

Global Decay for Medium Initial Data in the Inhomogeneous Muskat Problem

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Abstract

The inhomogeneous Muskat problem models the dynamics of an interface between two fluids of differing characteristics inside a non-uniform porous medium. In this work we assume the existence of global in time solutions to the inhomogeneous Muskat problem and we prove that medium sized initial data will experience global decay to an equilibrium. We rely on a family of Fourier norms $\|\cdot\|_{\mathcal{F}^{s,1}}$, previously introduced in [1], to demonstrate the long time behavior of the fluid interface.

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1 Introduction

The inhomogeneous Muskat problem models the dynamics of two incompressible, immiscible fluids in a non-uniform, porous medium. It is an extension of the classical Muskat problem which has been established and thoroughly studied in [1, 2, 3, 4]. In this paper we consider the two-dimensional case with fluids of identical constant viscosity, with a piecewise constant permeability, and with no surface tension. As is typical, we begin with Darcy's law which describes the flow of fluid within porous media

$$\frac{\mu}{\kappa}v = -\nabla p - g(0, \rho).$$

Above, μ is viscosity, κ is permeability, v is velocity, p is pressure, g is acceleration due to gravity, and ρ is fluid density. Let us parameterize the smooth, simple interface between the two fluids as

$$z(\alpha, t) = (z_1(\alpha, t), z_2(\alpha, t)) \quad \text{for } \alpha \in \mathbb{R}, t \in \mathbb{R}_{\geq 0},$$

and the smooth, simple interface between the distinct permeable regions as

$$h(\alpha) = (h_1(\alpha), h_2(\alpha)) \quad \text{for } \alpha \in \mathbb{R}.$$

Suppose that initially $z(\alpha, 0) \cap h(\alpha)$ is empty. Then, for $t \in [0, T)$ the two interfaces divide the plane into three connected regions $\Omega_1(t), \Omega_2(t)$, and Ω_3 . The physical quantities (μ, ρ) satisfy

$$(\mu, \rho)(x, t) = \begin{cases} (\mu_1, \rho_1) & \text{if } x \in \Omega_1(t) \\ (\mu_2, \rho_2) & \text{if } x \in \Omega_2(t) \cup \Omega_3 \end{cases}$$

and κ satisfies

$$\kappa(x) = \begin{cases} \kappa_1 & \text{if } x \in \Omega_1(t) \cup \Omega_2(t) \\ \kappa_2 & \text{if } x \in \Omega_3 \end{cases}$$

where we note that $\Omega_1(t) \cup \Omega_2(t) = \mathbb{R}^2 \setminus \Omega_3$ is independent of time.

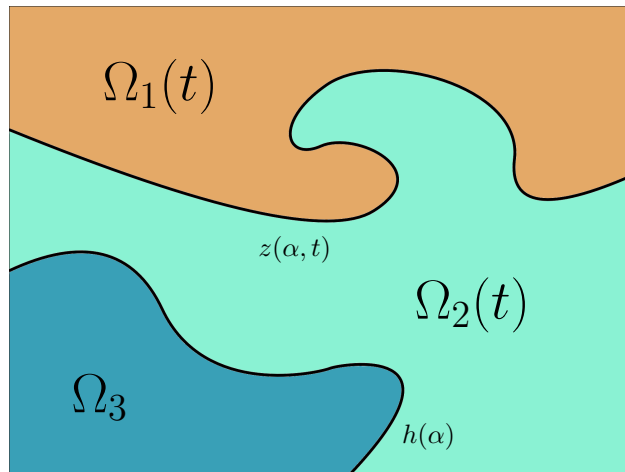


Figure 1: Inhomogenous Muskat Problem.

Darcy's law implies the vorticity ($\omega = \nabla \times v$) is limited to the interfaces,

$$\omega(x, t) = \omega_1(\alpha, t)\delta(x - z(\alpha, t)) + \omega_2(\alpha, t)\delta(x - h(\alpha)).$$

Then, assuming incompressibility ($\nabla \cdot v = 0$), Biot-Savart law dictates

$$\begin{aligned} v(x, t) &= BR(\omega_1, z)(x, t) + BR(\omega_2, h)(x, t) \\ &:= \frac{1}{2\pi} \text{p.v.} \int_{\mathbb{R}} \frac{(x - z(\beta, t))^\perp}{|x - z(\beta, t)|^2} \omega_1(\beta, t) d\beta + \frac{1}{2\pi} \text{p.v.} \int_{\mathbb{R}} \frac{(x - h(\beta))^\perp}{|x - h(\beta)|^2} \omega_2(\beta, t) d\beta. \end{aligned}$$

This velocity expression is instrumental to characterizing our main object of study, the fluid interface $z(\alpha, t)$. We refer to [5] for the derivation of the following system

$$\begin{aligned} \partial_t z(\alpha, t) &= BR(\omega_1, z)(z(\alpha, t), t) + BR(\omega_2, h)(z(\alpha, t), t) + c(\alpha, t) \partial_\alpha z(\alpha, t), \\ \omega_1(\alpha, t) &= -2 \frac{\mu_2 - \mu_1}{\mu_2 + \mu_1} (BR(\omega_1, z)(z(\alpha, t), t) + BR(\omega_2, h)(z(\alpha, t), t)) \cdot \partial_\alpha z(\alpha, t) \\ &\quad - 2\kappa_1 \frac{\rho_2 - \rho_1}{\mu_1 + \mu_2} g \partial_\alpha z_2(\alpha, t), \\ \omega_2(\alpha, t) &= -2 \frac{\kappa_1 - \kappa_2}{\kappa_1 + \kappa_2} (BR(\omega_1, z)(h(\alpha), t) + BR(\omega_2, h)(h(\alpha), t)) \cdot \partial_\alpha h(\alpha). \end{aligned}$$

The tangential term $c(\alpha, t)$ is chosen so that $\partial_\alpha z_1 \equiv 1$ implies $\partial_t z_1 \equiv 0$. It serves as a computational convenience which does not impact any physical behavior.

From this point onward we will write $f(\alpha)$ instead of $f(\alpha, t)$ wherever there is no danger of confusion. Enforcing the assumptions $\mu_1 = \mu_2 = 1$ and $\rho_2 > \rho_1$ allows us to untangle the implicit equation for ω_1 and ensures our system meets the Rayleigh-Taylor criterion and is therefore in the stable regime. Next, following [5, 6] by assuming $z(\alpha) = (\alpha, f(\alpha))$ and $h(\alpha) = (\alpha, -h_2)$ for $h_2 > 0$, we simplify the system into

$$\partial_t f(\alpha) = I_1(\alpha) + I_2(\alpha) \tag{1}$$

for

$$I_1(\alpha) = \frac{\rho}{\pi} \text{p.v.} \int_{\mathbb{R}} \frac{\beta (\partial_\alpha f(\alpha) - \partial_\alpha f(\alpha - \beta))}{\beta^2 + (f(\alpha) - f(\alpha - \beta))^2} d\beta, \tag{2}$$

$$I_2(\alpha) = \frac{1}{2\pi} \text{p.v.} \int_{\mathbb{R}} \frac{\beta + \partial_\alpha f(\alpha)(f(\alpha) + h_2)}{\beta^2 + (f(\alpha) + h_2)^2} \omega_2(\alpha - \beta) d\beta. \tag{3}$$

in which

$$\omega_2(x) = \frac{2}{\pi} \mathcal{K} \rho \text{p.v.} \int_{\mathbb{R}} \frac{f(x - \gamma) + h_2}{\gamma^2 + (f(x - \gamma) + h_2)^2} \partial_\alpha f(x - \gamma) d\gamma, \tag{4}$$

$$\mathcal{K} = \frac{\kappa_1 - \kappa_2}{\kappa_1 + \kappa_2}, \quad \rho = \frac{\kappa_1(\rho_2 - \rho_1)g}{2}.$$

Observe that $\rho > 0$, $|\mathcal{K}| < 1$. Finally, we assume the interfaces begin neither too close, nor too far apart,

$$0 < m \leq f(\alpha, 0) + h_2 \leq \|f(\alpha, 0) + h_2\|_{L^\infty} \leq \|f(\alpha, 0)\|_0 + h_2 < M < \infty, \tag{5}$$

where $\|f\|_0$ is defined in the following section.

In section 2 we outline our result that for a suitably small initial condition, the fluid interface will decay to an equilibrium $f \equiv 0$. In section 3 we characterize a Fourier transform pertinent to the estimates in section 4.

2 Global in time decay of medium sized f

In this paper, we aim to study the long-time behavior of solutions to the inhomogeneous Muskat problem described in the previous section.

To do so, we introduce the family of Fourier semi-norms

$$\|g\|_{\mathcal{F}^{s,p}} = \left(\int_{\mathbb{R}} |\xi|^{2sp} |\hat{g}(\xi)|^p d\xi \right)^{\frac{1}{p}}, \quad s \in \mathbb{R}, p \in [1, \infty) \quad (6)$$

and the shorthand

$$\|g\|_s = \|g\|_{\mathcal{F}^{s,1}}.$$

In the following theorems, we assume that $f(\alpha, t)$ is a global in time solution to (1) with initial data of medium size in the semi-norm $\mathcal{F}^{0,1} \cap \mathcal{F}^{2,1}$ (we do not currently have an argument to demonstrate global in time solutions for this class of initial data, it is a work in progress). Our estimate in section 4 shows

Lemma 2.1. *Fix $s = 0$ or $s = 1$ and let $\sigma_s(t)$ be defined as in (10), (11). Suppose $f_0 \in \mathbf{H}^3(\mathbb{R}) \cap \mathcal{F}^{0,1}(\mathbb{R}) \cap \mathcal{F}^{2,1}(\mathbb{R})$ such that $\sigma_s(0) > 0$ and (5) holds. Then there exists $T > 0$ so that the solution $f \in C([0, T]; \mathbf{H}^3(\mathbb{R}) \cap \mathcal{F}^{0,1}(\mathbb{R}) \cap \mathcal{F}^{1,1}(\mathbb{R}))$ satisfies*

$$\partial_t \|f\|_s(t) \leq -\sigma_s(t) \|f\|_{s+1}(t), \quad \text{for } t \in [0, T].$$

We will also need the Decay Lemma proved in [7], which we have restated for our setting below.

Lemma 2.2 (Decay Lemma). *Suppose $\|g\|_{s_1}(t) \leq C_0$ and*

$$\partial_t \|g\|_{s_2}(t) \leq -C \|g\|_{s_2+1}(t)$$

such that $s_1 < s_2$. Then

$$\|g\|_{s_2}(t) \lesssim (1+t)^{s_1-s_2}.$$

The above results imply the large time decay of solutions to the inhomogeneous Muskat problem.

Theorem 2.3. *Suppose $f_0 \in \mathbf{H}^3(\mathbb{R}) \cap \mathcal{F}^{0,1}(\mathbb{R}) \cap \mathcal{F}^{2,1}(\mathbb{R})$ such that $\sigma_s(0) > 0$ and (5) holds. Then the solution $f \in C([0, T]; \mathbf{H}^3(\mathbb{R}) \cap \mathcal{F}^{0,1}(\mathbb{R}) \cap \mathcal{F}^{1,1}(\mathbb{R}))$ decays as follows,*

$$\|f\|_1(t) \lesssim (1+t)^{-1}.$$

Proof. By lemma 2.1, for $s = 0$ and $s = 1$, the quantity $\|f\|_s$ is non-increasing on $[0, T]$. Combining the two estimates, equations (10), (11) support

$$\partial_t \|f\|_s(t) \leq -\sigma_s(0) \|f\|_{s+1}(t), \quad t \in [0, T].$$

Since it is additionally true that

$$\|f\|_{L^\infty} \leq \|f\|_0, \quad \|\partial_\alpha f\|_{L^\infty} \leq \|f\|_1,$$

we know f will continue to satisfy (1), (5) and $\|f\|_s(t)$ will remain small enough so that $\sigma_s(t) > 0$ for all $t \in \mathbb{R}_{\geq 0}$. Then we have the global in time inequalities

$$\partial_t \|f\|_0 \leq -\sigma_0(0) \|f\|_1, \quad \partial_t \|f\|_1 \leq -\sigma_1(0) \|f\|_2.$$

These conditions are sufficient to invoke the Decay Lemma, and so the proof is complete. ■

3 Determining \hat{I}_2 and $L(\hat{I}_2)$

Throughout this section we will use the following notation which arises naturally from the bound (5) and binomial expansions.

$$\begin{aligned}
\eta &= (r, s, t, u) & C_\eta &= \binom{r}{s} \binom{s}{t} \binom{2s-2t}{u} \frac{(-1)^{s+1}}{2rM^{2s}} h_2^{2s-2t-u} \\
\theta &= (r, s) & C_\theta &= \binom{2r+1}{s} (-1)^r h_2^{2r+1-s} \\
\lambda &= (n, k, p, q) & C_\lambda &= C_\eta \text{ whenever } \lambda = \eta \\
\nu &= (n, k) & C_\nu &= C_\theta \text{ whenever } \nu = \theta \\
\chi &= \chi_{[-M, M]} & & \\
\chi^c &= \chi_{\mathbb{R} \setminus [-M, M]} & &
\end{aligned}$$

Proposition 3.1. Fix $f \in C([0, T]; \mathbf{H}^3(\mathbb{R}) \cap \mathcal{F}^{0,1}(\mathbb{R}) \cap \mathcal{F}^{1,1}(\mathbb{R}))$, a solution to (1) such that (5) is true. Then,

$$\begin{aligned}
\hat{I}_2(\xi) &= \frac{1}{2\pi} \left[\sum_\lambda C_\lambda \int_{\mathbb{R}} \widehat{f^q}(\xi - \xi_1) \widehat{\omega}_2(\xi_1) (2p \mathcal{F}(\beta^{2p-1} \chi)(\xi_1) + i(\xi - \xi_1) \mathcal{F}(\beta^{2p} \chi)(\xi_1)) d\xi_1 \right. \\
&\quad + \sum_\nu C_\nu \int_{\mathbb{R}} \widehat{f^k}(\xi - \xi_1) \widehat{\omega}_2(\xi_1) \mathcal{F}(\frac{1}{\beta^{2n+1}} \chi^c)(\xi_1) d\xi_1 \\
&\quad \left. + \sum_\nu C_\nu \int_{\mathbb{R}} \left(\frac{i(\xi - \xi_1)}{k+2} \widehat{f^{k+2}}(\xi - \xi_1) + \frac{i(\xi - \xi_1) h_2}{k+1} \widehat{f^{k+1}}(\xi - \xi_1) \right) \widehat{\omega}_2(\xi_1) \mathcal{F}(\frac{1}{\beta^{2n+2}} \chi^c)(\xi_1) d\xi_1 \right], \tag{7}
\end{aligned}$$

where

$$\widehat{\omega}_2(\xi) = \frac{2}{\pi} \mathcal{K} \rho \left[\sum_\eta C_\eta i \xi \widehat{f^u}(\xi) \mathcal{F}(\gamma^{2t} \chi)(\xi) + \sum_\theta C_\theta \frac{i \xi}{s+1} \widehat{f^{s+1}}(\xi) \mathcal{F}(\frac{1}{\gamma^{2r+2}} \chi^c)(\xi) \right].$$

Proposition 3.2. The linear term in (7) is

$$L(\hat{I}_2)(\xi) = \mathcal{K} \rho |\xi| \hat{f}(\xi) e^{-2h_2|\xi|}. \tag{8}$$

The computation justifying proposition 3.1 is split over sections 3.3 and 3.4. Below we explain proposition 3.2. It is more straightforward to find the linear term in physical coordinates and take the Fourier transform afterwards. Looking to (4), we have

$$L(\omega_2)(x) = \frac{2}{\pi} \mathcal{K} \rho \text{ p.v.} \int_{\mathbb{R}} \frac{h_2}{\gamma^2 + h_2^2} \partial_x f(x - \gamma) d\gamma.$$

Then, analyzing (3),

$$L(I_2)(x) = \frac{1}{2\pi} \text{ p.v.} \int_{\mathbb{R}} \frac{\beta}{\beta^2 + h_2^2} L(\omega_2)(\alpha - \beta) d\beta.$$

We conclude

$$\begin{aligned}
L(\hat{I}_2)(\xi) &= \frac{1}{\pi^2} \mathcal{K} \rho \mathcal{F}\left(\left(\frac{x}{x^2+h_2^2} * \frac{h_2}{x^2+h_2^2} * \partial_\alpha f\right)(\alpha)\right)(\xi) \\
&= \frac{1}{\pi^2} \mathcal{K} \rho \mathcal{F}\left(\frac{x}{x^2+h_2^2}\right)(\xi) \mathcal{F}\left(\frac{h_2}{x^2+h_2^2}\right)(\xi) \mathcal{F}(\partial_\alpha f)(\xi) \\
&= \frac{1}{\pi^2} \mathcal{K} \rho (-i\pi \operatorname{sgn}(\xi) e^{-h_2|\xi|}) (\pi e^{-h_2|\xi|}) (i\xi \hat{f}(\xi)) \\
&= \mathcal{K} \rho |\xi| \hat{f}(\xi) e^{-2h_2|\xi|}.
\end{aligned}$$

3.3 Determining $\hat{\omega}_2$

We split (4) into two parts and take the transform of each part separately,

$$\omega_2(x) = \frac{2}{\pi} \mathcal{K} \rho(A_1(x) + A_2(x)).$$

We see

$$\begin{aligned} \hat{A}_1(\xi) &= \int_{\mathbb{R}} \text{p.v.} \int_{|\gamma| < M} \frac{f(x-\gamma) + h_2}{\gamma^2 + (f(x-\gamma) + h_2)^2} \partial_x f(x-\gamma) e^{-i\xi x} d\gamma dx \\ &= \text{p.v.} \int_{|\gamma| < M} \int_{\mathbb{R}} \frac{1}{2} \partial_x \log(1 + (\gamma^2/M^2 + (f(x-\gamma) + h_2)^2/M^2 - 1)) e^{-i\xi x} dx d\gamma. \end{aligned}$$

Then, since $|\gamma^2/M^2 + (f(x-\gamma) + h_2)^2/M^2 - 1| < 1$, we can make use of $\log(1+y) = -\sum_r \frac{(-y)^r}{r}$.

$$\hat{A}_1(\xi) = \text{p.v.} \int_{|\gamma| < M} \int_{\mathbb{R}} -\frac{1}{2} \partial_x \sum_{r=1}^{\infty} \frac{1}{r} (1 - \gamma^2/M^2 - (f(x-\gamma) + h_2)^2/M^2)^r e^{-i\xi x} dx d\gamma.$$

Taking a binomial expansion, we observe that the derivative annihilates any term with $u = 0$. Consequently,

$$\begin{aligned} \hat{A}_1(\xi) &= \text{p.v.} \int_{|\gamma| < M} -i\xi \sum_{r=1}^{\infty} \sum_{s=0}^r \sum_{t=0}^s \sum_{u=1}^{2s-2t} \binom{r}{s} \binom{s}{t} \binom{2s-2t}{u} \frac{(-1)^s}{2rM^{2s}} \gamma^{2t} h_2^{2s-2t-u} \widehat{\tau_{\gamma} f^u}(\xi) d\gamma \\ &= \sum_{\eta} C_{\eta} i\xi \widehat{f^u}(\xi) \mathcal{F}(\gamma^{2t} \chi)(\xi) \end{aligned}$$

Next, using $\frac{y}{1+y^2} = \sum_n (-1)^n y^{2n+1}$,

$$\begin{aligned} \hat{A}_2(\xi) &= \int_{\mathbb{R}} \int_{|\gamma| > M} \frac{f(x-\gamma) + h_2}{\gamma^2 + (f(x-\gamma) + h_2)^2} \partial_x f(x-\gamma) e^{-i\xi x} d\gamma dx \\ &= \int_{|\gamma| > M} \sum_{r=0}^{\infty} (-1)^r \int_{\mathbb{R}} \left(\frac{f(x-\gamma) + h_2}{\gamma} \right)^{2r+1} \frac{\partial_x f(x-\gamma)}{\gamma} e^{-i\xi x} dx d\gamma \\ &= \sum_{r=0}^{\infty} \sum_{s=0}^{2r+1} \binom{2r+1}{s} h_2^{2r+1-s} (-1)^r (\widehat{f^s} * \widehat{\partial_x f})(\xi) \int_{|\gamma| > M} \frac{1}{\gamma^{2r+2}} e^{-i\xi \gamma} d\gamma \\ &= \sum_{\theta} C_{\theta} \frac{i\xi}{s+1} \widehat{f^{s+1}}(\xi) \mathcal{F}\left(\frac{1}{\gamma^{2r+2}} \chi^c\right)(\xi) \end{aligned}$$

3.4 \hat{I}_2 in terms of $\hat{\omega}_2$

Similarly, splitting (3) into two parts,

$$I_2(\alpha) = \frac{1}{2\pi} (B_1(\alpha) + B_2(\alpha)).$$

Then, using the same Taylor series and binomial expansion as for \hat{A}_1 ,

$$\begin{aligned}
\hat{B}_1(\xi) &= \int_{\mathbb{R}} \text{p.v.} \int_{|\beta| < M} \frac{1}{2} (\partial_\beta + \partial_\alpha) \log(1 + (\beta^2/M^2 + (f(\alpha) + h_2)^2/M^2 - 1)) \omega_2(\alpha - \beta) e^{-i\xi\alpha} d\beta d\alpha \\
&= \text{p.v.} \int_{|\beta| < M} \sum_{n=1}^{\infty} \sum_{k=0}^n \sum_{p=0}^k \sum_{q=0}^{2k-2p} \binom{n}{k} \binom{k}{p} \binom{2k-2p}{q} \frac{(-1)^{k+1}}{2nM^{2k}} h_2^{2k-2p-q} \beta^{2p-1} ((2p + i\xi\beta) \widehat{f^q} * \widehat{\tau_\beta \omega_2})(\xi) d\beta \\
&= \sum_{\lambda} C_{\lambda} \text{p.v.} \int_{|\beta| < M} \beta^{2p-1} ((2p + i\xi\beta) \widehat{f^q} * \widehat{\tau_\beta \omega_2})(\xi) d\beta \\
&= \sum_{\lambda} C_{\lambda} \int_{\mathbb{R}} \widehat{f^q}(\xi - \xi_1) \widehat{\omega_2}(\xi_1) (2p \mathcal{F}(\beta^{2p-1} \chi)(\xi_1) + i(\xi - \xi_1) \mathcal{F}(\beta^{2p} \chi)(\xi_1)) d\xi_1.
\end{aligned}$$

And analogously to \hat{A}_2 ,

$$\begin{aligned}
\hat{B}_2(\xi) &= \int_{\mathbb{R}} \int_{|\beta| > M} \frac{\beta + \partial_\alpha f(\alpha)(f(\alpha) + h_2)}{\beta^2 + (f(\alpha) + h_2)^2} \omega_2(\alpha - \beta) e^{-i\xi\alpha} d\beta d\alpha \\
&= \int_{|\beta| > M} \int_{\mathbb{R}} \frac{1 + \partial_\alpha f(\alpha) \left(\frac{f(\alpha) + h_2}{\beta} \right)}{1 + \left(\frac{f(\alpha) + h_2}{\beta} \right)^2} \frac{\omega_2(\alpha - \beta)}{\beta} e^{-i\xi\alpha} d\alpha d\beta \\
&= \int_{|\beta| > M} \int_{\mathbb{R}} (1 + \partial_\alpha f(\alpha) \left(\frac{f(\alpha) + h_2}{\beta} \right)) \sum_{n=0}^{\infty} \frac{(-1)^n}{\beta^{2n+1}} (f(\alpha) + h_2)^{2n} \omega_2(\alpha - \beta) e^{-i\xi\alpha} d\alpha d\beta \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{2n} \binom{2n}{k} (-1)^n h_2^{2n-k} \int_{|\beta| > M} \frac{1}{\beta^{2n+1}} ((\widehat{f^k} + \frac{1}{\beta} \widehat{f^{k+1}} * \widehat{\partial_\alpha f} + \frac{h_2}{\beta} \widehat{f^k} * \widehat{\partial_\alpha f}) * \widehat{\tau_\beta \omega_2})(\xi) d\beta \\
&= \sum_{\nu} C_{\nu} \int_{|\beta| > M} \frac{1}{\beta^{2n+1}} ((\widehat{f^k} + \frac{1}{\beta} \widehat{f^{k+1}} * \widehat{\partial_\alpha f} + \frac{h_2}{\beta} \widehat{f^k} * \widehat{\partial_\alpha f}) * \widehat{\tau_\beta \omega_2})(\xi) d\beta \\
&= \sum_{\nu} C_{\nu} \int_{\mathbb{R}} \widehat{f^k}(\xi - \xi_1) \widehat{\omega_2}(\xi_1) \mathcal{F}(\frac{1}{\beta^{2n+1}} \chi^c)(\xi_1) d\xi_1 \\
&\quad + \sum_{\nu} C_{\nu} \int_{\mathbb{R}} \left(\frac{i(\xi - \xi_1)}{k+2} \widehat{f^{k+2}}(\xi - \xi_1) + \frac{i(\xi - \xi_1) h_2}{k+1} \widehat{f^{k+1}}(\xi - \xi_1) \right) \widehat{\omega_2}(\xi_1) \mathcal{F}(\frac{1}{\beta^{2n+2}} \chi^c)(\xi_1) d\xi_1.
\end{aligned}$$

4 Estimating $\partial_t \|f\|_s$

Well,

$$\begin{aligned}
\partial_t \|f\|_s &= \partial_t \int_{\mathbb{R}} |\xi|^s |\hat{f}(\xi)| d\xi = \int_{\mathbb{R}} |\xi|^s \partial_t (\hat{f}(\xi) \overline{\hat{f}(\xi)})^{1/2} d\xi \\
&= \int_{\mathbb{R}} |\xi|^s \frac{1}{2} \frac{\partial_t \hat{f}(\xi) \overline{\hat{f}(\xi)} + \hat{f}(\xi) \partial_t \overline{\hat{f}(\xi)}}{|\hat{f}(\xi)|} d\xi = \int_{\mathbb{R}} |\xi|^s \text{Re} \left(\overline{\text{sgn}(\hat{f}(\xi))} \widehat{\partial_t f}(\xi) \right) d\xi,
\end{aligned}$$

where we take the convention $\text{sgn}(z) = z/|z|$ for $z \in \mathbb{C}$. Due to [3] and proposition 3.2 we have

$$\begin{aligned}
\partial_t f(\alpha) &= I_1(\alpha) + I_2(\alpha) \\
&= \rho(-\Lambda f(\alpha) + N_1(\alpha)) + (L(I_2)(\alpha) + N_2(\alpha))
\end{aligned}$$

where Λ is the square root Laplacian, and N_1 is nonlinear in f . Furthermore, the following estimates are proven in, or follow from work in [3].

$$\begin{aligned}\|N_1\|_0 &\leq \|f\|_1 \frac{2\|f\|_1}{1 - \|f\|_1^2}, \\ \|N_1\|_1 &\leq \|f\|_2 \frac{2\|f\|_1^2(3 - \|f\|_1^2)}{(1 - \|f\|_1^2)^2}.\end{aligned}\tag{9}$$

So,

$$\begin{aligned}\partial_t \|f\|_s &= \rho \int_{\mathbb{R}} |\xi|^s \operatorname{Re} \left(\overline{\operatorname{sgn}(\hat{f}(\xi))} (-\widehat{\Lambda}f(\xi) + \rho^{-1}L(\hat{I}_2)(\xi) + \hat{N}_1(\xi) + \rho^{-1}\hat{N}_2(\xi)) \right) d\xi \\ &\leq \rho \left(-\|f\|_{s+1} + \mathcal{K} \int_{\mathbb{R}} |\xi|^{s+1} |\hat{f}(\xi)| e^{-2h_2|\xi|} d\xi + \|N_1\|_s + \rho^{-1}\|N_2\|_s \right).\end{aligned}$$

We will show

$$\partial_t \|f\|_s(t) \leq -\sigma_s(t) \|f\|_{s+1}(t)$$

where $\sigma_s(t)$ depends on $\operatorname{sgn}(\mathcal{K})$.

Case 1 ($\mathcal{K} < 0$):

In this case the term $\mathcal{K} \int_{\mathbb{R}} |\xi|^{s+1} |\hat{f}(\xi)| e^{-2h_2|\xi|} d\xi$ aids the decay of the interface, yet it can be arbitrarily small. We must show that $\|N_1\|_s + \rho^{-1}\|N_2\|_s$ is bounded by $\|f\|_{s+1}$ for some space of initial conditions. Here,

$$\sigma_s(t) = \rho \left(1 - \|f\|_{s+1}^{-1} \|N_1\|_s - \sum_{n=1}^{\infty} (a_n + b_n \|f\|_1) \|f\|_0^n \right).\tag{10}$$

Case 2 ($\mathcal{K} > 0$):

In this case the term $\mathcal{K} \int_{\mathbb{R}} |\xi|^{s+1} |\hat{f}(\xi)| e^{-2h_2|\xi|} d\xi$ hinders the decay of the interface. Its magnitude is bounded above by $\mathcal{K}\|f\|_{s+1}$, so we must show $\|N_1\|_s + \rho^{-1}\|N_2\|_s$ is bounded by $(1 - \mathcal{K})\|f\|_{s+1}$ for some space of initial conditions. (Note $\mathcal{K} < 1$). Here,

$$\sigma_s(t) = \rho \left(1 - \mathcal{K} - \|f\|_{s+1}^{-1} \|N_1\|_s - \sum_{n=1}^{\infty} (a_n + b_n \|f\|_1) \|f\|_0^n \right).\tag{11}$$

We observe that for $\|f_0\|_0$ and $\|f_0\|_1$ small enough,

$$\sigma_s(0) > 0$$

By continuity in time, we will then have

$$\partial_t \|f\|_s(t) \leq -\sigma_s(t) \|f\|_{s+1}(t) \leq 0, \quad \text{for } t \in [0, T].$$

Equations (10), (11) are justified by the following estimate on $\|N_2\|_s$.

4.1 Estimating $\rho^{-1}\|N_2\|_s$

We will estimate $\|N_2\|_1$ and argue that $\|N_2\|_0$ is perfectly analogous. Let us write

$$\hat{N}_2(\xi) = \hat{I}_2(\xi) - L(\hat{I}_2)(\xi) = \frac{1}{\pi^2} \mathcal{K} \rho \sum_{\ell=1}^{10} \hat{J}_\ell$$

where the \hat{J}_ℓ are described below. We aim to show that

$$\rho^{-1}\|N_2\|_1 \leq \frac{1}{\pi^2}|\mathcal{K}| \sum_{\ell=1}^{10} \|\xi \hat{J}_\ell\|_{L^1} \leq \|f\|_2 \sum_{n=1}^{\infty} (a_n + b_n \|f\|_1) \|f\|_0^n < \varepsilon \|f\|_2.$$

for sufficiently small $\|f\|_0$ and $\|f\|_1$ where $a_n, b_n \geq 0$. Throughout the following estimates we will use the notation $d_j = \|x^{-j}\chi^c\|_{L^1}$ and $D_j = \|x^j\chi\|_{L^1}$. We will also use the interpolation result $\|g\|_1^2 \leq \|g\|_2\|g\|_0$ and Young's inequality.

$$\begin{aligned} \|J_1\|_1 &= \left\| \xi \sum_{\lambda} \sum_{\eta} C_{\lambda} C_{\eta} \int_{\mathbb{R}} \widehat{f^q}(\xi - \xi_1) \xi_1 \widehat{f^u}(\xi_1) \mathcal{F}(\gamma^{2t}\chi)(\xi_1) 2p \mathcal{F}(\beta^{2p-1}\chi)(\xi_1) d\xi_1 \right\|_{L^1} \\ &\leq \sum_{\lambda} \sum_{\eta} |C_{\lambda} C_{\eta}| 2p D_{2p-1} D_{2t} u \|\xi(|\widehat{\partial_{\alpha} f}| * \widehat{f} * \dots * \widehat{f})\|_{L^1} \\ &\leq \sum_{\lambda} \sum_{\eta} |C_{\lambda} C_{\eta}| 2p D_{2p-1} D_{2t} u (\|\widehat{\partial_{\alpha}^2 f}\|_{L^1} \|\widehat{f}\|_{L^1}^{u+q-1} + (u+q-1) \|\widehat{\partial_{\alpha} f}\|_{L^1}^2 \|\widehat{f}\|_{L^1}^{u+q-2}) \\ &\leq \|f\|_2 \sum_{\lambda} \sum_{\eta} |C_{\lambda} C_{\eta}| 2p D_{2p-1} D_{2t} u(u+q) \|f\|_0^{u+q-1} \end{aligned}$$

Since we excised any linear part it must be that $u+q \geq 2$. Similar things will be true for every $\|J_\ell\|_1$, i.e. none of the final estimates will contain terms linear in f .

$$\begin{aligned} \|J_2\|_1 &= \left\| \xi \sum_{\lambda} \sum_{\eta} C_{\lambda} C_{\eta} \int_{\mathbb{R}} (\xi - \xi_1) \widehat{f^q}(\xi - \xi_1) \xi_1 \widehat{f^u}(\xi_1) \mathcal{F}(\gamma^{2t}\chi)(\xi_1) \mathcal{F}(\beta^{2p}\chi)(\xi_1) d\xi_1 \right\|_{L^1} \\ &\leq \sum_{\lambda} \sum_{\eta} |C_{\lambda} C_{\eta}| D_{2p} D_{2t} uq \|\xi(|\widehat{\partial_{\alpha} f}| * \widehat{\partial_{\alpha} f} * \widehat{f} * \dots * \widehat{f})\|_{L^1} \\ &\leq \|f\|_2 \sum_{\lambda} \sum_{\eta} |C_{\lambda} C_{\eta}| D_{2p} D_{2t} uq(u+q) \|f\|_1 \|f\|_0^{u+q-2} \end{aligned}$$

$$\begin{aligned} \|J_3\|_1 &= \left\| \xi \sum_{\lambda} \sum_{\theta} C_{\lambda} C_{\theta} \int_{\mathbb{R}} \widehat{f^q}(\xi - \xi_1) \frac{\xi_1}{s+1} \widehat{f^{s+1}} \mathcal{F}(\frac{1}{\gamma^{2r+2}}\chi^c)(\xi_1) 2p \mathcal{F}(\beta^{2p-1}\chi)(\xi_1) d\xi_1 \right\|_{L^1} \\ &\leq \|f\|_2 \sum_{\lambda} \sum_{\theta} |C_{\lambda} C_{\theta}| 2p D_{2p-1} d_{2r+2} (s+q+1) \|f\|_0^{s+q} \end{aligned}$$

$$\begin{aligned} \|J_4\|_1 &= \left\| \xi \sum_{\lambda} \sum_{\theta} C_{\lambda} C_{\theta} \int_{\mathbb{R}} (\xi - \xi_1) \widehat{f^q}(\xi - \xi_1) \frac{\xi_1}{s+1} \widehat{f^{s+1}} \mathcal{F}(\frac{1}{\gamma^{2r+2}}\chi^c)(\xi_1) \mathcal{F}(\beta^{2p}\chi)(\xi_1) d\xi_1 \right\|_{L^1} \\ &= \|f\|_2 \sum_{\lambda} \sum_{\theta} |C_{\lambda} C_{\theta}| D_{2p} d_{2r+2} (s+q+1) \|f\|_1 \|f\|_0^{s+q-1} \end{aligned}$$

$$\begin{aligned} \|J_5\|_1 &= \left\| \xi \sum_{\nu, n=1} \sum_{\eta} C_{\nu} C_{\eta} \int_{\mathbb{R}} \widehat{f^k}(\xi - \xi_1) \xi_1 \widehat{f^u}(\xi_1) \mathcal{F}(\gamma^{2t}\chi) \mathcal{F}(\frac{1}{\beta}\chi^c)(\xi_1) d\xi_1 \right\|_{L^1} \\ &\leq \|f\|_2 \sum_{\nu, n=1} \sum_{\eta} |C_{\nu} C_{\eta}| \pi D_{2t} u(u+k) \|f\|_0^{u+k-1} \end{aligned}$$

$$\begin{aligned} \|J_6\|_1 &= \left\| \xi \sum_{\nu, n=1} \sum_{\theta} C_{\nu} C_{\theta} \int_{\mathbb{R}} \widehat{f^k}(\xi - \xi_1) \frac{\xi_1}{s+1} \widehat{f^{s+1}}(\xi_1) \mathcal{F}\left(\frac{1}{\gamma^{2r+2}} \chi\right) \mathcal{F}\left(\frac{1}{\beta} \chi^c\right)(\xi_1) d\xi_1 \right\|_{L^1} \\ &\leq \|f\|_2 \sum_{\nu, n=1} \sum_{\theta} |C_{\nu} C_{\theta}| \pi d_{2r+2} (s+k+1) \|f\|_0^{s+k} \end{aligned}$$

$$\begin{aligned} \|J_7\|_1 &= \left\| \xi \sum_{\nu, n>1} \sum_{\eta} C_{\nu} C_{\eta} \int_{\mathbb{R}} \widehat{f^k}(\xi - \xi_1) \xi_1 \widehat{f^u}(\xi_1) \mathcal{F}(\gamma^{2t} \chi) \mathcal{F}\left(\frac{1}{\beta^{2n+1}} \chi^c\right)(\xi_1) d\xi_1 \right\|_{L^1} \\ &\leq \|f\|_2 \sum_{\nu, n>1} \sum_{\eta} |C_{\nu} C_{\eta}| d_{2n+1} D_{2t} u(u+k) \|f\|_0^{u+k-1} \end{aligned}$$

$$\begin{aligned} \|J_8\|_1 &= \left\| \xi \sum_{\nu, n>1} \sum_{\theta} C_{\nu} C_{\theta} \int_{\mathbb{R}} \widehat{f^k}(\xi - \xi_1) \frac{\xi_1}{s+1} \widehat{f^{s+1}}(\xi_1) \mathcal{F}\left(\frac{1}{\gamma^{2r+2}} \chi\right) \mathcal{F}\left(\frac{1}{\beta^{2n+1}} \chi^c\right)(\xi_1) d\xi_1 \right\|_{L^1} \\ &\leq \|f\|_2 \sum_{\nu, n>1} \sum_{\theta} |C_{\nu} C_{\theta}| d_{2n+1} d_{2r+2} (s+k+1) \|f\|_0^{s+k} \end{aligned}$$

$$\begin{aligned} \|J_9\|_1 &= \left\| \xi \sum_{\nu} \sum_{\eta} C_{\nu} C_{\eta} \int_{\mathbb{R}} \left(\frac{\xi - \xi_1}{k+2} \widehat{f^{k+2}}(\xi - \xi_1) + \frac{(\xi - \xi_1) h_2}{k+1} \widehat{f^{k+1}}(\xi - \xi_1) \right) \xi_1 \widehat{f^u}(\xi_1) \mathcal{F}(\gamma^{2t} \chi)(\xi_1) \mathcal{F}\left(\frac{1}{\beta^{2n+2}} \chi^c\right)(\xi_1) d\xi_1 \right\|_{L^1} \\ &\leq \|f\|_2 \sum_{\nu} \sum_{\eta} |C_{\nu} C_{\eta}| d_{2n+2} D_{2t} u \|f\|_1 \left((u+k+2) \|f\|_0^{u+k} + h_2(u+k+1) \|f\|_0^{u+k-1} \right) \end{aligned}$$

$$\begin{aligned} \|J_{10}\|_1 &= \left\| \xi \sum_{\nu} \sum_{\theta} C_{\nu} C_{\theta} \int_{\mathbb{R}} \left(\frac{\xi - \xi_1}{k+2} \widehat{f^{k+2}}(\xi - \xi_1) + \frac{(\xi - \xi_1) h_2}{k+1} \widehat{f^{k+1}}(\xi - \xi_1) \right) \right. \\ &\quad \cdot \left. \frac{\xi_1}{s+1} \widehat{f^{s+1}}(\xi_1) \mathcal{F}\left(\frac{1}{\gamma^{2r+2}} \chi^c\right)(\xi_1) \mathcal{F}\left(\frac{1}{\beta^{2n+2}} \chi^c\right)(\xi_1) d\xi_1 \right\|_{L^1} \\ &\leq \|f\|_2 \sum_{\nu} \sum_{\theta} |C_{\nu} C_{\theta}| d_{2r+2} d_{2n+2} \|f\|_1 \left((s+k+3) \|f\|_0^{s+k+1} + h_2(s+k+2) \|f\|_0^{s+k} \right) \end{aligned}$$

To see that these power series converge we must show that the coefficients in the summations do not grow too quickly. We will do so explicitly for the series associated with J_3 since it is especially generic.

$$\begin{aligned} &\sum_{\lambda} \sum_{\theta} |C_{\lambda} C_{\theta}| 2^p D_{2p-1} d_{2r+2} (s+q+1) \|f\|_0^{s+q} \\ &= \sum_{\lambda} \sum_{\theta} \binom{n}{k} \binom{k}{p} \binom{2k-2p}{q} \frac{h_2^{2k-2p-q}}{2nM^{2k}} \binom{2r+1}{s} h_2^{2r+1-s} 2^p \|x^{2p-1} \chi\|_{L^1} \|x^{-2r-2} \chi^c\|_{L^1} (s+q+1) \|f\|_0^{s+q} \\ &= \sum_{\lambda} \sum_{\theta} \binom{n}{k} \binom{k}{p} \binom{2k-2p}{q} \binom{2r+1}{s} \frac{h_2^{2k-2p-q} h_2^{2r+1-s}}{2nM^{2k}} \frac{2M^{2p} 2M^{-2r-1}}{2r+1} (s+q+1) \|f\|_0^{s+q} \\ &\leq \sum_{(n,k,r)} \binom{n}{k} \binom{k}{p} \frac{(\|f\|_0 + h_2)^{2k-2p}}{nM^{2k-2p}} \frac{2(\|f\|_0 + h_2)^{2r+1}}{(2r+1)M^{2r+1}} 2(r+k+1) \\ &= \sum_{(n,k,r)} \binom{n}{k} \binom{k}{p} \frac{4(r+k+1)}{n(2r+1)} \left(\frac{\|f\|_0 + h_2}{M} \right)^{2k-2p+2r+1}. \end{aligned}$$

The above converges whenever $\|f\|_0 + h_2 \ll M$, a bound we assumed in (5).

A Elementary definitions and lemmas

In the process of proving this decay estimate I came across many mathematical ideas which, although common knowledge to practitioners of parabolic PDEs, were sources of bewilderment for me. This section will discuss a few of these standard definitions and results in roughly the order that they appear in the paper.

A.1 Biot-Savart Law

A cursory perusal of Biot-Savart Law may leave one with the impression that it is an intrinsically electromagnetic result. When deriving the interface equation we use it in a more general setting as the solution to a system of equations.

Suppose we know some smooth vector field $\mathbf{v}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ satisfies

$$\begin{aligned}\nabla \times \mathbf{v} &= \mathbf{w}, \\ \nabla \cdot \mathbf{v} &= 0.\end{aligned}$$

for a given, rapidly decaying $\mathbf{w}: \mathbb{R}^3 \rightarrow \mathbb{R}^3$. We will solve for \mathbf{v} in terms of \mathbf{w} .

In Cartesian coordinates the vector Laplacian is defined as

$$\Delta \mathbf{v} = \nabla(\nabla \cdot \mathbf{v}) - \nabla \times (\nabla \times \mathbf{v}) = (\Delta \mathbf{v}_1, \Delta \mathbf{v}_2, \Delta \mathbf{v}_3).$$

This implies

$$\Delta \mathbf{v} = -\nabla \times \mathbf{w} \quad \implies \quad \Delta \mathbf{v}_k = -(\nabla \times \mathbf{w})_k, \quad k = 1, 2, 3.$$

So each \mathbf{v}_k is the solution to a Poisson equation.

By our assumption on the decay rate of \mathbf{w} , we additionally have that $-(\nabla \times \mathbf{w})$ decays rapidly. Then, classical theory says $\mathbf{v}_k = -(\Phi * (\nabla \times \mathbf{w})_k)$ is the unique (up to a constant), bounded solution of the system, where Φ is the fundamental solution to Laplace's equation.

A.2 Semi-Norms

A semi-norm has all the properties of a norm besides non-degeneracy.

Definition. A semi-norm $\|\cdot\|: S \rightarrow \mathbb{R}$, on an appropriate set of functions S , satisfies

1. $\|f\| \geq 0$ for every $f \in S$,
2. $\|cf\| = |c| \|f\|$ for $c \in \mathbb{R}$, $f \in S$,
3. $\|f + g\| \leq \|f\| + \|g\|$ for $f, g \in S$.

The Fourier semi-norms can be strengthened to norms if we choose S to eliminate degeneracy, or if we use equivalence classes to identify degenerate functions. The former is more common, for example,

$$\|\cdot\|_{\mathcal{F}^{1,1}}: L^2(\mathbb{R}) \rightarrow \mathbb{R}.$$

The latter can be accomplished as follows,

$$\|\cdot\|_{\mathcal{F}^{1,1}}: \mathcal{F}^{1,1}(\mathbb{R}) / \sim := \{f: \mathbb{R} \rightarrow \mathbb{R} \mid \|f\|_{\mathcal{F}^{1,1}} < \infty\} / \sim \rightarrow \mathbb{R}$$

where $f \sim g$ if $g(x) = f(x) + \chi(x)$ almost everywhere for $\partial_x \chi \equiv 0$ almost everywhere.

A.3 Fourier Inequalities and Identities

Note, we use the symbols $\mathcal{F}(f)$ and \hat{f} interchangeably for the Fourier transform. Similarly, $\mathcal{F}^{-1}(f)$ and \check{f} for the inverse Fourier transform.

A.3.1 Inequalities

A standard inequality is

$$\|\hat{g}\|_{L^\infty} \leq \|g\|_{L^1}.$$

When Fourier inversion is allowed, let $g = \hat{f}$ and then

$$\|\partial_\alpha f\|_{L^\infty} = \|\widehat{\partial_\alpha f}\|_{L^\infty} \leq \|\widehat{\partial_\alpha f}\|_{L^1} = \|f\|_1.$$

The above inequality was useful in proving the main result, similar statements are true for any $\|\cdot\|_s$.

In addition, we should prove the useful interpolation

$$\|g\|_1^2 \leq \|g\|_2 \|g\|_0.$$

Note, by Cauchy-Schwartz

$$\|g\|_1^2 = \left(\int_{\mathbb{R}} |\xi| |\hat{g}| d\xi \right)^2 = \left\langle |\xi| |\hat{g}|^{1/2}, |\hat{g}|^{1/2} \right\rangle^2 \leq \|\xi \hat{g}^{1/2}\|_{L^2}^2 \|\hat{g}^{1/2}\|_{L^2}^2 = \|g\|_2 \|g\|_0.$$

A.3.2 Fractional Laplacian

The square root Laplacian $\Lambda = (-\Delta)^{1/2}$ can be defined in many ways. We provide two useful definitions,

$$\widehat{\Lambda u} = |\xi| \hat{u},$$

and

$$(\Lambda u)(x) := \frac{1}{\pi} \text{p.v.} \int_{\mathbb{R}} \frac{\partial_\tau u(\tau)}{x - \tau} d\tau.$$

The first definition generalizes into the fractional Laplacian far more easily as

$$\widehat{\Lambda^s u} := |\xi|^s \hat{u}, \quad \text{for } s \in \mathbb{R}.$$

A.3.3 Convolutions

In section 3 there is a subtlety in calculating $L(\hat{I}_2)$. To be precise, the function defined by the convolution

$$L(I_2)(\alpha) = \frac{1}{\pi^2} \mathcal{K} \rho \left(\frac{x}{x^2+h_2^2} * \frac{h_2}{x^2+h_2^2} * \partial_\alpha f \right)(\alpha)$$

is in $L^\infty(\mathbb{R})$. Therefore, we must be careful to interpret the Fourier transform as a distribution until we prove that it is actually a function. The following lemma suffices.

Lemma. If $f, g \in L^2(\mathbb{R})$ then $\widehat{(f * g)} = \hat{f} \hat{g}$ almost everywhere.

Note that $\hat{f} \hat{g} \in L^1(\mathbb{R})$ and additionally that the below maps are continuous

$$(f, g) \mapsto f * g, \quad (f, g) \mapsto \hat{f} \hat{g}, \quad \mathcal{F}^{-1}: L^1 \rightarrow L^\infty.$$

We can consider sequences of Schwartz functions $f_n \rightarrow f$ and $g_n \rightarrow g$, and argue by continuity that

$$\mathcal{F}^{-1}(\hat{f} \hat{g}) = f * g.$$

The result follows.

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