x \in S\}$, so π naturally induces a coloring π' on S by:

$$\begin{aligned} \pi' \colon S &\to r \\ y &\mapsto \pi(\{y\}) \end{aligned}$$

According to the Infinite Pigeonhole Principle, since S is infinite, there is an infinite monochromatic set $H \subseteq S$ for π'. H is homogeneous for π.

Note that H can be selected constructively: the set $\{c \in r \colon \pi'^{-1}(c)$ is infinite$\}$ is non-empty, so we can choose the minimum element t of this set and let $H = \pi'^{-1}(t)$.

Induction Case: Suppose for any $S \in [\omega]^\omega$ and for any colouring $\pi : [S]^n \to r$, there is an $H \in [S]^\omega$ such that H is homogeneous for π. Fix $S \in [\omega]^\omega$ and a colouring $\pi : [S]^{n+1} \to r$. For each $a \in S$, let π_a be an r-coloring of $[S - \{a\}]^n$ defined by:

$$\begin{aligned} \pi_a \colon [S - \{a\}]^n &\to r \\ x &\mapsto \pi(x \cup \{a\}) \end{aligned}$$

We will inductively construct an infinite sequence $a_0 < a_1 < a_2 < ...$ of finite ordinals and an infinite sequence $S_0 \supseteq S_1 \supseteq S_2 \supseteq ...$ of infinite subsets of S as follows:

$S_0 = S$ and $a_0 = \min(S_0)$.

If S_k and a_k are defined, $\pi_{a_k} \restriction [S_k - \{a_k\}]^n$ is an r-coloring of $[S_k - \{a_k\}]^n$. By induction hypothesis, there is an infinite homogeneous set $H_k \subseteq S_k - \{a_k\}$ with respect to this coloring. Let $S_{k+1} = H_k$ and $a_{k+1} = \min\{a \in H_k : a > a_k\}$.

By construction, for each $i \in \omega$, $[\{a_j : j > i\}]^n$ is monochromatic with respect to π_{a_i}; let $\tau(a_i)$ be that color. τ is an r-coloring of $\{a_i : i \in \omega\}$, so by the infinite Pigeonhole Principle, we can constructively choose an infinite monochromatic set $H \subseteq \{a_i : i \in \omega\}$ for τ (just as what we did in the base case). For any $h \in [H]^{n+1}$, $\pi(h) = \pi_{\min(h)}(h - \min(h)) = \tau(\min(h))$. Therefore H is homogeneous for π. □

This proof is constructive and requires no choice. At first glance, it may seem that choice is used to pick an infinite sequence of homogeneous sets in the induction step. However, since a homogeneous set in the base case is purely constructive, we are able to inductively construct homogeneous sets for colorings over n-tuples for any natural number n.

Remark 2.3. The proof of Theorem 2.2 works for every well-orderable set X, by taking a countably infinite subset X' of X and restricting every coloring on $[X]^n$ to $[X']^n$.

2.2. **Open Ramsey Theorem.**

Definition 2.4. Let X be an infinite set. A set $S \subseteq 2^X$ is **Ramsey over** X (or simply **Ramsey** if it makes no confusion) if there is an infinite subset M of X such that $[M]^\infty \subseteq S$ or $[X]^\infty - S$.

Theorem 2.5 (Open Ramsey Theorem)**.** *For any infinite set X and for any set $S \subseteq 2^X$, if S is open (with respect to the product topology on 2^X), then S is Ramsey over X.*

Theorem 2.6 (Open Ramsey Theorem on ω)**.** *For any set $S \subseteq 2^\omega$, if S is open (with respect to the product topology on 2^ω), then S is Ramsey over ω.*

Galvin and Prikry [2] gave the usual proof of Theorem 2.6 in 1973. The proof, however, uses the Axiom of Dependent Choice to choose an infinite sequence of infinite subsets of ω. In this subsection we will give an alternative proof of Theorem 2.6 without using any choice.

Definition 2.7. For $S_1, S_2 \in 2^\omega$, $S_1 < S_2$ if for all $s_1 \in S_1$ and $s_2 \in S_2$, $s_1 < s_2$.

Definition 2.8. For $k \in \omega$ and $S \in 2^\omega$, $k < S$ if $k < s$ for all $s \in S$; $k > S$ if $k > s$ for all $s \in S$.

Definition 2.9. For $k \in \omega$ and $A \in \text{fin}(\text{fin}(\omega))$, $k < A$ if $k < a$ for all $a \in A$; $k > A$ if $k > a$ for all $a \in A$.

For Definition 2.10 and 2.11, S is a fixed subset of 2^ω.

Definition 2.10. For any $M \in [\omega]^\omega$ and $m \in [\text{fin}(M)]^\omega$, for any $A \in \text{fin}(\text{fin}(\omega))$, we say that m **welcomes** A if for every $y \in [m]^\omega$, $\bigcup y \cup \bigcup A \in S$.

Definition 2.11. For any $M \in [\omega]^\omega$ and any $A \in \text{fin}(\text{fin}(\omega))$, we say that M **accepts** A if every $m \in [\text{fin}(M^{>A})]^\omega$ welcomes A; M **rejects** A if no $m \in [\text{fin}(M^{>A})]^\omega$ welcomes A.

Definition 2.12. Suppose x is a subset of ω and I is a finite partial function from ω to 2. Let $L : \omega \to 2$ be the characteristic function of x. I is a **first initial segment** of x if $I = L \restriction l$ for some $l \in \omega$.

Proof of Theorem 2.6. Note: the structure of this proof is an analogue to that of the proof of [2, Theorem 1].

Fix an open $S \subseteq 2^\omega$. If there is an $M \in [\omega]^\omega$ that accepts $\varnothing$, then $\{\{x\} : x \in M\}$ welcomes $\varnothing$. Thus, for any $N \in [M]^\omega$, $N = N \cup \varnothing = \bigcup\{\{x\} : x \in N\} \cup \bigcup \varnothing$, which is in S. Therefore $[M]^\omega \subseteq S$ and we are done.

So suppose that no $M \in [\omega]^\omega$ accepts $\varnothing$. We will inductively construct an infinite sequence $M_0 \supseteq M_1 \supseteq M_2 \supseteq ...$ of infinite subsets of ω and an infinite sequence $a_0 < a_1 < a_2 < ...$ of singletons such that for all i, $a_i \subseteq M_i$ and M_i rejects all subsets of $\{a_0, a_1, ..., a_{i-1}\}$. The construction goes as follows:

$M_0 = \omega$.

Having chosen $M_0 \supseteq M_1 \supseteq ... \supseteq M_n$ and $a_0 < a_1 < ... < a_{n-1}$, we look for a singleton $a_n \in \text{fin}(M_n)$ with $a_n > a_{n-1}$ and an $M_{n+1} = \{x \in M_n : x > t\}$ for some $t \in \omega$ such that M_{n+1} rejects all subsets of $\{a_0, a_1, ..., a_n\}$. Suppose this is impossible. We will inductively construct an infinite sequence $M_n \supseteq N_0 \supseteq N_1 \supseteq N_2 \supseteq ...$ (each N_i is in the form of $M_n^{>t}$ for some $t \in \omega$), an infinite sequence $b'_0 < b'_1 < b'_2 < ...$ (each b'_i is a finite subset of N_{i-1}), and an infinite sequence $E_0, E_1, E_2, ...$ (each E_i is a subset of $\{a_0, a_1, ..., a_{n-1}\}$) such that N_i accepts $E_i \cup \{b'_i\}$ for all i. The construction goes as follows:

Base Case: Let $b_0 = \{\min(M_n^{>a_{n-1}})\}$. We cannot take $a_n = b_0$ and $M_{n+1} = M_n$, so there is some $A \subseteq \{a_0, a_1, ..., a_{n-1}, b_0\}$ such that some $x \in [\text{fin}(M_n^{>A})]^\omega$ welcomes A. Since M_n rejects all subsets of $\{a_0, a_1, ..., a_{n-1}\}$, A must be the form of $E_0 \cup \{b_0\}$ with $E_0 \subseteq \{a_0, a_1, ..., a_{n-1}\}$.

Thus $\bigcup x \cup \bigcup(E_0 \cup \{b_0\}) \in S$. Since S is open, there is a first initial segment I of $\bigcup x \cup \bigcup(E_0 \cup \{b_0\})$ such that every $y \in 2^\omega$ with I as a first initial segment is in S.

Let $b'_0 = I^{-1}(1) \cap M_n^{>a_{n-1}}$ and $N_0 = \{y \in M_n : y > \max(\text{Dom}(I))\}$. We want to show that N_0 accepts $E_0 \cup \{b'_0\}$. Fix $T \in [\text{fin}(N_0^{>E_0 \cup \{b'_0\}})]^\omega$ and fix $t \in [T]^\omega$.

If $y \in I^{-1}(1)$, then $y \in \bigcup x \cup \bigcup(E_0 \cup \{b_0\})$, so $y \in \bigcup x$ or $y \in \bigcup E_0$ or $y \in b_0$.

(i) If $y \in \bigcup x$, then $y \in M_n^{>E_0 \cup \{b_0\}}$. In particular, $y > b_0 > a_{n-1}$. So

$y \in I^{-1}(1) \cap M_n^{>a_{n-1}} = b_0' \subseteq \bigcup t \cup \bigcup(E_0 \cup \{b_0'\})$.
(ii) If $y \in \bigcup E_0$, then $y \in \bigcup t \cup \bigcup(E_0 \cup \{b_0'\})$ automatically.
(iii) If $y \in b_0$, then $y = \min(M_n^{>a_{n-1}})$. So $y \in I^{-1}(1) \cap M_n^{>a_{n-1}} = b_0' \subseteq \bigcup t \cup \bigcup(E_0 \cup \{b_0'\})$.
Therefore $y \in \bigcup t \cup \bigcup(E_0 \cup \{b_0'\})$

If $y \in I^{-1}(0)$, then $y \notin \bigcup x \cup \bigcup(E_0 \cup \{b_0\})$.
(i) $y \notin \bigcup E_0$ obviously.
(ii) Since $y \in \mathrm{Dom}(I)$, $y \notin M_n^{>\max(\mathrm{Dom}(I))} = N_0$. Thus $y \notin \bigcup t$.
(iii) Since $b_0' \subseteq I^{-1}(1)$, $y \notin b_0'$.
Therefore $y \notin \bigcup t \cup \bigcup(E_0 \cup \{b_0'\})$.

We have shown that I is a first initial segment of $\bigcup t \cup \bigcup(E_0 \cup \{b_0'\})$, which concludes that N_0 accepts $E_0 \cup \{b_0'\}$.

Induction Case: Having chosen $M_n \supseteq N_0 \supseteq N_1 \supseteq ... \supseteq N_k$, $b_0' < b_1' < ... < b_k'$, and $E_0, E_1, ..., E_k$ as desired, let $b_{k+1} = \{\min(N_k^{>b_k'})\}$. We cannot take $a_n = b_{k+1}$ and $M_{n+1} = N_k$, so there is some $A \subseteq \{a_0, a_1, ..., a_{n-1}, b_{k+1}\}$ such that some $x \in [\mathrm{fin}(N_k^{>A})]^\omega$ welcomes A. Since M_n rejects all subsets of $\{a_0, a_1, ..., a_{n-1}\}$, so does $N_k \subseteq M_n$. A must be the form of $E_{k+1} \cup \{b_{k+1}\}$ with $E_{k+1} \subseteq \{a_0, a_1, ..., a_{n-1}\}$.

Thus $\bigcup x \cup \bigcup(E_{k+1} \cup \{b_{k+1}\}) \in S$. Since S is open, there is a first initial segment I of $\bigcup x \cup \bigcup(E_{k+1} \cup \{b_{k+1}\})$ such that every $y \in 2^\omega$ with I as a first initial segment is in S.

Let $b_{k+1}' = I^{-1}(1) \cap N_k^{>b_k'}$ and $N_{k+1} = \{y \in N_k : y > \max(\mathrm{Dom}(I))\}$. We want to show that N_{k+1} accepts $E_{k+1} \cup \{b_{k+1}'\}$.

Fix $T \in [\mathrm{fin}(N_{k+1}^{>E_{k+1} \cup \{b_{k+1}'\}})]^\omega$ and fix $t \in [T]^\omega$.

If $y \in I^{-1}(1)$, then $y \in \bigcup x \cup \bigcup(E_{k+1} \cup \{b_{k+1}\})$, so $y \in \bigcup x$ or $y \in \bigcup E_{k+1}$ or $y \in b_{k+1}$.
(i) If $y \in \bigcup x$, then $y \in N_k^{>E_{k+1} \cup \{b_{k+1}\}}$. In particular, $y > b_{k+1} > b_k'$. So $y \in I^{-1}(1) \cap N_k^{>b_k'} = b_{k+1}' \subseteq \bigcup t \cup \bigcup(E_{k+1} \cup \{b_{k+1}'\})$.
(ii) If $y \in \bigcup E_{k+1}$, then $y \in \bigcup t \cup \bigcup(E_{k+1} \cup \{b_{k+1}'\})$ automatically.
(iii) If $y \in b_{k+1}$, then $y = \min(N_k^{>b_k'})$. So $y \in I^{-1}(1) \cap N_k^{>b_k'} = b_{k+1}' \subseteq \bigcup t \cup \bigcup(E_{k+1} \cup \{b_{k+1}'\})$.
Therefore $y \in \bigcup t \cup \bigcup(E_{k+1} \cup \{b_{k+1}'\})$

If $y \in I^{-1}(0)$, then $y \notin \bigcup x \cup \bigcup(E_{k+1} \cup \{b_{k+1}\})$.
(i) $y \notin \bigcup E_{k+1}$ obviously.
(ii) Since $y \in \mathrm{Dom}(I)$, $y \notin N_k^{>\max(\mathrm{Dom}(I))} = N_{k+1}$. Thus $y \notin \bigcup t$.
(iii) Since $b_{k+1}' \subseteq I^{-1}(1)$, $y \notin b_{k+1}'$.
Therefore $y \notin \bigcup t \cup \bigcup(E_{k+1} \cup \{b_{k+1}'\})$.

We have shown that I is a first initial segment of $\bigcup t \cup \bigcup(E_{k+1} \cup \{b_{k+1}'\})$, which concludes that N_{k+1} accepts $E_{k+1} \cup \{b_{k+1}'\}$.

After this construction, we now have $M_n \supseteq N_0 \supseteq N_1 \supseteq N_2 \supseteq ...$, $b'_0 < b'_1 < b'_2 < ...$, and $E_0, E_1, E_2, ...$ Since each E_i is a subset of $\{a_0, a_1, ..., a_{n-1}\}$ and there are only finitely many subsets of $\{a_0, a_1, ..., a_{n-1}\}$, *without loss of generality*, we may assume that $E = E_i$ for all $i \in \omega$.

Fix y to be an infinite subset of $\{b'_i : i \in \omega\}$; there is an $b'_j = \min(y)$. By construction, $y - \{b'_j\} \in [\mathrm{fin}(N_j)]^\omega$. Since N_j accepts $E_j \cup \{b'_j\} = E \cup \{b'_j\}$ and $y - \{b'_j\} > E \cup \{b'_j\}$, we have $y - \{b'_j\}$ welcomes $E \cup \{b'_j\}$, and thus $\bigcup(y - \{b'_j\}) \cup \bigcup(E \cup \{b'_j\}) \in S$. Note that $\bigcup(y - \{b'_j\}) \cup \bigcup(E \cup \{b'_j\}) = \bigcup y \cup \bigcup E$, so $\bigcup y \cup \bigcup E \in S$. Therefore, $\{b'_i : i \in \omega\}$ welcomes E. However, M_n rejects E and $\{b'_i : i \in \omega\} \in [\mathrm{fin}(M_n^{>E})]^\omega$, which is a contradiction.

Therefore we know that a_n and M_{n+1} with the desired properties exist. Let a_n be the smallest such (the well-ordering on ω naturally induces a well-ordering on the singletons of elements of ω) and choose the smallest $t \in \omega$ such that $M_{n+1} = \{x \in M_n : x > t\}$ works with a_n.

Let $M = \bigcup\{a_i : i \in \omega\}$. We will show that M rejects every element in $\mathrm{fin}(\mathrm{fin}(M))$. Fix an $m \in \mathrm{fin}(\mathrm{fin}(M))$ and fix an $x \in [\mathrm{fin}(M^{>m})]^\omega$. Since every a_i is a singleton, there is a finite $m' = \{a_{q_1}, a_{q_2}, ..., a_{q_k}\} \subseteq \{a_i : i \in \omega\}$ such that $\bigcup m' = \bigcup m$. Every element in x is a finite subset of $M^{>m} = M^{>m'} = \bigcup\{a_i : i > \max\{q_1, q_2, ..., q_k\}\} \subseteq M^{>m}_{\max\{q_1,q_2,...,q_k\}+1}$. $M_{\max\{q_1,q_2,...,q_k\}+1}$ rejects m', and thus rejects m, so $x \in [\mathrm{fin}(M^{>m}_{\max\{q_1,q_2,...,q_k\}+1})]^\omega$ cannot welcome m. Therefore, M rejects every element in $\mathrm{fin}(\mathrm{fin}(M))$.

Finally we will show that $[M]^\omega \subseteq [\omega]^\omega - S$. Suppose there is an $Y \in [M]^\omega$ such that $Y \in S$. Since S is open, there is a first initial segment I of Y such that every $y \in 2^\omega$ with I as a first initial segment is in S. Let $l = \min(M^{>\max(\mathrm{Dom}(I))})$. l exists because M is infinite.

Then M accepts $\{\{y\} : y \in I^{-1}(1) \cup \{l\}\}$ because for any $x \in [\mathrm{fin}(M^{>I^{-1}(1)\cup\{l\}})]^\omega$ and any $x' \in [x]^\omega$, $x' > l > \max(\mathrm{Dom}(I))$, which implies that $\bigcup x' \cup \bigcup\{\{y\} : y \in I^{-1}(1) \cup \{l\}\}$ has I as a first initial segment and thus is in S. Since $I^{-1}(1) \subseteq Y \subseteq M$ and $l \in M$, we know that $I^{-1}(1) \cup \{l\} \in \mathrm{fin}(M)$. Thus $\{\{y\} : y \in I^{-1}(1) \cup \{l\}\} \in \mathrm{fin}(\mathrm{fin}(M))$, but that is a contradiction because M rejects every element in $\mathrm{fin}(\mathrm{fin}(M))$.

Therefore $[M]^\omega \subseteq [\omega]^\omega - S$, which completes the proof. □

This proof is constructive and requires no choice.

Remark 2.13. The proof of Theorem 2.6 works for every well-orderable set X, by taking a countably infinite subset X' of X and restricting every $S \in 2^X$ to $2^{X'}$ (note that $S \cap 2^{X'}$ is Ramsey over $X' \Rightarrow S$ is Ramsey over X).

3. The Basic Cohen Model

In this section, we will first give a brief introduction to the basic Cohen model. Then we will prove some properties of the basic Cohen model, which will be used to investigate Ramsey's theorem and open Ramsey theorem in this model. For readers who are not familiar with forcing, please refer to [3, Part III] and [6, Section 10].

3.1. **Introduction to the Basic Cohen Model.**

This subsection is a summary of [3, Chapter 17.1 & 17.2] and [6, Section 12].

Let M be a countable transitive model of ZFC and let $\mathbb{P}$ be the set of all finite partial functions from $\omega \times \omega$ to 2, ordered by reverse inclusion. Let $\mathbb{1}$ denote the greatest element in $\mathbb{P}$, which is $\varnothing$. Since $M \models$ ZFC, we have $\mathbb{P} \in M$. Suppose G is a $\mathbb{P}$-generic filter over M. The generic extension $M[G]$ is a transitive model of ZFC.

For each $x \in M$, let $\dot{x} = \{\langle \dot{y}, \mathbb{1}\rangle : y \in x\}$ be the inductively defined $\mathbb{P}$-name for x.

For each $n \in \omega$, we define a $\mathbb{P}$-name $\underset{\sim}{a}_n = \{\langle \dot{k}, p\rangle : p(\langle n, k\rangle) = 1\}$, and let $\underset{\sim}{A} = \{\langle \underset{\sim}{a}_n, \mathbb{1}\rangle : n \in \omega\}$. For each $\underset{\sim}{a}_n$, let a_n denote $\underset{\sim}{a}_n[G]$ and a_n is called a **Cohen real**. Let A denote $\underset{\sim}{A}[G] = \{a_n : n \in \omega\}$. By an easy density argument, $a_i \vartriangle a_j \neq \varnothing$ for all $a_i, a_j \in A$.

Now we construct a symmetric submodel $M(G)$ of $M[G]$. If $s : \omega \to \omega$ is a bijection, we define an automorphism $i_s : \mathbb{P} \to \mathbb{P}$ by stipulating:

$$i_s(p)(\langle n, k\rangle) = p(\langle s^{-1}(n), k\rangle)$$

Let $H = \{i_s : s \text{ is a permutation of } \omega\}$. H is a group of automorphisms of $\mathbb{P}$. For each $E \in \mathrm{fin}(\omega)$, let $\mathrm{fix}_H(E) = \{i_s \in H : s(n) = n \text{ for all } n \in E\}$. Let F be the filter on H generated by the subgroups $\{\mathrm{fix}_H(E) : E \in \mathrm{fin}(\omega)\}$. F is a normal filter.

Note that every automorphism i of $\mathbb{P}$ induces an automorpism of $\mathbb{P}$-names by the inductive definition: $i(\sigma) = \{\langle i(\tau), i(p)\rangle : \langle \tau, p\rangle \in \sigma\}$. For each permutation i of $\mathbb{P}$, we use the same symbol i for the corresponding automorpism of $\mathbb{P}$-names.

An $\underset{\sim}{x} \in M^{\mathbb{P}}$ (the class of $\mathbb{P}$-names in M) is said to be **symmetric** if $\mathrm{sym}_H(\underset{\sim}{x}) = \{i_s \in H : i_s(\underset{\sim}{x}) = \underset{\sim}{x}\} \in F$, i.e., there is an $E \in \mathrm{fin}(\omega)$ such that $\mathrm{fix}_H(E) \subseteq \mathrm{sym}_H(\underset{\sim}{x})$.

We inductively define $M^{\mathrm{HS}(\mathbb{P})}$, the class of all **hereditarily symmetric** names, as follows: $\underset{\sim}{x} \in M^{\mathrm{HS}(\mathbb{P})}$ if $\underset{\sim}{x}$ is symmetric and $\mathrm{Dom}(\underset{\sim}{x}) = \{\underset{\sim}{y} : \exists p \in \mathbb{P}\ (\langle \underset{\sim}{y}, p\rangle \in \underset{\sim}{x})\} \subseteq M^{\mathrm{HS}(\mathbb{P})}$.

Let $M(G) := \{\underset{\sim}{x}[G] : \underset{\sim}{x} \in M^{\mathrm{HS}(\mathbb{P})}\}$, which is the so-called basic Cohen model. $M(G)$ is a symmetric submodel of the generic extension and it is a model of ZF. A and all Cohen reals are in $M(G)$. The following two theorems of $M(G)$ are true:

Theorem 3.1 ([3, Proposition 17.1][6, Theorem 12.2]). *$M(G)$ is a countable transitive model of ZF. $M \subseteq M(G) \subseteq M[G]$.*

Theorem 3.2 ([3, pp. 386-388][6, Theorem 12.3]). *$M(G) \vDash \underset{\sim}{A}[G] \subseteq [\omega]^{\omega}$ and $M(G) \vDash$ "$\underset{\sim}{A}[G]$ is infinite". $M(G) \vDash$ "$\underset{\sim}{A}[G]$ is Dedekind-finite". In particular, there are infinite Dedekind-finite sets in $M(G)$, so $M(G)$ is not a model of the Axiom of Choice.*

Also note that we have the natural analogue of the forcing theorems for symmetric submodels. Let $\Vdash^{\mathrm{Sym}}$ denote the forcing relation.

3.2. **Some Properties of the Basic Cohen Model.**

Blass [1] proved in 1977 that Ramsey's theorem for pairs is false in the basic Cohen model. His proof uses some results from a paper by Halpern and Levy [4], which is a very old paper and does not use the modern simplified forcing language. In this subsection, we will present and prove these results in the modern forcing language.

Definition 3.3. For $E_1, E_2 \in \mathrm{fin}(\omega)$, $E_1 <^{\mathrm{fin}} E_2$ if $|E_1| < |E_2|$ or ($|E_1| = |E_2|$ and $\min(E_1 \triangle E_2) \in E_1$).

Fact 3.4. *$<^{fin}$ is a well-ordering on fin(ω).*

Definition 3.5. Let $\underset{\sim}{x} \in M^{\mathrm{HS}(\mathbb{P})}$ and let $E \in \mathrm{fin}(\omega)$. E is said to be a **support** of $\underset{\sim}{x}$ if $\mathrm{fix}_H(E) \subseteq \mathrm{sym}_H(\underset{\sim}{x})$. Let **support($\underset{\sim}{x}$)** be the least such E with respect to $<^{\mathrm{fin}}$.

Notation 3.6. For $p \in \mathbb{P}$ and $E \in \mathrm{fin}(\omega)$, let $p : E = p \restriction (E \times \omega)$.

Lemma 3.7 ([5, Lemma 5.24]). *Let $\varphi(x_1, x_2, ..., x_n)$ be a formula. If $\underset{\sim}{x}_1, \underset{\sim}{x}_2, ..., \underset{\sim}{x}_n \in M^{HS(\mathbb{P})}$ and if E is a support of each of the $\underset{\sim}{x}_1, \underset{\sim}{x}_2, ..., \underset{\sim}{x}_n$, then for each condition $p \in \mathbb{P}$, if $p \Vdash^{Sym} \varphi(\underset{\sim}{x}_1, \underset{\sim}{x}_2, ..., \underset{\sim}{x}_n)$ then $p : E \Vdash^{Sym} \varphi(\underset{\sim}{x}_1, \underset{\sim}{x}_2, ..., \underset{\sim}{x}_n)$.*

Lemma 3.8. *Suppose G is a $\mathbb{P}$-generic filter and $X \in M(G)$ is an infinite subset of A. Then there is an $a_n \in X$ and there is a first initial segment I of a_n such that $\{a_i \in A : I$ is a first initial segment of $a_i\} \subseteq X$.*

Proof. Let $\underset{\sim}{X}$ be a hereditarily symmetric name of X. Since X is infinite, there is an $k \in \omega - \operatorname{support}(\underset{\sim}{X})$ such that $a_k \in X$ because $\operatorname{support}(\underset{\sim}{X})$ is finite.

By forcing theorem, there is a $p \in G$ such that $p \Vdash^{\mathrm{Sym}} \underset{\sim}{a_k} \in \underset{\sim}{X}$. By Lemma 3.7, $p : (\operatorname{support}(\underset{\sim}{X}) \cup \{k\}) \Vdash^{\mathrm{Sym}} \underset{\sim}{a_k} \in \underset{\sim}{X}$. Let $p' = p : (\operatorname{support}(\underset{\sim}{X}) \cup \{k\})$.

Let $L : \omega \to 2$ be the characteristic function of a_k and let $l = \max(\{\max\{t : \langle k, t\rangle \in \operatorname{Dom}(p')\}\} \cup \{\min(a_k \triangle a_j) : j \in \operatorname{support}(\underset{\sim}{X})\}) + 1$. We will show that $L \restriction l$ is the first initial segment that we want.

If $a_i \in A$ and $L \restriction l$ is a first initial segment a_i, there are two cases:

(i) $i \in \operatorname{support}(\underset{\sim}{X})$. Note that $l > \min(a_k \triangle a_i)$, so $\min(a_k \triangle a_i) \in \operatorname{Dom}(L \restriction l)$, which implies that $\min(a_k \triangle a_i) \in a_k$ iff $\min(a_k \triangle a_i) \in a_i$. But this is a contradiction, so this case is impossible.

(ii) $i \notin \operatorname{support}(\underset{\sim}{X})$. Let s be the permutation of ω mapping i to k, k to i, and fixing everything else. Since $p' \Vdash^{\mathrm{Sym}} \underset{\sim}{a_k} \in \underset{\sim}{X}$, we have $i_s(p') \Vdash^{\mathrm{Sym}} i_s(\underset{\sim}{a_k}) \in i_s(\underset{\sim}{X})$. Note that $i_s(\underset{\sim}{a_k}) = \underset{\sim}{a_i}$ and $i_s(\underset{\sim}{X}) = \underset{\sim}{X}$. $i_s(p') = (p' : \operatorname{support}(\underset{\sim}{X})) \cup \{\langle i, j\rangle : \langle k, j\rangle \in p'\}$. $(p' : \operatorname{support}(\underset{\sim}{X})) \in G$ because $p' \in G$. Since $L \restriction l$, where $l > \max\{t : \langle k, t\rangle \in \operatorname{Dom}(p')\}$, is a first initial segment of a_i, we have $\langle i, j\rangle \in G$ iff $\langle k, j\rangle \in G$ for all $j \leq \max\{t : \langle k, t\rangle \in \operatorname{Dom}(p')\}$, so $\{\langle i, j\rangle : \langle k, j\rangle \in p'\} \in G$. Thus $i_s(p') \in G$, which concludes that $a_i \in X$.

Therefore we have $\{a_i \in A : I \text{ is a first initial segment of } a_i\} \subseteq X$. $\square$

Remark 3.9. The proof of Lemma 3.8 shows that for every infinite subset X of A, the set of all such a_n's is co-finite, which is in fact the complement of the the least support of X (a support for a value [6, Definition 10.8] is defined in [5, p. 72] and the existence of a least support is proved in [5, Lemma 5.22]).

3.3. **Ramsey's Theorem in the Basic Cohen Model.**

In this subsection we will slightly modify Blass' proof [1] to show that Ramsey's theorem for n-tuples for any $n \geq 2$ fails in the basic Cohen model.

Theorem 3.10. *Suppose G is a $\mathbb{P}$-generic filter. Ramsey's theorem is false in $M(G)$.*

Proof. Note: this proof is an analogue to that of [1, Theorem 1].

Fix $n \geq 2$. For any $x \in [A]^n$, let $\triangle x = \{k \in \omega : \exists y_1, y_2 \in x(k \in y_1 \wedge k \notin y_2)\}$. Let π be a 2-coloring on $[A]^n$ defined by:

$$\pi : [A]^n \to 2$$
$$x \mapsto \begin{cases} 0 & \text{if } \min(\triangle x) \text{ is even} \\ 1 & \text{otherwise} \end{cases}$$

If there is an infinite subset H of A that is homogeneous for π, then *without loss of generality*, suppose $\pi(x) = 0$ for all $x \in [H]^n$. Since H is an infinite subset of A, by Lemma 3.8, there is an $a_m \in H$ and there is a first initial segment I of a_m such that $\{a_i \in A : I$ is a first initial segment of $a_i\} \subseteq H$.

(i) If $\mathrm{Dom}(I)$ is odd, then by an easy density argument, there are an $a_k \supseteq I \cup \{\langle \mathrm{Dom}(I), 0\rangle\}$ and an $a_{k'} \supseteq I \cup \{\langle \mathrm{Dom}(I), 1\rangle\}$. Let z be an n-element subset of $\{a_i \in A : I$ is a first initial segment of $a_i\}$ (which is infinite by an easy density argument) with $a_k, a_{k'} \in z$. $z \in [H]^n$ and $\min(\triangle z) = \mathrm{Dom}(I)$, which is odd. This is a contradiction.

(ii) If $\mathrm{Dom}(I)$ is even, then by an easy density argument, there are an $a_k \supseteq I \cup \{\langle \mathrm{Dom}(I), 0\rangle, \langle \mathrm{Dom}(I) + 1, 0\rangle\}$ and an $a_{k'} \supseteq I \cup \{\langle \mathrm{Dom}(I), 0\rangle, \langle \mathrm{Dom}(I) + 1, 1\rangle\}$. Let z be an n-element subset of $\{a_i \in A : I \cup \{\langle \mathrm{Dom}(I), 0\rangle\}$ is a first initial segment of $a_i\}$ (which is infinite by an easy density argument) with $a_k, a_{k'} \in z$. $z \in [H]^n$ and $\min(\triangle z) = \mathrm{Dom}(I) + 1$, which is odd. This is a contradiction.

Therefore π does not have a homogeneous set. □

3.4. **Open Ramsey Theorem in the Basic Cohen Model.**

In this subsection we will prove that open Ramsey theorem fails in the basic Cohen model by giving an example.

Definition 3.11. Suppose $a_i, a_j \in A$, we say that $a_i < a_j$ if $\min(a_i \triangle a_j) \in a_j$.

Theorem 3.12. *Suppose G is a $\mathbb{P}$-generic filter. Open Ramsey theorem is false in M(G).*

Proof. Let $S = \{X \in 2^A :$ there are $a_{n_1} < a_{n_2} < a_{n_3} \in A$ such that $a_{n_1}, a_{n_3} \in X$ and $a_{n_2} \notin X\}$. For any $X \in S$, fix $a_{n_1} < a_{n_2} < a_{n_3}$ as stipulated. Let $L : A \to 2$ be the characteristic function of X and let $I = L \restriction \{a_{n_1}, a_{n_2}, a_{n_3}\}$. Every $X' \in 2^A$ whose characteristic function is an extension of I is in S by definition. Thus S is open.

Suppose S is Ramsey. Then there is an infinite subset M of A such that $[M]^\infty \subseteq S$ or $[A]^\infty - S$. Since M is an infinite subset of A, then by Lemma 3.8, there is an $a_n \in M$ and a first initial segment I_n of a_n such that $\{a_i \in A : I_n$ is a first initial segment of $a_i\} \subseteq M$.

(i) If $[M]^\infty \subseteq S$, then let $M' = \{a_i \in A : I_n$ is a first initial segment of $a_i\}$. M' is an infinite subset of M by an easy density argument. If there are $a_{m_1} < a_{m_2} < a_{m_3} \in A$ such that $a_{m_1}, a_{m_3} \in M'$, then $\min(a_{m_1} \triangle a_{m_2}) \in a_{m_2}$ and $\min(a_{m_2} \triangle a_{m_3}) \in a_{m_3}$. If $\min(a_{m_2} \triangle a_{m_3}) \in \mathrm{Dom}(I_n)$, then since I_n is a first initial segment of both a_{m_1} and a_{m_3}, we have $a_{m_1} \cap \mathrm{Dom}(I_n) = a_{m_3} \cap \mathrm{Dom}(I_n)$, which implies that $\min(a_{m_1} \triangle a_{m_2}) = \min(a_{m_2} \triangle a_{m_3}) \in a_{m_3} \cap \mathrm{Dom}(I_n) = a_{m_1} \cap \mathrm{Dom}(I_n) \subseteq a_{m_1}$. This is a contradiction. So $\min(a_{m_2} \triangle a_{m_3}) \notin \mathrm{Dom}(I_n)$, which means that $a_{m_2} \cap \mathrm{Dom}(I_n) = a_{m_3} \cap \mathrm{Dom}(I_n)$. I_n is a first initial segment of a_{m_2}, so $a_{m_2} \in M'$. Thus, $M' \notin S$, which is a contradiction.

(ii) If $[M]^\infty \subseteq [A]^\infty - S$, then let $M' = \{a_i \in A : I_n$ is a first initial segment of $a_i\}$. Fix an $a_m \in A$ such that $I_n \cup \{\langle \mathrm{Dom}(I_n), 0\rangle, \langle \mathrm{Dom}(I_n) + 1, 1\rangle\}$ is a first initial segment of a_m. The existence of a_m is guaranteed by an easy density argument. Let $M'' = M' - \{a_m\}$. M'' is an infinite subset of M. Fix an $a_l \in A$ such that $I_n \cup \{\langle \mathrm{Dom}(I_n), 0\rangle, \langle \mathrm{Dom}(I_n) + 1, 0\rangle\}$ is a first initial segment of a_l and fix an $a_r \in A$ such that $I_n \cup \{\langle \mathrm{Dom}(I_n), 1\rangle, \langle \mathrm{Dom}(I_n) + 1, 1\rangle\}$ is a first initial segment of a_r. $a_l < a_m < a_r$. $a_l, a_r \in M''$ and $a_m \notin M''$. Thus $M'' \in S$, which is a contradiction.

Therefore S is not Ramsey. □

4. The Basic Fraenkel Model

In this section, we will first give a brief introduction to the basic Fraenkel model. Then we will present some properties of the basic Fraenkel model, which will be used to investigate Ramsey's theorem and open Ramsey theorem in this model. For readers who are not familiar with ZFA, please refer to [3, Chapter 8].

4.1. **Introduction to the Basic Fraenkel Model.**

This subsection is a summary of [3, Chapter 8.1 & 8.2.1].

Let A, the set of atoms, be a countably infinite set and let M be the union of the cumulative hierarchy with respect to A defined as:

$$\begin{aligned} M_0 &:= A; \\ M_{\alpha+1} &:= \mathcal{P}(M_\alpha); \\ M_\alpha &:= \bigcup_{\beta<\alpha} M_\beta \text{ if } \alpha \text{ is a limit ordinal;} \\ \text{and } M &:= \bigcup_{\alpha<\Omega} M_\alpha \end{aligned}$$

M is a transitive model of ZFA. Let H be the group of all permutations of A. For each $E \in \text{fin}(A)$, let $\text{fix}_H(E) = \{i \in H : i(a) = a \text{ for all } a \in E\}$. Let F be the filter on H generated by the subgroups $\{\text{fix}_H(E) : E \in \text{fin}(A)\}$. F is a normal filter.

Note that every permutation i of A induces an $\in$-automorphism of M by the inductive definition:

$$\begin{aligned} i : M &\to M \\ x &\mapsto \begin{cases} \varnothing & \text{if } x = \varnothing \\ i(x) & \text{if } x \in A \\ \{i(y) : y \in x\} & \text{otherwise} \end{cases} \end{aligned}$$

For each permutation i of A, we use the same symbol i for the corresponding $\in$-automorphism of M.

An $x \in M$ is said to be **symmetric** if $\text{sym}_H(x) = \{i \in H : i(x) = x\} \in F$, i.e., there is an $E \in \text{fin}(A)$ such that $\text{fix}_H(E) \subseteq \text{sym}_H(x)$.

An $x \in M$ is said to be **hereditarily symmetric** if x and every element in the transitive closure of x is symmetric. Notice that for every $x \in M$ and every $i \in H$, x is hereditarily symmetric iff $i(x)$ is hereditarily symmetric.

Let V_{F_0} be the class of all hereditarily symmetric objects in M. V_{F_0} is called the basic Fraenkel model. It is a permutation model and a model of ZFA. A and all atoms are in V_{F_0}. The following theorem of V_{F_0} is true:

Theorem 4.1 ([3, Proposition 8.3])**.** *Let $m = |A|$. Then $V_{F_0} \vDash \aleph_0 \not\leq m$. In particular, there are infinite Dedekind-finite sets in V_{F_0}, so V_{F_0} is not a model of the Axiom of Choice.*

4.2. **Some Properties of the Basic Fraenkel Model.**

In this subsection, we will present some useful results about the basic Fraenkel model.

Definition 4.2. Let $x \in V_{F_0}$ and let $E \in \text{fin}(A)$. E is said to be a **support** of x if $\text{fix}_H(E) \subseteq \text{sym}_H(x)$.

Definition 4.3. Let $\hat{V}_{F_0} := \bigcup_{\alpha\in\Omega} \mathcal{P}^\alpha(\varnothing)$ be the the union of the cumulative hierarchy built from $\varnothing$. $\hat{V}_{F_0}$, which is a submodel of V_{F_0}, is a model of ZF and is called the **kernel** of V_{F_0}.

Fact 4.4 ([3, Fact 8.1])**.** *For any $x \in \hat{V}_{F_0}$ and $\pi \in H$, $\pi(x) = x$.*

Remark 4.5. Definition 4.3 and Fact 4.4 work in general for all permutation models.

Lemma 4.6 ([3, Lemma 8.2])**.** *Let $S \subseteq A$ in V_{F_0} and let $E \in$ fin(A) be a support of S. Then S is either finite or co-finite. Furthermore, if S is finite, then $S \subseteq E$, and if S is co-finite, then $A - S \subseteq E$.*

Lemma 4.7 ([1, Lemma])**.** *Let $x \in V_{F_0}$ be a non-well-orderable set. Then there is an infinite subset x' of x and an infinite subset S of A such that there is bijection from x' to S.*

4.3. **Ramsey's Theorem in the Basic Fraenkel Model.**

Blass [1] proved that Ramsey's theorem for pairs holds in the basic Fraenkel model. In this subsection we will slightly modify Blass' proof to show that the full Ramsey's theorem also holds.

Theorem 4.8. *Ramsey's theorem is true in V_{F_0}.*

Proof. Note: this proof is an analogue to that of [1, Theorem 2].

Suppose X is an infinite set in V_{F_0} and $n, r \in \omega - \{0\}$. Fix $\pi : [X]^n \to r$. We want to show that there is an infinite subset H of X such that H is homogeneous for π.

(i) If X is well-orderable, then by Remark 2.3, we are done.

(ii) If X is non-well-orderable, then by Lemma 4.7, there is an infinite subset X' of X and an infinite subset S of A such that there is bijection f from X' to S. Let π' be an r-coloring on $[S]^n$ defined as:

$$\pi' : [S]^n \to r$$
$$x \mapsto \pi(f^{-1}[x])$$

Let $E \in \text{fin}(A)$ be a support of π'. We will show that $S - E$ is homogeneous for π'. Fix $S_1, S_2 \in [S - E]^n$. Let $i : A \to A$ be a permutation of A mapping S_1 to S_2 bijectively and fixing everything else. Suppose $\pi'(S_1) = r_0 \in r$; then $\langle S_1, r_0\rangle \in \pi'$. Since i is an $\in$-automorphism of M, $i(\langle S_1, r_0\rangle) \in i(\pi')$. $i(\langle S_1, r_0\rangle) = \langle i(S_1), i(r_0)\rangle$ by the definition of i; $i(S_1) = S_2$ because i maps S_1 to S_2 bijectively; $i(r_0) = r_0$ because of Fact 4.4 and the fact that r_0 is in the kernel of V_{F_0}; $i(\pi') = \pi'$ because i fixes E. As a result, $\langle S_2, r_0 \rangle \in \pi'$, so $\pi'(S_2) = r_0 = \pi'(S_1)$. Thus $S - E$ is homogeneous for π'.

Let $H = f^{-1}[S - E]$. Since f maps X' to S bijectively, H is an infinite subset of $X' \subseteq X$. Fix $H_1, H_2 \in [H]^n$; $f[H_1], f[H_2] \in [S - E]^n$. Since $S - E$ is homogeneous for π', we have $\pi'(f[H_1]) = \pi'(f[H_2])$. $\pi'(f[H_1]) = \pi'(f[H_2]) \Rightarrow \pi(f^{-1}[f[H_1]]) = \pi(f^{-1}[f[H_2]]) \Rightarrow \pi(H_1) = \pi(H_2)$. Thus, H is homogeneous for π. □

4.4. **Open Ramsey Theorem in the Basic Fraenkel Model.**

In this subsection we will show that open Ramsey theorem holds in the basic Fraenkel model.

Theorem 4.9. *Open Ramsey theorem is true in V_{F_0}.*

Proof. Suppose X is an infinite set in V_{F_0}. Let Y be an open subset of 2^X. We want to show that Y is Ramsey over X, i.e., there is an infinite subset M of X such that $[M]^\infty \subseteq Y$ or $[X]^\infty - Y$.

(i) If X is well-orderable, then by Remark 2.13, we are done.

(ii) If X is non-well-orderable, then by Lemma 4.7, there is an infinite subset X' of X and an infinite subset S of A such that there is bijection f from X' to S. Let $Y' = Y \cap 2^{X'}$. Let $P = \{f[y] : y \in Y'\} \subseteq 2^S$. We will show that P is Ramsey over S.

Let $E \in \text{fin}(A)$ be a support of P. If $[S - E]^\infty \subseteq [S]^\infty - P$, we are done. So suppose $[S - E]^\infty \nsubseteq [S]^\infty - P$; then there is an $x \in [S - E]^\infty \cap P$. Since f is bijective, $f^{-1}[x]$ is an infinite subset of $f^{-1}[S - E] \subseteq X'$ and is also in $\{f^{-1}[y] : y \in P\} = Y' \subseteq Y$. Let $L : X \to 2$ be the characteristic function of $f^{-1}[x]$. Since Y is open, there is a finite subfunction l of L such that every $y \in 2^X$ whose characteristic function from X to 2 is an extension of l is in Y. Note that $l^{-1}(1)$ is a finite subset of $f^{-1}[x]$. Let $I = f[l^{-1}(1)]$. Let $n = |l^{-1}(1)| = |I|$.

Fix $x' \in [x]^\infty$ and fix $I' \in [x']^n$. Let $i : A \to A$ be a permutation of A mapping I to I' bijectively and fixing everything else. $i(x') = (x' - I') \cup I$. Since $I = f[l^{-1}(1)]$, we have $l^{-1}(1) \subseteq f^{-1}[I] \subseteq f^{-1}[i(x')]$. Since $x' - I' \subseteq x$ and $I \subseteq x$, we have $i(x') \subseteq x$ and $f^{-1}[i(x')] \subseteq f^{-1}[x]$, which implies that $l^{-1}(0) \cap f^{-1}[i(x')] = \varnothing$. Thus the characteristic function of $f^{-1}[i(x')]$ extends l, so $f^{-1}[i(x')] \in Y$. Moreover, since $f^{-1}[i(x')] \subseteq \text{Dom}(f) = X'$, we have $f^{-1}[i(x')] \in 2^{X'}$. Consequently, $f^{-1}[i(x')] \in Y \cap 2^{X'} = Y'$. Thus $i(x') = f[f^{-1}[i(x')]] \in \{f[y] : y \in Y'\} = P$.

Since i^{-1} is an $\in$-automorphism of M, we have $i^{-1}(i(x')) \in i^{-1}(P)$. $i^{-1}(i(x')) = x'$; $i^{-1}(P) = P$ because i^{-1} fixes E. Thus $x' \in P$, which implies that $[x]^\infty \subseteq P$. Therefore P is Ramsey over S.

Fix an $M_0 \in [S]^\infty$ such that $[M_0]^\infty \subseteq P$ or $[S]^\infty - P$. Let $M = f^{-1}[M_0]$.

If $[M_0]^\infty \subseteq P$, then fix $m \in [M]^\infty$. $f[m] \in [M_0]^\infty \subseteq P$, so $m = f^{-1}[f[m]] \in \{f^{-1}[y] : y \in P\} = Y'$. Thus $[M]^\infty \subseteq Y' \subseteq Y$.

If $[M_0]^\infty \subseteq [S]^\infty - P$, then fix $m \in [M]^\infty$. $f[m] \in [M_0]^\infty \subseteq [S]^\infty - P$, so $f[m] \notin P$, i.e., $f[m] \neq f[y]$ for some $y \in Y'$. It follows that $m \notin Y'$. Since $M \subseteq \mathrm{Dom}(f) = X'$, we have $m \in [M]^\infty \subseteq 2^{X'}$, so $m \notin Y - 2^{X'} = Y - Y'$. Thus $m \notin Y$. Consequently, $[M]^\infty \subseteq [X']^\infty - Y \subseteq [X]^\infty - Y$.

Therefore, $[M]^\infty \subseteq Y$ or $[X]^\infty - Y$. Y is Ramsey over X. □

5. The Ordered Mostowski Model

In this section, we will first give a brief introduction to the ordered Mostowski model. Then we will prove some properties of the ordered Mostowski model, which will be used to investigate Ramsey's theorem and open Ramsey theorem in this model.

5.1. **Introduction to the Ordered Mostowski Model.**

This subsection is a summary of [3, Chapter 8.3].

Let A, the set of atoms, consist of a countably infinite set together with an ordering $<^M$ such that A is order-isomorphic to $\mathbb{Q}$. Let M be the union of the cumulative hierarchy with respect to A. Let H be the group of all order-preserving permutations of A. The rest of the construction (including the construction of the normal filter F on H and the definitions of symmetry and hereditary symmetry) is exactly the same as what we did for the basic Fraenkel model.

Let V_M be the corresponding permutation model, the so-called ordered Mostowski model. V_M is a model of ZFA. A and all atoms are in V_M. The following results of V_M are true:

Lemma 5.1 ([3, Lemma 8.10])**.** *The set $R = \{\langle a_1, a_2\rangle \ : a_1 <^M a_2\} \subseteq A \times A$ belongs to V_M.*

Theorem 5.2 ([3, Lemma 8.13])**.** *Let $m = |A|$. Then $V_M \vDash \aleph_0 \not\leq 2^m$. In particular, there are infinite sets in V_M whose power sets are Dedekind-finite, so V_M is not a model of the Axiom of Choice.*

5.2. **Some Properties of the Ordered Mostowski Model.**

In this subsection, we will present and prove some useful results about the ordered Mostowski model.

Notation 5.3. For our convenience, let ∞^A and $-\infty^A$ be two symbols such that $-\infty^A <^M a <^M \infty^A$ for all $a \in A$. For $a_1 <^M a_2$, let $(a_1, a_2) := \{a \in A : a_1 <^M a <^M a_2\}$ and we call (a_1, a_2) an **interval of atoms**. For $s \in \text{fin}(A) - \{\varnothing\}$, let $\max_A(s)$ and $\min_A(s)$, respectively, be the maximum and the minimum elements of s with respect to $<^M$.

Lemma 5.4 ([5, p. 47]). *For every $x \in V_M$, $V_M \vDash$ (x can be well-ordered) iff $\text{fix}_H(x) = \{i \in H : i(y) = y$ for all $y \in x\} \in F$.*

Remark 5.5 ([5, p. 47]). Lemma 5.4 holds for every permutation model.

Lemma 5.6 ([3, Lemma 8.11]).
(a) For $x \in V_M$, if E_1 and E_2 are supports of x, then so is $E_1 \cap E_2$.
(b) Every $x \in V_M$ has a least support.
(c) The class of all pairs $\langle x, E\rangle$, where $x \in V_M$ and E is the least support of x, is symmetric.

Lemma 5.7. *Suppose $x \in V_M$ is a non-well-orderable set. Then there is an infinite subset x' of x and an infinite subset S of A such that there is bijection from x' to S.*

Proof. Note: the structure of this proof is an analogue to that of the proof of [1, Lemma].

Let $E \in \text{fin}(A)$ be a support of x. Since x is not well-orderable, by Lemma 5.4, $\text{fix}_H(x) \notin F$, so $\text{fix}_H(E) \nsubseteq \text{fix}_H(x)$. As a result, there is an i in $\text{fix}_H(E)$ but not $\text{fix}_H(x)$, and there is a $y \in x$ such that $i(y) \neq y$. Thus, E is not a support of y. By Lemma 5.6 (b), let $F \cup \{a_0\}$ be the least support of y with $a_0 \in A - (E \cup F)$.

Let $a_1 = \max_A\{a \in E \cup F : a <^M a_0\}$ or $-\infty^A$ if $\{a \in E \cup F : a <^M a_0\} = \varnothing$ and let $a_2 = \min_A\{a \in E \cup F : a_0 <^M a\}$ or ∞^A if $\{a \in E \cup F : a_0 <^M a\} = \varnothing$. Note that $a_0 \in (a_1, a_2)$

Let $f = \{\langle \pi(a_0), \pi(y)\rangle : \pi \in \text{fix}_H(E \cup F)\}$. For each $\pi' \in \text{fix}_H(E \cup F)$, $\pi'(f) = \{\langle \pi'(\pi(a_0)), \pi'(\pi(y))\rangle : \pi \in \text{fix}_H(E \cup F)\} = f$, so $\text{fix}_H(E \cup F) \subseteq \text{sym}_H(f)$ and as a result, $f \in V_M$. We will show that f is an injective function from (a_1, a_2) to x.

We first show that f is a function. If $\pi_1, \pi_2 \in \text{fix}_H(E \cup F)$ and $\pi_1(a_0) = \pi_2(a_0)$, then $\pi_1^{-1}\pi_2(a_0) = \pi_1^{-1}\pi_1(a_0) = a_0$, which implies that $\pi_1^{-1}\pi_2 \in \text{fix}_H(E \cup F \cup \{a_0\})$. Consequently, $\pi_1^{-1}\pi_2(y) = y \Rightarrow \pi_2(y) = \pi_1(y)$. Thus f is a function.

We then show that $\text{Dom}(f) = (a_1, a_2)$. *Without loss of generality*, suppose $a_1 \neq -\infty$ and $a_2 \neq \infty$. For each $\pi \in \text{fix}_H(E \cup F)$, since $a_0 \in (a_1, a_2)$ and π is order-preserving, we have $\pi(a_0) \in (\pi(a_1), \pi(a_2))$, where $(\pi(a_1), \pi(a_2)) = (a_1, a_2)$ because $a_1, a_2 \in E \cup F$. For each $a \in (a_1, a_2)$, let π_a be an order-preserving permutation of A mapping (a_1, a_0) to

(a_1, a), a_0 to a, (a_0, a_2) to (a, a_2), and fixing everything else; $\pi_a \in \text{fix}_H(E \cup F)$ and $\pi_a(a_0) = a$. Thus, $\text{Dom}(f) = (a_1, a_2)$.

Next we show that Range(f) $\subseteq x$. For each $\pi \in \text{fix}_H(E \cup F)$, $\pi(y) \in \pi(x) = x$ because π is an $\in$-automorphism of M, $y \in x$, and $\pi \in \text{fix}_H(E) \subseteq \text{sym}_H(x)$.

Finally it remains to show that f is injective. Suppose not; then there are $\pi_1, \pi_2 \in \text{fix}_H(E \cup F)$ such that $\pi_1(a_0) \neq \pi_2(a_0)$ and $\pi_1(y) = \pi_2(y)$. *Without loss of generality*, suppose $\pi_1(a_0) <^M \pi_2(a_0)$

Let $\pi^* = \pi_1^{-1}\pi_2$. $\pi^*(y) = \pi_1^{-1}\pi_2(y) = \pi_1^{-1}\pi_1(y) = y$. Let $\pi^*(a_0) = b_0$. Note that since $\pi_1 \in \text{fix}_H(E \cup F)$, so is π_1^{-1}. $\pi_1(a_0) <^M \pi_2(a_0) \Rightarrow \pi_1^{-1}\pi_1(a_0) <^M \pi_1^{-1}\pi_2(a_0) \Rightarrow a_0 <^M \pi^*(a_0) = b_0$. Since π_1^{-1} and π_2 both fixes $E \cup F$ and a_2 either is in $E \cup F$ or is ∞, $a_0 <^M a_2 \Rightarrow b_0 = \pi_1^{-1}\pi_2(a_0) <^M a_2$. Fix $t \in (a_0, a_2)$. Let π_t^* be an order-preserving permutation of A mapping (a_0, b_0) to (a_0, t), b_0 to t, (b_0, a_2) to (t, a_2), and fixing everything else. $\pi_t^* \in \text{fix}_H(E \cup F \cup \{a_0\})$. $\pi_1, \pi_2 \in \text{fix}_H(E \cup F)$ implies that $\pi^* = \pi_1^{-1}\pi_2 \in \text{fix}_H(E \cup F)$, so $\langle \pi^*(a_0), \pi^*(y) \rangle \in f$. Since π_t^* is an $\in$-automorphism of M, we have $\pi_t^*(\langle \pi^*(a_0), \pi^*(y) \rangle) \in \pi_t^*(f)$. $\pi_t^*(\langle \pi^*(a_0), \pi^*(y) \rangle) = \pi_t^*(\langle b_0, y \rangle) = \langle \pi_t^*(b_0), \pi_t^*(y) \rangle = \langle t, y \rangle$ because π_t^* maps b_0 to t and fixes $F \cup \{a_0\}$; $\pi_t^*(f) = f$ because π_t^* fixes $E \cup F$. So $\langle t, y \rangle \in f$. Thus, f is constant on (a_0, a_2) with value y.

Let $\pi^{*\prime} = \pi_2^{-1}\pi_1$. By a similar argument, f is constant on (a_1, a_0) with value y.

Let π_{id} be the identity function on A. $\pi_{id} \in \text{fix}_H(E \cup F)$ so $\langle a_0, y \rangle = \langle \pi_{id}(a_0), \pi_{id}(y) \rangle \in f$. Therefore we have that f is constant on its entire domain, which is (a_1, a_2), with value y. As a result, we have $\pi(y) = y$ for all $\pi \in \text{fix}_H(E \cup F)$, which means that $E \cup F$ is a support of y. $E \cup F$ and $F \cup \{a_0\}$ are two supports of y so by Lemma 5.6 (a), $(E \cup F) \cap (F \cup \{a_0\})$ is a support of y. Note that $a_0 \notin E \cup F$, so $(E \cup F) \cap (F \cup \{a_0\}) \subseteq F$ and this contradicts the assumption that $F \cup \{a_0\}$ is the least support of y. Therefore f is injective, and f is a bijection from $\text{Dom}(f) = (a_1, a_2)$, which is an infinite subset of A, to Range(f), which is an infinite subset of x. ☐

Lemma 5.8. *For every $x \in V_M$, if x is a subset of A, then there is a finite union U of intervals of atoms and a finite subset T of A such that $x = U \cup T$.*

Proof. Suppose $x \in V_M$ is a subset of A. Let $E = \{a_1 <^M a_1 <^M ... <^M a_n\}$ be a support of x. For our convenience, let $a_0 = -\infty^A$ and $a_{n+1} = \infty^A$.

Fix $0 \leq i \leq n$. If $x \cap (a_i, a_{i+1}) \neq \varnothing$, then there is a $y \in x \cap (a_i, a_{i+1})$. Fix $y' \in (a_i, a_{i+1})$. Let π be an order-preserving permutation of A mapping (a_i, y) to (a_i, y'), y to y', (y, a_{i+1}) to (y', a_{i+1}), and fixing everything else. Since $y \in x$ and π is an $\in$-automorphism of M, we have $\pi(y) \in \pi(x)$. $\pi(y) = y'$ because π maps y to y'; $\pi(x) = x$ because $\pi \in \text{fix}_H(E)$. So $y' \in x$ and as a result, $(a_i, a_{i+1}) \subseteq x$. Thus either $(a_i, a_{i+1}) \subseteq x$ or $x \cap (a_i, a_{i+1}) = \varnothing$.

Therefore, $x = \bigcup\{(a_i, a_{i+1}) : 0 \leq i \leq n \text{ and } x \cap (a_i, a_{i+1}) \neq \varnothing\} \cup \{a_i : 1 \leq i \leq n \text{ and } a_i \in x\}$. □

5.3. **Ramsey's Theorem in the Ordered Mostowski Model.**

In this subsection we will show that Ramsey's theorem holds in the ordered Mostowski model. The proof will be similar to that of Theorem 4.8.

Theorem 5.9. *Ramsey's theorem is true in V_M.*

Proof. Note: the structure of this proof is an analogue to that of the proof of [1, Theorem 2].

Suppose X is an infinite set in V_M and $n, r \in \omega - \{0\}$. Fix $\pi : [X]^n \to r$. We want to show that there is an infinite subset H of X such that H is homogeneous for π.

(i) If X is well-orderable, then by Remark 2.3, we are done.

(ii) If X is non-well-orderable, then by Lemma 5.7, there is an infinite subset X' of X and an infinite subset S of A such that there is bijection f from X' to S. Let π' be an r-coloring on $[S]^n$ defined as:

$$\pi' : [S]^n \to r$$
$$x \mapsto \pi(f^{-1}[x])$$

Let $E \in \text{fin}(A)$ be a support of π'. We will show that $(\max_A(E), \infty^A)$ is homogeneous for π'. Fix $S_1, S_2 \in [(\max_A(E), \infty^A)]^n$. Suppose $S_1 = \{a_1 <^M a_2 <^M ... <^M a_n\}$ and $S_2 = \{a'_1 <^M a'_2 <^M ... <^M a'_n\}$. For our convenience, let $a_0 = a'_0 = \max_A(E)$ and let $a_{n+1} = a'_{n+1} = \infty^A$. Let $i : A \to A$ be an order-preserving permutation of A mapping (a_i, a_{i+1}) to (a'_i, a'_{i+1}) for all $0 \leq i \leq n$, a_i to a'_i for all $1 \leq i \leq n$, and fixing everything else.

Suppose $\pi'(S_1) = r_0 \in r$; then $\langle S_1, r_0\rangle \in \pi'$. Since i is an $\in$-automorphism of M, $i(\langle S_1, r_0\rangle) \in i(\pi')$. $i(\langle S_1, r_0\rangle) = \langle i(S_1), i(r_0)\rangle$ by the definition of i; $i(S_1) = S_2$ because i maps S_1 to S_2 bijectively; $i(r_0) = r_0$ because of Remark 4.5 and the fact that r_0 is in the kernel of V_M; $i(\pi') = \pi'$ because i fixes E. As a result, $\langle S_2, r_0\rangle \in \pi'$, so $\pi'(S_2) = r_0 = \pi'(S_1)$. Thus $(\max_A(E), \infty^A)$ is homogeneous for π'.

Let $H = f^{-1}[(\max_A(E), \infty^A)]$. Since f maps X' to S bijectively, H is an infinite subset of $X' \subseteq X$. Fix $H_1, H_2 \in [H]^n$; $f[H_1], f[H_2] \in [(\max_A(E), \infty^A)]^n$. Since $(\max_A(E), \infty^A)$ is homogeneous for π', we have $\pi'(f[H_1]) = \pi'(f[H_2])$. $\pi'(f[H_1]) = \pi'(f[H_2]) \Rightarrow \pi(f^{-1}[f[H_1]]) = \pi(f^{-1}[f[H_2]]) \Rightarrow \pi(H_1) = \pi(H_2)$. Thus, H is homogeneous for π. □

5.4. **Open Ramsey Theorem in the Ordered Mostowski Model.**

In this subsection we will show that open Ramsey theorem fails in the ordered Mostowski model by giving an example. The example will be similar to that of Theorem 3.12.

Theorem 5.10. *Open Ramsey theorem is false in V_M.*

Proof. Let $S = \{X \in 2^A :$ there are $a_{n_1} <^M a_{n_2} <^M a_{n_3} \in A$ such that $a_{n_1}, a_{n_3} \in X$ and $a_{n_2} \notin X\}$. For any $X \in S$, fix $a_{n_1} <^M a_{n_2} <^M a_{n_3}$ as stipulated. Let $L : A \to 2$ be the characteristic function of X and let $I = L \restriction \{a_{n_1}, a_{n_2}, a_{n_3}\}$. Every $X' \in 2^A$ whose characteristic function is an extension of I is in S by definition. Thus S is open.

Suppose S is Ramsey. Then there is an infinite subset H of A such that $[H]^\infty \subseteq S$ or $[A]^\infty - S$. H is a subset of A, so by Lemma 5.8, there is a finite union U of intervals of atoms and a finite subset T of A such that $H = U \cup T$. Since H is infinite, U cannot be $\varnothing$ so there must be $a_1 <^M a_2$ such that $(a_1, a_2) \subseteq H$.

(i) If $[H]^\infty \subseteq S$, then let $H' = (a_1, a_2)$. $H' \in [H]^\infty$ and if there are $a_{m_1} <^M a_{m_2} <^M a_{m_3} \in A$ such that $a_{m_1}, a_{m_3} \in H'$, then $a_1 <^M a_{m_1} <^M a_{m_2} <^M a_{m_3} <^M a_2$, which means that $a_{m_2} \in H'$. So $H' \notin S$, which is a contradiction.

(ii) If $[H]^\infty \subseteq [A]^\infty - S$, then fix $a_m \in (a_1, a_2)$ and let $H' = (a_1, a_m) \cup (a_m, a_2)$. Fix $a_l \in (a_1, a_m)$ and $a_r \in (a_m, a_2)$. We have $a_l <^M a_m <^M a_r$; $a_l, a_r \in H'$; $a_m \notin H'$. So $H' \in S$, which is a contradiction.

Therefore S is not Ramsey. □

Corollary 5.11. *Theorem 5.9 and 5.10 show that Ramsey's theorem does not imply open Ramsey theorem in ZFA.*

6. Possible Future Work

This paper proves some consistency and independence results about Ramsey's theorem and open Ramsey theorem. There are a number of other problems and conjectures related to these two theorems that are still remaining open. For example, whether Ramsey's theorem implies open Ramsey theorem in ZF and whether open Ramsey theorem implies Ramsey's theorem in ZF or ZFA. Also, it is an open problem whether Ramsey's theorem for pairs implies Ramsey's theorem for triples in ZF, and even the converse is open.

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