THE UNIFORMIZATION THEOREM

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Abstract. Riemann surfaces lie at the intersection of many areas of math. The Uniformization theorem is a major result in Riemann surface theory. This paper, written at the 2019 Michigan REU, gives a modern proof of the Uniformization theorem, investigating a lot of interesting math along the way.

Contents

Introduction

In Section 1, we give basic definitions and constructions relating to Riemann surfaces. In Section 2, we state and interpret the Uniformization theorem and develop some of the machinery needed to prove the theorem, including the Riemann-Roch theorem and cohomology. Finally we tie everything together in Section 3 with a proof of the Uniformization theorem via the Riemann-Roch theorem and the Hodge Decomposition theorem for Riemann surfaces, along with a few analysis results which lie at the heart of Riemann surface theory.

1. Riemann Surfaces and Covering Theory

Definition 1.1 (Riemann Surface). *A Riemann surface is a 1-dimensional complex manifold. That is, it is a Hausdorff space R such that for all $p \in R$, there is an open neighborhood U_p of p and a homeomorphism φ_p (called a "chart") from U_p to an open subset of $\mathbb{C}$ satisfying the following compatibility criterion: For all $p, q \in R$, if $U_p \cap U_q \neq \emptyset$, then the transition function $\varphi_p \circ \varphi_q^{-1} : \varphi_q(U_p \cap U_q) \to \varphi_p(U_p \cap U_q)$ is holomorphic.*

The technical details of this definition are a little unwieldy. Intuitively, a Riemann surface is a topological space that, near every point, "looks like" part of the complex plane. We only consider connected Riemann surfaces in this paper.

Example 1.1. *$\mathbb{C}$ is a Riemann surface. We can take $\mathbb{C}$ as U_p and $id : \mathbb{C} \to \mathbb{C}$ as φ_p for all $p \in \mathbb{C}$. Using the same construction, all open subsets of $\mathbb{C}$ are also Riemann surfaces. One that will be of particular interest is the open unit disk $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$.*

Example 1.2. *The Riemann Sphere $\hat{\mathbb{C}}$, the complex plane with a point at infinity, is another example of a Riemann surface. For any point p besides ∞, we take $U_p = \hat{\mathbb{C}} \setminus \{\infty\} = \mathbb{C}$ and $\varphi_p(z) = z$. We also set $U_\infty = \hat{\mathbb{C}} \setminus \{0\}$ and $\varphi_\infty(z) = \frac{1}{z}$, where $\frac{1}{\infty}$ is taken to be 0. The transition functions $\varphi_p \circ \varphi_q^{-1}$ are the identity if $U_p = U_q$ and the function $\varphi_p \circ \varphi_q^{-1}(z) = \frac{1}{z}$ on $\mathbb{C} \setminus \{0\}$ otherwise (in which case exactly one of p or q is ∞). In both cases, the transition function is holomorphic.*

1.1. **Maps between Riemann Surfaces.** Maps between Riemann surfaces have many interesting properties, and several will be important to this paper.

Definition 1.2 (Holomorphism)**.** *Let X and Y be Riemann surfaces. A function $f : X \to Y$ is called holomorphic if, for all charts g of X and h of Y, $h \circ f \circ g^{-1}$ is holomorphic (as a complex-valued function) wherever this expression makes sense. If no codomain is specified, a holomorphic function on X is taken to be a holomorphism $X \to \mathbb{C}$.*

Definition 1.3 (Biholomorphism)**.** *A biholomorphism is a bijective holomorphism with holomorphic inverse.*

Remark. *The definition of a biholomorphism is equivalent to the condition of being a holomorphic homeomorphism. Since a biholomorphism preserves complex structure, two spaces are considered "the same" in the category of Riemann surfaces if they are biholomorphic.*

Definition 1.4 (Covering)**.** *For topological spaces X, Y, and a continuous surjection $p : X \to Y$, p is a covering of Y by X if there exists a discrete space S such that for all $y \in Y$, there exists a neighborhood V of y such that $p^{-1}(V)$ is homeomorphic to $V \times S$. If S is finite, $|S|$ is called the degree of the covering.*

Example 1.3. *One example, which we will come back to, is the covering of the torus $\mathbb{C}/\langle 1, i\rangle$ by $\mathbb{C}$, where $\langle 1, i\rangle$ acts on $\mathbb{C}$ by addition in $\mathbb{C}$. See Figure 1.*

We can think of $p^{-1}(V)$ as several copies of V in the space X. It is also useful to define a relative to the covering that behaves somewhat more irregularly than a covering.

Definition 1.5 (Ramified Covering)**.** *A continuous surjection $p : X \to Y$ is a ramified covering if it is a covering except on a discrete set of points, called the ramification locus.*

If a ramified covering has finite degree d away from the ramification locus, the points in the ramification locus also have d preimages, counting multiplicity.

Theorem 1.1. *Every non-constant holomorphic function $f : X \to Y$ between connected Riemann surfaces, where X is compact, is a ramified covering.*

Proof. The image of a compact space under a continuous function is compact, and the Open Mapping Theorem states that every non-constant holomorphic function is an open map, so $f(X)$ is compact (therefore closed) and open in Y. Since $f(X)$ is nonempty, this means $f(X) = Y$, so f is a continuous surjection. For a proof that f has the remaining properties of a ramified cover, see [FK92], §I.1.6. □

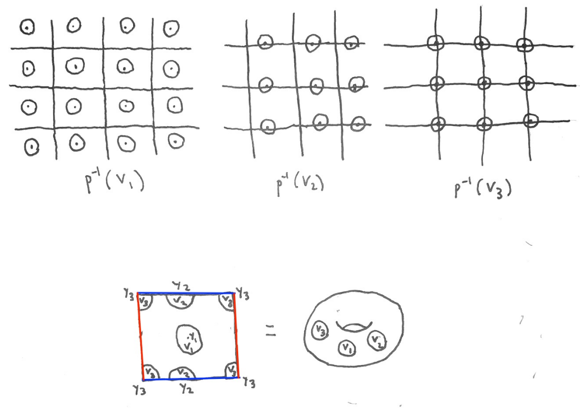

FIGURE 1. This shows neighborhoods V_1, V_2, V_3 of three points y_1, y_2, y_3 so that V_1, V_2, V_3 behave as V does in the definition of the covering. Here, S is $\mathbb{Z} \times \mathbb{Z}$.

The proof of Theorem 1.1 quickly gives another important result.

Theorem 1.2 (Liouville's Theorem for Compact Riemann Surfaces)**.** *If X is a compact Riemann surface, any holomorphic function $f : X \to \mathbb{C}$ is constant.*

Proof. By the proof of Theorem 1.1, if f were nonconstant, the image $f(X)$ would be compact and open in $\mathbb{C}$. This implies $f(X) = \emptyset$, which is absurd. So f must be a constant function. $\square$

By Theorem 1.2, holomorphic maps from compact Riemann surfaces to $\mathbb{C}$ do not give us any useful information about compact Riemann surfaces. We will later have to consider nonconstant meromorphic functions on compact Riemann surfaces, which are the same as holomorphic functions from those surfaces to $\hat{\mathbb{C}}$.

1.2. **Coverings and the Fundamental Group.** In what follows, "covering" means an unramified covering. Coverings interact nicely with many objects in the relevant topological spaces. For instance, coverings transfer paths in the base Y to the cover X in a clean and natural way.

Theorem 1.3 (Path Lifting Property)**.** *Let $p : X \to Y$ be a covering, let $\gamma : [0,1] \to Y$ be a path, and let $p(x_0) = \gamma(0)$. Then, there is a unique path $\tilde{\gamma} : [0,1] \to X$ such that $p(\tilde{\gamma}) = \gamma$ and $\tilde{\gamma}(0) = x_0$. We call $\tilde{\gamma}$ a lift of γ.*

This is proven in [Hat02]. The path lifting property is especially interesting in the way it pertains to loops, which may or may not themselves lift to loops. It is helpful to define the fundamental group, a group of loops on a space X.

Definition 1.6 (Fundamental Group)**.** *For a topological space X and a point $x_0 \in X$, the fundamental group $\pi_1(X, x_0)$ is the set of homotopy classes of paths in X $\gamma : [0,1] \to X$ with $\gamma(0) = \gamma(1) = x_0$ such that every path reached in the homotopy also starts and ends at x_0. The homotopy class of the path γ is denoted $[\gamma]$, and the operation on the fundamental group is loop concatenation, that is, $[\gamma_1] * [\gamma_2]$ is the homotopy class $[\gamma_{(1,2)}]$ where*

$$\gamma_{(1,2)}(t) = \begin{cases} \gamma_1(2t) & t \in [0, \frac{1}{2}] \\ \gamma_2(2t-1) & t \in [\frac{1}{2}, 1] \end{cases}$$

There are a few things to note here. The loop concatenation operation is simply first following the first loop, then the second loop, and adjusting the "speed" to keep the domain of the path as $[0,1]$. The identity of the fundamental group is the path e with $e \equiv x_0$. Also, for continuous functions $f, g : X \to Y$, a homotopy H from f to g is a continuous function $X \times [0,1] \to Y$ such that for all $x \in X$, $H(x,0) = f(x)$ and $H(x,1) = g(x)$. This can be thought of as a continuous deformation from f to g. If a homotopy from X to Y exists, f and g are said to be homotopic. This is an equivalence relation on continuous functions from X to Y, and the equivalence classes under this relation are called homotopy classes.

The definition of the fundamental group using homotopy classes of loops instead of just loops allows the fundamental group to have the properties of a group. If the fundamental group was defined as a group of loops instead of as a group of homotopy classes, it would have neither associativity nor the inverse property. The inverse of a path $\alpha : [0,1] \to X$ is the path β such that $\beta(x) = \alpha(1-x)$. It should also be noted that the operation is well-defined, and does not depend on the representative chosen for the equivalence class. For proofs, see [Hat02].

Finally, the fundamental group seems like a curious definition, because it is defined on a topological space and a point in that space, rather than just a topological space. In fact, if for $x_0, x_1 \in X$ there is a path $\alpha : [0,1] \to X$ with $\alpha(0) = x_0$ and $\alpha(1) = x_1$, then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$. The group isomorphism takes $[\gamma] \in \pi_1(X, x_1)$ to $[\gamma']$, where

$$\gamma'(t) = \begin{cases} \alpha(3t) & t \in [0, \frac{1}{3}] \\ \gamma(3t-1) & t \in [\frac{1}{3}, \frac{2}{3}] \\ \alpha(3-3t) & t \in [\frac{2}{3}, 1] \end{cases}$$

This takes a loop starting and ending at x_1 and takes it to a loop starting and ending at x_0 by following a path from x_0 to x_1, then following the loop starting and ending at x_1, then following a path back to x_0. For a proof that this is an isomorphism, see [Hat02].

Since if x_0 and x_1 are connected by a path, then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$, a path connected space X has the property that its fundamental group does not depend on base point. In this case, we refer to $\pi_1(X)$ as the fundamental group of X.

We can refer to the fundamental groups of connected Riemann surfaces without specifying base points, since they are path connected. This is because they are connected and locally path connected and together these two properties imply path connectedness. (A space X is locally path connected if, for all $x \in X$, for all neighborhoods U of x, there is a path-connected neighborhood $V \subset U$ of x, and Riemann surfaces satisfy this property because there are by definition neighborhoods of every point homeomorphic to open sets in the complex plane.)

Example 1.4. *The torus T has a fundamental group generated by two generators, a and b. Because $aba^{-1}b^{-1}$ deforms to a single point (a^{-1} and b^{-1} are the paths a and b followed in the reverse direction) the word $aba^{-1}b^{-1}$ is trivial in fundamental group for the torus. In addition, there are no other relations on a and b. This fundamental group is in fact isomorphic to $\mathbb{Z} \times \mathbb{Z}$. See Figure 2.*

The fundamental group and coverings are linked by the following theorem, sometimes called the Fundamental Theorem of Galois Theory for Covering Spaces:

Theorem 1.4. *If X, Y are path connected, and $p : X \to Y$ is a regular covering, then:*

1. *$p_*(\pi_1(X))$ is normal in $\pi_1(Y)$.*
2. *The Deck group $D(p)$, the group of automorphisms $d : X \to X$ such that $p \circ d = p$, is isomorphic to $\pi_1(Y)/p_*(\pi_1(X))$.*

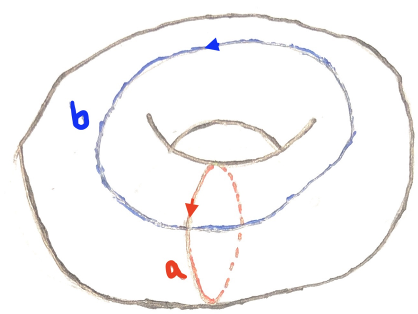

FIGURE 2. The paths a and b are the two generators.

A covering $p : X \to Y$ is called regular if, for every loop $\gamma : [0,1] \to X$ with $\gamma(0) = \gamma(1)$, all lifts $\widetilde{p \circ \gamma}$ of the loop $p \circ \gamma$ in Y are loops, that is, they satisfy $\widetilde{p \circ \gamma}(0) = \widetilde{p \circ \gamma}(1)$. The induced map on the fundamental group p_* takes $[\alpha] \in \pi_1(X)$ to $[p \circ \alpha] \in \pi_1(Y)$. This is well-defined regardless of both base point for the fundamental group and representative for the class $[\alpha]$. Also, the deck group can be thought of as shuffling the preimages of a neighborhood under a covering.

Proof. This proof uses ideas from [Kug93].

The key observation is that the path lifting property can be used to relate elements of $\pi_1(Y)$ to elements of $D(p)$. Given $[\alpha] \in \pi_1(Y)$, for each $y \in Y$ let

$$\alpha_{y_0} : [0,1] \to Y$$

be a representative of the class that corresponds to $[\alpha]$ in $\pi_1(Y, y_0)$. Also, if $p(x_0) = y_0$, let γ_{y_0,x_0} be the unique lift of α_{y_0} with $\gamma_{y_0,x_0}(0) = x_0$. Then, define the map $\psi : \pi_1(Y) \to D(p)$ in the following way:

$$\psi([\alpha]) : X \to X$$
$$x \mapsto \gamma_{p(x),x}(1)$$

ψ is a group homomorphism. Its kernel is the elements $[\alpha]$ such that $\psi([\alpha])$ is the identity, that is, elements that lift to loops regardless of local representation. The regular covering property is very useful in determining which elements these are, since it guarantees that any images of loops in X under the covering map p only lift to loops in X. This holds for representatives of classes of loops as well, so $[\alpha] \in \ker \psi$ iff $\alpha = p(\beta)$ for some $\beta \in \pi_1(X)$, which is exactly the condition $[\alpha] \in p_*(\pi_1(X))$, so

$$\ker \psi = p_*(\pi_1(X)).$$

The kernel of a group homomorphism is always normal, and by the First Isomorphism Theorem,

$$\psi(\pi_1(Y)) = \pi_1(Y)/\ker \psi = \pi_1(Y)/p_*(\pi_1(X)).$$

All that remains to be shown is that ψ is a surjection.

To do this, let d be a deck transformation. Note d is continuous. Then, let $x_0, x_1 \in X$ such that $d(x_0) = x_1$. Using the path connectedness of X, let $\gamma : [0,1] \to X$ be a path such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$. Then, $\psi([p \circ \gamma]) = d$. First, note $p \circ \gamma$ is a loop, since $p(\gamma(1)) = p(x_1) = p(d(x_0)) =$

$p(x_0) = p(\gamma(0))$. So, $\psi([p \circ \gamma]) \in D(p)$. Also, $\psi([p \circ \gamma])(x_0) = x_1$, since γ is the only lift of $p \circ \gamma$ that is x_0 at 0.

In general, a deck transformation's image at one point completely determines the entire deck transformation. This is because, given $d(x_0) = x_1$, for all $x \in X$, there exists a path γ_x with $\gamma_x(0) = x_0$ and $\gamma_x(1) = x$. Then, $d \circ \gamma_x$ is a path from $d(x_0) = x_1$ to $d(x)$. So, there is only one option for $d(x)$.

Therefore, since $\psi([p \circ \gamma])$ and d are deck transformations that agree on the image of a point, they are the same deck transformation. So $d \in \psi(\pi_1(Y))$, so since d was arbitrary $\psi(\pi_1(Y)) = D(p)$. Therefore $D(p) = \psi(\pi_1(Y) = \pi_1(Y)/p_*(\pi_1(X))$. □

With this theorem, we can consider normal subgroups of the fundamental group of a space Y in correspondence with some covers X of Y. In turn, if these spaces have nontrivial fundamental group, the normal subgroups of their fundamental groups correspond to covers of them. With the right conditions, all of these spaces are covered by a cover with trivial fundamental group.

Example 1.5. *The fundamental group of the torus $\mathbb{C}/\langle 1, i\rangle$ is isomorphic to $\mathbb{Z} \times \mathbb{Z}$. This group is abelian, so all of its subgroups are therefore normal. Figure 3 shows the correspondence of various covers of the torus with subgroups of its fundamental group. Each covering map takes a point to its equivalence class under a quotient.*

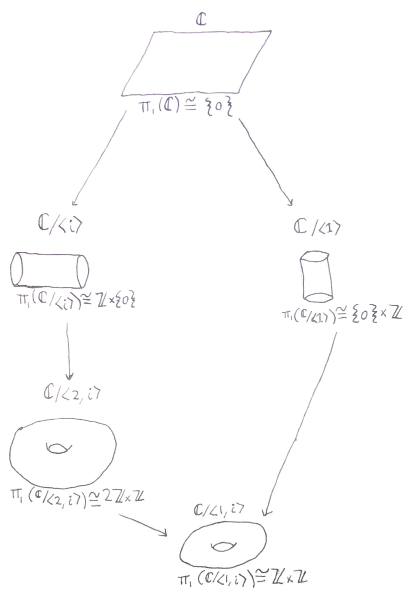

FIGURE 3. This shows several covers of a torus, all covered by a space ($\mathbb{C}$) with trivial fundamental group.

1.3. **Universal Covers.** In the previous example, $\mathbb{C}$ is a cover for all of the other covers of the torus. This is a property of covering spaces with trivial fundamental group, and for this reason, such covers are called "universal."

Definition 1.7 (Universal Cover)**.** *A path-connected space X is called simply connected if it has trivial fundamental group (so every loop is homotopic to a point). A simply connected cover is called a universal cover.*

Remark. *If a space has a universal cover, it has a unique universal cover.*

In fact, a few connectivity conditions are sufficient for a space to have a universal cover.

Theorem 1.5. *A locally path connected space X has a universal cover if and only if X is connected and semi-locally simply connected.*

X is semi-locally simply connected if for all $x \in X$, there exists a neighborhood U of x such that every loop in U is homotopic to a single point in X. Note the homotopy is not required to take place within U, hence the "semi-local" condition. Most even vaguely well-behaved spaces are semi-locally simply connected - the standard counterexample is the Hawaiian earring (Figure 4), which consists of the union of all circles in $\mathbb{R}^2$ that are centered at $(\frac{1}{n}, 0)$ with radius $\frac{1}{n}$ for each $n \in \mathbb{N}$, with the subspace topology from the Euclidean topology on $\mathbb{R}^2$.

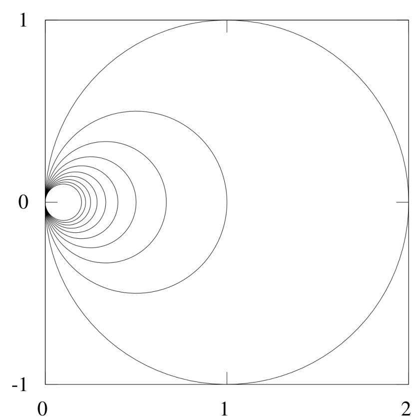

Figure 4. This space is not semi-locally simply connected, since no neighborhood of $(0, 0)$ has the property that all loops in the neighborhood are contractible to a single point in the space.

Proof. This proof includes ideas from [Hat02].

First, we show that these conditions are necessary for a locally path connected space to have a universal cover. Suppose X has a universal cover $\tilde{X}$, and let $p : \tilde{X} \to X$ be the covering map.

We first show X is connected. Let $x_0, x_1 \in X$, and let $\tilde{x}_0$ and $\tilde{x}_1$ be preimages of x_0 and x_1 under p. Then, since $\tilde{X}$ is path connected, there is a path $\alpha : [0, 1] \to \tilde{X}$ such that $\alpha(0) = \tilde{x}_0$ and $\alpha(1) = \tilde{x}_1$. Then, $p \circ \alpha$ is a path in X from x_0 to x_1, so X is path connected and therefore also connected.

Next, we show X is semi-locally simply connected. Let $x \in X$. Then, let U be an open neighborhood of X such that the preimage of U is homeomorphic to $U \times S$, where S has the discrete topology. Then, let γ be a loop in U with $\gamma(0) = \gamma(1) = x_0$ and lift γ to $\tilde{\gamma}$, such that $\tilde{\gamma}(0) = \tilde{x}_0$ for some preimage $\tilde{x}_0$ of x_0. This lift is contained entirely in one of the copies of U in $\tilde{X}$, since S

is discrete, so $\tilde{\gamma}$ is a loop. Since $\tilde{X}$ is simply connected, there is a homotopy $H : [0,1] \times [0,1] \to Y$ such that $H(x,0) = \tilde{\gamma}(x)$, $H(x,1)$ is constantly $\tilde{x}_0$, and $H(0,x) = H(1,x) = \tilde{x}_0$ for all x. Then, $p \circ H$ is a homotopy in X from γ to a single point that fixes the base point of the loop, so γ is contractible in X and X is semi-locally simply connected.

Given a connected, locally path connected, and semi-locally simply connected space has a universal cover. Let X be such a space. The universal cover of X is constructed by taking a point $x_0 \in X$, and letting

$$\tilde{X} = \{[\gamma] \mid \gamma : [0,1] \to X \text{ is a path with } \gamma(0) = x_0\}$$

where the homotopy class $[\gamma]$ only includes homotopies that fix the base point and final point of the path γ. This allows for a very natural covering map $p([\gamma]) = \gamma(1)$. See [**?**] for a proof that this space is indeed a universal cover for X. $\square$

All connected Riemann surfaces satisfy the connectivity conditions for Theorem 1.5, since each point has a neighborhood homeomorphic to an open subset of $\mathbb{C}$, so they all have universal covers. See [IT] for a proof that the universal covers of Riemann surfaces have Riemann surface structure. There is an immensely powerful and important theorem concerning universal covers of Riemann surfaces, which is the main purpose of this paper.

2. The Uniformization Theorem

The Uniformization Theorem can be stated in two forms. The first form emphasizes the universal covers and the second suggests the importance of the theorem by describing how the three universal covers give us information about every Riemann surface.

Theorem 2.1 (The Uniformization Theorem, version 1)**.** *Up to biholomorphism, there are just three simply connected Riemann surfaces: the complex plane $\mathbb{C}$, the Riemann sphere $\hat{\mathbb{C}}$, and the open disk $\mathbb{D}$.*

Theorem 2.2 (The Uniformization Theorem, version 2)**.** *Every connected Riemann surface X is biholomorphic to a quotient of one of $\hat{\mathbb{C}}$, $\mathbb{C}$, or $\mathbb{D}$ by the covering space action of a subgroup Γ of its automorphism (self-biholomorphism) group.*

Remark. *$\mathbb{D}$ is biholomorphic to the upper half-plane $\mathbb{H} = \{z \in \mathbb{C} \mid \mathrm{Re}\,(z) > 0\}$, which is sometimes used in the statement of the Uniformization Theorem.*

In the second form of the Uniformization Theorem, the term "covering space action" is defined to be an action of a group G on a space X such that for all $x \in X$, there exists a neighborhood U of x such that $gU \cap U \neq \emptyset$ if and only if g is the identity. See [Hat02] for a proof that the map $p : X \to X/G$ taking points in X to their equivalence class is a covering and that G is closely related to the group of deck transformations. In our case, this means the second form of the Uniformization Theorem is describing the universal cover of X. It is worth verifying that the three universal covers are indeed meaningfully distinct.

Theorem 2.3. *The three spaces $\hat{\mathbb{C}}$, $\mathbb{C}$, and $\mathbb{D}$ are mutually non-biholomorphic.*

Proof. Any biholomorphism is a homeomorphism, so since $\hat{\mathbb{C}}$ is compact and $\mathbb{C}$ and $\mathbb{D}$ are not, $\hat{\mathbb{C}}$ is not biholomorphic to either $\mathbb{C}$ or $\mathbb{D}$. Furthermore, any holomorphism $\mathbb{C} \to \mathbb{D}$ must be constant by Liouville's Theorem from complex analysis, so it cannot be invertible. $\square$

So, we can understand the complex structure on a Riemann surface by understanding the surface as a quotient of one of these three Riemann surfaces by a group of its automorphisms. The automorphisms of the three universal covers are easy to describe, and each admits a particularly nice metric.

Theorem 2.4 (Automorphisms of the universal covers)**.**

- *The automorphisms of* $\hat{\mathbb{C}}$ *are* $f(z) = \frac{az+b}{cz+d}$ *where* $ad - bc \neq 0$ *and* $a, b, c, d \in \mathbb{C}$.
- *The automorphisms of* $\mathbb{D}$ *are of the form* $f(z) = \frac{az+b}{\bar{b}z+\bar{a}}$ *where* $|a|^2 - |b|^2 = 1$ *and* $a, b, c, d \in \mathbb{R}$.
- *The automorphisms of* $\mathbb{C}$ *are of the form* $f(z) = az + b$ *where* $a, b \in \mathbb{C}$ *and* $a \neq 0$.

For a proof, see [Bra]. For a details on the construction of the metrics on the universal covers and quotient spaces, see [Bon09].

Definition 2.1 (Quotient Metric)**.** *Let* X *be a topological space, let* $d : X \times X \to \mathbb{R}$ *be a metric on* X*, let* $\sim$ *be an equivalence relation on* X*, and let* $[x]$ *denote the* $\sim$*-equivalence class of* x*. Then, the quotient metric* $\bar{d} : (X/\sim) \times (X/\sim) \to \mathbb{R}$ *is defined such that*

$$\bar{d}([P], [Q]) = \inf\{d(P, Q_1) + d(P_2, Q_2) + \cdots + d(P_n, Q_n) \mid Q_i \in [P_{i+1}] \forall 1 \leq i \leq n-1, Q_n \in [Q]\},$$

if this construction makes $\bar{d}$ *a metric.*

The metrics on Riemann surfaces are one of the miracles of Uniformization. The three universal covers have very well-behaved and understood metrics, in particular, the three metrics all have constant curvature. Because of natural correspondences between the automorphism and isometry groups of the three universal covers, the quotient metrics on Riemann surfaces inherit the constant curvature of the metrics on the universal covers. So, the Uniformization Theorem gives a correspondence between complex structures and metric structures on a Riemann surface!

2.1. **The Riemann-Roch Theorem.** Our next goal is to understand the proof of the Uniformization Theorem. We will use the Riemann-Roch Theorem, an important theorem about the existence of meromorphic functions on a compact Riemann surface. To state the theorem, it is first necessary to make some definitions.

Definition 2.2. *Let* X *be a Riemann surface, and let* $p_1, \ldots, p_d \in X$ *be a set of distinct points. Then,* $D = \{p_1, \ldots, p_d\}$ *is called a divisor. We define* $H^0(D)$ *to be the complex vector space of meromorphic functions on* X *which have no poles away from* D *and at most simple poles in* D*. For ease of notation, we define* $h^0(D) = \dim H^0(D)$.

Definition 2.3 (Differential Forms)**.** *For a Riemann Surface* X*, a differential one-form is an element of the cotangent space. A differential 1-form is a holomorphic differential form if it is represented as* $f(z)dz$*, where* f *is holomorphic, in every coordinate patch. A meromorphic differential form is defined to be a holomorphic differential form except at a set of isolated points* $\{a_1, \ldots, a_m\}$*, where the local representation is given by* $g(z)dz$ *where* g *is meromorphic with a pole at* a_i.

Remark. *All holomorphic differential forms are meromorphic.*

The cotangent space is defined to be the dual of the tangent space, which is an object attached to each point that essentially consists of local objects ("point derivations") that behave like derivatives. In the case of Riemann surfaces, a basis for the tangent space over $\mathbb{R}$ is given by $(\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}})$, where z is the local coordinate. The differentials dz and $d\bar{z}$ are defined as the dual vectors of the corresponding partial derivatives, so they are both linear functionals on the tangent space and

$$dz(\frac{\partial}{\partial z}) = 1$$

$$dz(\frac{\partial}{\partial \bar{z}}) = 0$$
$$d\bar{z}(\frac{\partial}{\partial z}) = 0$$
$$d\bar{z}(\frac{\partial}{\partial \bar{z}}) = 1$$

We also define d, ∂, and $\bar{\partial}$ operators on smooth functions in local coordinates. Let f be such a function. Then, we let

$$df := \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z}$$
$$\partial f := \frac{\partial f}{\partial z}dz$$
$$\bar{\partial} f := \frac{\partial f}{\partial \bar{z}}d\bar{z}$$

All 1-forms are locally of the form

$$f(z)dz + g(z)d\bar{z},$$

and we call a form a $(1,0)$-form if g is zero and a $(0,1)$-form if f is zero.

Similarly, given a one-form $\omega = f(z)dz + g(z)d\bar{z}$, we can define $d\omega, \partial\omega$ and $\bar{\partial}\omega$. We define $d\omega := df \wedge dz + dg \wedge d\bar{z}$, and $\partial\omega$ and $\bar{\partial}\omega$ similarly, and in general, we have $d = \partial + \bar{\partial}$. See [Sch07], Chapter 4 for details and more explanation of differential forms.

We define $H^0(K-D)$ to be the space of holomorphic 1-forms on X that vanish at every point in D, and as before let $h^0(K-D) = \dim H^0(K-D)$. We are now in a position to state a form of the Riemann-Roch Theorem:

Theorem 2.5 (Riemann-Roch, form 1)**.** *Let X be a Riemann surface of genus g, let $p_1, \dots, p_d \in X$ be distinct, and let $D = \{p_1, \dots, p_d\}$ be a divisor. Then,*

$$h^0(D) - h^0(K-D) = d - g + 1.$$

We investigate this in the case where $X = \hat{\mathbb{C}}$ to see the limitations of this form of the theorem. First, we check that the theorem holds in this form in this case.

Theorem 2.6. *Let $p_1, \dots, p_d \in \hat{\mathbb{C}}$ be distinct, and let $D = \{p_1, \dots, p_d\}$ be a divisor. The Riemann-Roch theorem holds in this case.*

Proof. First, $h^0(D) = d + 1$. We can assume $p_1, \dots, p_d \in \mathbb{C}$. Then,

$$H^0(D) = \text{span}(1, \frac{1}{z-p_1}, \frac{1}{z-p_2}, \dots, \frac{1}{z-p_d}).$$

All functions of this form are certainly in $H^0(D)$, so we must show that all functions in $H^0(D)$ are in this span. Indeed, let $F \in H^0(D)$. Then, let $c_1, \dots, c_d$ be the residues of F when expressed in z at $p_1, \dots, p_d$. Now, consider the function

$$f(z) = F(z) - \frac{c_1}{z-p_1} - \frac{c_2}{z-p_2} - \dots - \frac{c_d}{z-p_d}$$

where z is the canonical coordinate chart on $\hat{\mathbb{C}}$ centered at 0.

F only has simple poles since it is in $H^0(D)$, so around every point p_i, its Laurent expansion is of the form

$$\frac{c_i}{z-p_i} + \sum_{j=0}^{\infty} a_j (z-p_i)^j.$$

Therefore, the Laurent expansion of f (which is meromorphic because it is the sum of meromorphic functions) around every point p_i is of the form

$$\sum_{j=0}^{\infty} a_j (z-p_i)^j,$$

so f is holomorphic near every point in $\hat{\mathbb{C}}$.

Therefore f is holomorphic on $\hat{\mathbb{C}}$, so by Theorem 1.2 f is constant. So, there exists $c_0 \in \mathbb{C}$ such that

$$\begin{aligned} c_0 &= F(z) - \frac{c_1}{z-p_1} - \frac{c_2}{z-p_2} - \cdots - \frac{c_d}{z-p_d} \\ \implies F(z) &= c_0 + \frac{c_1}{z-p_1} + \frac{c_2}{z-p_2} + \cdots + \frac{c_d}{z-p_d} \end{aligned}$$

So, $H^0(D) = \mathrm{span}(1, \frac{1}{z-p_1}, \frac{1}{z-p_2}, \dots, \frac{1}{z-p_d})$, so $h^0(D) = d+1 = d-g+1$, since $\hat{\mathbb{C}}$ has genus 0. So, for the Riemann-Roch Theorem to hold in this case, we must now verify $h^0(K-D) = 0$.

By 1.1, every meromorphic function f on $\hat{\mathbb{C}}$ besides the zero function has the same number of zeroes as it has poles (counting multiplicity), since a meromorphic function on $\hat{\mathbb{C}}$ is a holomorphic function $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$. We use this to show a lemma about meromorphic one-forms on $\hat{\mathbb{C}}$.

Let ω, ω' be two meromorphic one-forms on $\hat{\mathbb{C}}$, and let $f = \frac{\omega}{\omega'}$. Then, f is a meromorphic function, so it has the same number of zeroes as it has poles. Now, the number of zeroes f has is equal to the number of zeroes of ω plus the number of poles of ω', and the number of poles of f is the number of poles of ω plus the number of zeroes of ω'. So,

$$\begin{aligned} \mathrm{zeroes}(\omega) + \mathrm{poles}(\omega') &= \mathrm{zeroes}(\omega') + \mathrm{poles}(\omega) \\ \implies \mathrm{zeroes}(\omega) - \mathrm{poles}(\omega) &= \mathrm{zeroes}(\omega') - \mathrm{poles}(\omega'). \end{aligned}$$

So, the number of zeroes minus the number of poles is constant across all (nonzero) meromorphic one-forms on the Riemann sphere. Now, consider the meromorphic one-form dz on $\hat{\mathbb{C}}$. It is a holomorphic one-form with no zeros on $\mathbb{C}$. At ∞, we use a change of coordinates to understand its behavior. Let the local coordinate at ∞ be w, and note $w = \frac{1}{z}$. Then,

$$dz = d(\frac{1}{w}) = -\frac{1}{w^2} dw,$$

so at $z = \infty$, $w = 0$ and dz has a double pole. Since dz has no zeroes,

$$\mathrm{zeroes}(dz) - \mathrm{poles}(dz) = -2.$$

So, all meromorphic one-forms ω on the Riemann sphere that are not identically zero have

$$\mathrm{zeroes}(\omega) - \mathrm{poles}(\omega) = -2.$$

In particular, there are no holomorphic one-forms on the Riemann sphere besides the zero form, so $h^0(K-D) = 0$. □

The example of dz also hints at a limitation of our current approach. It has a pole with multiplicity greater than one, which is counted in considerations with zeroes and poles but not treated by our current definition of divisors. The Riemann-Roch theorem is made more general and powerful by defining divisors in a less intuitive, but more general, way [Sch07].

Definition 2.4 (Divisor, version 2)**.** *The group of divisors of a Riemann surface X is the free abelian group generated by points of X.*

We denote divisors with a formal sum, so we write a divisor D as

$$D = \sum_{a \in X} n_a a$$

where $n_a \in \mathbb{Z}$ for all a and $n_a = 0$ for all but finitely many a. We define addition of divisors pointwise, so that the sum of two divisors $D = \sum_{a \in X} n_a a$ and $D' = \sum_{a \in X} m_a a$ is

$$D + D' = \sum_{a \in X} (m_a + n_a) a.$$

Similarly, we can define a difference between divisors, take products of a divisor with a scalar, and so on. We also define a partial order on divisors where we say $D \geq D'$ if $n_a \geq m_a$ for all $a \in X$.

Some divisors are special. For instance, every nonzero meromorphic function f on X has a divisor associated to it. The order $\mathrm{ord}_a(f)$ of f at a point a is k if f has a zero of multiplicity k at a, $-k$ if f has a pole of multiplicity k, and zero otherwise. The divisor associated to f, denoted (f), is

$$\sum_{a \in X} \mathrm{ord}_a(f) a.$$

Similarly, every nonzero meromorphic differential form ω has a divisor associated to it. Near every point a, ω can be represented as $g_a(z)dz$, Then, we define

$$\mathrm{ord}_a(\omega) = \mathrm{ord}_a(g_a)$$

(it can be checked that this is independent of local coordinate) and we let the divisor associated to ω be denoted

$$(\omega) = \sum_{a \in X} \mathrm{ord}_a(\omega) a.$$

For meromorphic functions or forms a, b,

$$(a \cdot b) = (a) + (b) \text{ and } \left(\frac{1}{a}\right) = -(a).$$

Two divisors are said to be equivalent if their difference is (f) for some meromorphic f. All nonzero meromorphic differential forms are equivalent since, given two such forms ω, ω', the ratio $\frac{\omega}{\omega'}$ is some meromorphic function f, so

$$(f) = (\frac{\omega}{\omega'}) = (\omega) - (\omega').$$

The equivalence class of divisors corresponding to meromorphic one-forms is called K.

In this more general setting, the divisor $D = \{p_1, \ldots, p_d\}$ is denoted $p_1 + \cdots + p_d$. We also redefine $H^0(D)$ in a more general way using the partial ordering of the divisors. We define

$$H^0(D) := \{f \text{ meromorphic} \mid (f) \geq -D\}$$

Let D, D' be equivalent, so that $D - D' = (f)$ for some meromorphic f. Then, $H^0(D) \cong H^0(D')$, and the isomorphism is given by $g \mapsto f \cdot g$.

So, if $D = -47p_1 + 2p_2$ for some $p_1, p_2 \in X$, then all $f \in H^0(D)$ must have a zero of at least order 47 at p_1, can have a pole of at most order 2 at p_2, and must be holomorphic on the rest of X. So, the new definition of $H^0(D)$ is a generalization compatible with the old definition. The other term on the left-hand side of the Riemann-Roch Theorem, $H^0(K - D)$, is defined in the same way:

$$H^0(K - D) := \{f \text{ meromorphic} \mid (f) \geq D - k\}$$

where k is taken as any representative of the divisor class K (all representatives will give the same space, up to isomorphism). $H^0(K-D)$ is more naturally interpreted as a space of forms than as a space of functions:

Theorem 2.7. *$H^0(K-D)$ is isomorphic to the space of meromorphic differential forms ω such that $(\omega) \geq D$.*

Proof. Let ω_0 be a nonzero meromorphic differential form, so that K is the equivalence class of (ω_0). Then, for all meromorphic forms ω, $\frac{\omega}{\omega_0}$ is some meromorphic function f, so $\omega = f\omega_0$. We have

$$(\omega) \geq D$$

$$\iff (f\omega_0) = (f) + (\omega_0) = (f) + k \geq D$$

$$\iff (f) \geq D - k \iff f \in H^0(K-D).$$

So, $f \mapsto f\omega_0$ is an isomorphism between the two spaces. $\square$

Hence, the new definition of $H^0(K-D)$ is compatible with the previous one, and $H^0(K-D)$ is a very natural object to consider.

We also need to reconsider one more term in the Riemann-Roch Theorem: We replace the term d with the degree $\deg D$ of the divisor D, where $\deg D$ is defined as the sum of the coefficients of points in D.

Theorem 2.8 (Riemann-Roch, more general)**.** *Let X be a compact Riemann surface of genus g, and let $D = \sum_{a \in x} n_a a$ be a divisor for X. Then,*

$$h^0(D) - h^0(K-D) = \deg D - g + 1.$$

Remark. *We can quickly adjust the proof of 2.6 to account for multiplicity, as long as there are no terms in the divisor with negative coefficient. If the point p_i has multiplicity m in the divisor, we add the terms $\frac{1}{z-p_i}, \frac{1}{(z-p_i)^2}, \ldots, \frac{1}{(z-p_i)^m}$ into the basis for $H^0(D)$ and a similar argument shows this is a basis for $H^0(D)$, which therefore has dimension* $\deg D + 1$.

In this paper, we only prove Riemann-Roch as stated in Theorem 2.5. To do this, we are interested in constructing meromorphic functions with poles only at certain points. Following [Don11], we start by letting X be an Riemann surface of genus g. Then, we let $p \in X$, and attempt to construct a meromorphic function with simple pole at p and no other poles.

We can let z_p be a local coordinate in a neighborhood U of p such that $z_p(p) = 0$. Then, $\frac{1}{z_p}$ is a meromorphic function on U with simple pole at p. To extend this function to be a global function on X, we let B be a bump function such that B is smooth, B is 1 in a neighborhood of p, and B is zero outside of U. Then, $B \cdot \frac{1}{z_p}$ is a smooth function on $U \setminus \{p\}$ that extends to a smooth function on $X \setminus \{p\}$ that is zero outside of U. We also use $B \cdot \frac{1}{z_p}$ to denote the extension of this function when unambiguous.

This function is smooth but not holomorphic on $X \setminus \{p\}$, so we wish to find a smooth f on X such that $f + B \cdot \frac{1}{z_p}$ is holomorphic. One criterion for a function to be holomorphic will be helpful.

Theorem 2.9. *Let g be a smooth function on a Riemann surface X. g is holomorphic if and only if $\bar{\partial} g = 0$.*

Proof. g is of the form $g(x+iy) = u(x,y) + iv(x,y)$, where u and v are real-valued differentiable functions. By the Chain Rule, the partial derivative with respect to $\bar{z}$ is

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\frac{\partial}{\partial x} + \frac{i}{2}\frac{\partial}{\partial y},$$

so

$$\bar{\partial} g = \frac{\partial g}{\partial \bar{z}} d\bar{z} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial x} - \frac{\partial v}{\partial y}\right).$$

By the Cauchy-Riemann Equations, g is holomorphic if and only if $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. From the expression for $\bar{\partial} g$, these are exactly the conditions for the terms in the parentheses to cancel.

□

There is a similar theorem for anti-holomorphic functions, which we state without proof.

Theorem 2.10. *Let g be a smooth function on a Riemann Surface X. g is anti-holomorphic, that is, g is such that*

$$\begin{aligned} \bar{g} : X &\to \mathbb{C} \\ x &\mapsto \overline{g(x)} \end{aligned}$$

is a holomorphic function, if and only if $\partial g = 0$.

We can use Theorem 2.9 to extend $\bar{\partial}(B \cdot \frac{1}{z_p})$ to a smooth $(0,1)$-form on all of X. Since $B \equiv 1$ in a small neighborhood V of p, and $\frac{1}{z_p}$ is holomorphic in $V \setminus \{p\}$, we have $\bar{\partial}(B \cdot \frac{1}{z_p}) = 0$ on $V \setminus \{p\}$. So, we can extend $\bar{\partial}(B \cdot \frac{1}{z_p})$ to a $(0,1)$-form on all of X by defining

$$A = \begin{cases} \bar{\partial}(B \cdot \frac{1}{z_p}) & z \neq p \\ 0 & z = p \end{cases}$$

So, we now have a smooth $(0,1)-$form A defined globally on X, and we are looking for a smooth function f on X such that $\bar{\partial}(f + B \cdot \frac{1}{z_p}) = 0$ on $X \setminus \{p\}$. So, on $X \setminus \{p\}$, we want $\bar{\partial}(f) = -\bar{\partial}(B \cdot \frac{1}{z_p}) = -A$, and since $\bar{\partial} f$ is globally defined, we must have $\bar{\partial} f = -A$. Since A is not defined globally as the image of a certain function under the $\bar{\partial}$ operator, it is not clear that such f exists. Fortunately, cohomology provides us with tools that help measure whether differential one-forms are the images of functions under differential operators.

2.2. Cohomology.

The following two definitions are given in [Lov].

Definition 2.5 (Vector space complex)**.** *A vector space complex* **A** *is a (possibly finite) sequence of vector spaces A^i connected with linear transformations d^i such that $d^{i+1} \circ d^i = 0$ for all i :*

$$0 \xrightarrow{d^{-1}} A^0 \xrightarrow{d^0} A^1 \xrightarrow{d^1} A^2 \cdots$$

When there is no ambiguity, we refer to the d^i operators simply as d.

$\operatorname{Im}(d^{i-1})$ must be a subspace of $\ker(d^i)$ for all i in order for the $d^{i+1} \circ d^i = 0$ condition to be satisfied. This property allows us to define cohomology.

Definition 2.6 (Cohomology)**.** *Let* $\mathbf{A}$ *be a vector space complex. Then, the* k*th cohomology vector space corresponding to* $\mathbf{A}$, H^k, *is*

$$H^k = \ker(d^k)/\operatorname{Im}(d^{k-1}).$$

Based on the type of complex, different notation will be assigned to H^k.

The reason we have gone to all the trouble of defining cohomology is because the forms on a Riemann Surface form a complex, with the d operator as the transformations. Indeed, recall from our definition of forms that, for a smooth f,

$$df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z}$$

so

$$d(df) = d\frac{\partial f}{\partial z} \wedge dz + d\frac{\partial f}{\partial \bar{z}} \wedge d\bar{z} = \left(\frac{\partial}{\partial z}\frac{\partial f}{\partial z}dz + \frac{\partial}{\partial \bar{z}}\frac{\partial f}{\partial z}d\bar{z}\right) \wedge dz + \left(\frac{\partial}{\partial z}\frac{\partial f}{\partial \bar{z}}dz + \frac{\partial}{\partial \bar{z}}\frac{\partial f}{\partial \bar{z}}d\bar{z}\right) \wedge d\bar{z}$$

Since the wedge product is alternating and bilinear, several terms cancel, and we are left with

$$\frac{\partial}{\partial \bar{z}}\frac{\partial f}{\partial z}d\bar{z} \wedge dz + \frac{\partial}{\partial z}\frac{\partial f}{\partial \bar{z}}dz \wedge d\bar{z} = \frac{\partial f}{\partial z \partial \bar{z}}d\bar{z} \wedge dz - \frac{\partial f}{\partial z \partial \bar{z}}d\bar{z} \wedge dz = 0.$$

Also, the d operator annihilates all two-forms in the case of Riemann surfaces. This fact, and the fact that $d^2 = 0$ allows us to define a complex where the transformations are the differential operator d.

Definition 2.7 (De Rham Complex)**.** *Let* X *be a Riemann Surface, and* $\Omega^0(X)$ *be the smooth functions on* X, $\Omega^1(X)$ *be the smooth 1-forms* $fdz + gd\bar{z}$ *on* X, *and* $\Omega^2(X)$ *be the smooth 2-forms, then* $fdz \wedge d\bar{z}$ *on* X,

$$0 \to \Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \Omega^2(X) \xrightarrow{d} 0$$

is called the De Rham complex.

The associated De Rham cohomology spaces are denoted

$$H^0_{dR}(X) = \ker(d : \Omega^0(X) \to \Omega^1(X))$$
$$H^1_{dR}(X) = \ker(d : \Omega^1(X) \to \Omega^2(X))/\operatorname{Im}(d : \Omega^0(X) \to \Omega^1(X))$$
$$H^2_{dR}(X) = \Omega^2(X)/\operatorname{Im}(d : \Omega^1(X) \to \Omega^2(X).$$

However, the De Rham cohomology is not precise enough for Riemann surfaces. It exists on any smooth manifold, but ignores the additional complex structure that Riemann surfaces have. For this reason, we use the $\bar{\partial}$ operator to define other cohomology spaces. Like d, $\bar{\partial}^2 = 0$.

In addition to $\Omega^0(X)$, $\Omega^1(X)$, and $\Omega^2(X)$ as defined in Definition 2.7, we define $\Omega^{0,1}(X)$ to be the space of $(0,1)$-forms on X and $\Omega^{1,0}(X)$ to be the space of $(1,0)$-forms on X. Note $\bar{\partial}(\Omega^{0,1}(X)) = \bar{\partial}(\Omega^2(X))\{0\}$. We now define some cohomology spaces:

Definition 2.8 (Dolbeault cohomology)**.** *We define four Dolbeault cohomology vector spaces, coming from the two complexes*

$$0 \to \Omega^0(X) \xrightarrow{\bar{\partial}} \Omega^{0,1}(X) \xrightarrow{\bar{\partial}} 0 \tag{1}$$

$$0 \to \Omega^{1,0}(X) \xrightarrow{\bar{\partial}} \Omega^2(X) \xrightarrow{\bar{\partial}} 0 \tag{2}$$

The cohomology spaces we define are

$$
\begin{aligned}
H^{0,0}(X) &:= \ker(\bar\partial : \Omega^0(X) \to \Omega^{0,1}(X)) \\
H^{0,1}(X) &:= \Omega^{0,1}(X)/\operatorname{Im}(\bar\partial : \Omega^0(X) \to \Omega^{0,1}(X)) \\
H^{1,0}(X) &:= \ker(\bar\partial : \Omega^{1,0}(X) \to \Omega^2(X)) \\
H^{1,1}(X) &:= \Omega^2(X)/\operatorname{Im}(\bar\partial : \Omega^{1,0}(X) \to \Omega^2(X)
\end{aligned}
$$

$H^{0,0}(X)$ and $H^{0,1}(X)$ come from the first complex and $H^{1,0}(X)$ and $H^{1,1}(X)$ come from the second complex.

Remark. *These are the only nontrivial Dolbeault cohomology spaces in the case of one complex dimension. In higher dimensions, there is a Dolbeault space $H^{p,q}(X)$ for larger p, q.*

The group $H^{0,0}(X)$ is just the space of holomorphic functions, and the group $H^{1,0}(X)$ is the space of holomorphic one-forms. However, the space $H^{0,1}(X)$ is in fact the main reason we made our excursion into cohomology in the first place.

We were trying to determine whether there was a smooth function f such that $\bar\partial f = -A$, where $-A$ is a given smooth $(0,1)$-form. This is equivalent to the cohomology class of $-A$ being 0 in the space $H^{0,1}(X)$. A powerful theorem will let us count the dimension of $H^{0,1}(X)$.

Theorem 2.11 (Hodge Decomposition for Riemann Surfaces). *Let X be a Riemann surface. Then,*

$$H^1_{dR}(X) \cong H^{1,0}(X) \oplus H^{0,1}(X).$$

We need to assume one major analysis theorem before proving this theorem.

Theorem 2.12 (Donaldson's"Main Analysis Theorem" [Don11]). *Let X be a compact Riemann surface. If, for ρ a 2-form, there is a solution to $\partial\bar\partial\phi = \rho$, then $\int_X \rho = 0$, and the solution ϕ is unique up to addition of a constant. Also, if ρ is a 2-form such that $\int_X \rho = 0$, then there is a ϕ such that $\partial\bar\partial\phi = \rho$.*

For a proof, see [Don11].

We are now ready to prove Theorem 2.11.

Proof. This proof uses ideas from [Fil].

Recall

$$H^{0,1}(X) := \Omega^{0,1}(X)/\operatorname{Im}(\bar\partial : \Omega^0(X) \to \Omega^{0,1}(X))$$

and

$$H^{1,0}(X) := \ker(\bar\partial : \Omega^{1,0}(X) \to \Omega^2(X)).$$

We first show the space $H^{0,1}(X)$ is isomorphic to the space $\overline{\Omega^1(X)}$ of anti-holomorphic one-forms, that is, one-forms that are represented as $g(z)d\bar z$ in each coordinate chart where $z \mapsto \overline{g(z)}$ is a holomorphic function. Let $\omega \in \overline{\Omega^1(X)}$, and note since ω is a $(0,1)$-form, $[\omega]$ is an element of $H^{0,1}(X)$. Consider the map

$$
\begin{aligned}
i : \overline{\Omega^1(X)} &\to H^{0,1}(X) \\
\omega &\mapsto [\omega]
\end{aligned}
$$

We have that i is an isomorphism. First, it is an injection, since if $[\omega] = 0$ in $H^{0,1}(X)$, then there is a smooth f such that $\bar\partial f = \omega$. Then, $\partial\bar\partial f = \partial\omega = 0$. This is because in every local coordinate, $\partial\omega$ is $\frac{\partial g}{\partial z} dz \wedge d\bar z$ for anti-holomorphic g and, by Theorem 2.10, $\frac{\partial g}{\partial z} = 0$. By Theorem 2.12, f must be constant, as the solution to $\partial\bar{\partial}\phi = 0$ must be unique up to addition by a constant and $\phi \equiv 0$ is one solution. Since f is constant, $\bar{\partial}f = \omega = 0$. So, $\ker i$ is trivial, so i is injective.

Next, i is a surjection. Let $[\theta] \in H^{0,1}(X)$. Then, we are looking for an anti-holomorphic one-form ω such that $\omega = \theta + \bar{\partial}\phi$ for some smooth ϕ. By Theorem 2.10, this is equivalent to

$$0 = \partial\omega = \partial\theta + \partial\bar{\partial}\phi.$$

So, we wish to show there is ϕ such that $\partial\bar{\partial}\phi = -\partial\theta$, and since

$$\int_X -\partial\theta = \int_{\delta X = \emptyset} -\theta = 0$$

by Stokes' Theorem, such ϕ exists by Theorem 2.12. So, i is surjective.

Therefore $\overline{\Omega^1(X)} \cong H^{0,1}(X)$.

Next, we show

$$H^{1,0}(X) \oplus \overline{\Omega^1(X)} \cong H^1_{dR}(X).$$

Consider the following map:

$$\begin{aligned} v : H^{1,0}(X) \oplus \overline{\Omega^1(X)} &\to H^1_{dR}(X) \\ v(\omega_1, \omega_2) &= [\omega_1 + \omega_2] \end{aligned}$$

v is an isomorphism.

First, v is injective. Suppose $[\omega_1 + \omega_2] = 0$. Then, $\omega_1 + \omega_2 = df$ for some smooth f on X. Since ω_1 is a $(1,0)$-form and ω_2 is a $(0,1)$-form, this means $\omega_1 = \partial f$ and $\omega_2 = \bar{\partial}f$. So, by Theorem 2.10, $\partial\omega_2 = 0$, so $\partial\bar{\partial}f = 0$. Therefore, by Theorem 2.12, f must be a constant function. So, $\partial f = \bar{\partial}f = 0$, so $\omega_1 = \omega_2 = 0$.

So, $\ker v$ is trivial, so v is injective. Next, we show v is surjective.

Let $[\alpha] \in H^1_{dR}(X)$, so that we must have $d\alpha = 0$. Then, in every local coordinate, α is represented as $a_1(z)dz + a_2(z)d\bar{z}$ for smooth a_1, a_2. Let the form represented as $a_1(z)dz$ in every local coordinate be α_1, and let the form represented as $a_2(z)d\bar{z}$ in every local coordinate be α_2.

Since α_1 is a $(1,0)$-form and α_2 is a $(0,1)$-form, we have $d\alpha_1 = \bar{\partial}\alpha_1$ and $d\alpha_2 = \partial\alpha_2$. So, by Stokes' Theorem,

$$\int_X \bar{\partial}\alpha_1 = \int_X \partial\alpha_2 = 0.$$

So, by Theorem 2.12, there exist smooth f_1, f_2 such that $\partial\bar{\partial}f_1 = \bar{\partial}\alpha_1$ and $\partial\bar{\partial}f_2 = -\partial\alpha_2$.

So, we have

$$\partial(\bar{\partial}f_2 + \alpha_2) = 0$$

and

$$\bar{\partial}(\partial f_1 + \alpha_1) = 0$$

(for the latter, note $\partial\bar{\partial} = -\bar{\partial}\partial$) so $\bar{\partial}f_2 + \alpha_2 \in \overline{\Omega^1(X)}$ and $\partial f_1 + \alpha_1 \in H^{1,0}(X)$. Therefore,

$$v(\partial f_1 + \alpha_1, \bar{\partial}f_2 + \alpha_2) = [\alpha + \partial f_1 + \bar{\partial}f_2]$$

is in the image of v. So, if we can show $\partial f_1 + \bar{\partial}f_2$ is of the form df for some smooth f, we will show v is surjective.

Since

$$0 = d\alpha = d\alpha_1 + d\alpha_2 = \bar{\partial}\alpha_1 + \partial\alpha_2,$$

we have

$$\partial\bar{\partial}(f_1 - f_2) = \bar{\partial}\alpha_1 + \partial\alpha_2 = 0$$

so by Theorem 2.12, f_1-f_2 is a constant function. Therefore there exists $c \in \mathbb{C}$ such that $f_1 = f_2+c$. Then,

$$df_1 = \partial f_1 + \bar{\partial} f_1 = \partial f_1 + \bar{\partial}(f_2 + c) = \partial f_1 + \bar{\partial} f_2.$$

Therefore $\partial f_1 + \bar{\partial} f_2$ is equivalent to 0 in the de Rham cohomology, so

$$v(\partial f_1 + \alpha_1, \bar{\partial} f_2 + \alpha_2) = [\alpha + \partial f_1 + \bar{\partial} f_2] = [\alpha]$$

and v is surjective.

So, v is an isomorphism, and $H^{1,0}(X) \oplus \overline{\Omega^1(X)} \cong H^1_{dR}(X)$. Because $H^{0,1}(X) \cong \overline{\Omega^1(X)}$, this gives

$$H^{1,0}(X) \oplus H^{0,1}(X) \cong H^1_{dR}(X).$$

□

In fact, we can quickly determine one more important relation involving the spaces in the Hodge Decomposition.

Theorem 2.13. *Let X be a compact Riemann surface. Then, $H^{1,0}(X) \cong H^{0,1}(X)$.*

Proof. We have $H^{1,0}(X)$ is the space of holomorphic one-forms, and that $H^{0,1}(X)$ is isomorphic to the space $\overline{\Omega^1(X)}$, the space of anti-holomorphic one-forms. The map

$$c : H^{1,0}(X) \to \overline{\Omega^1(X)}$$
$$\omega \mapsto \bar{\omega}$$

is an isomorphism, since it is an invertible linear transformation. So, $H^{1,0}(X) \cong \overline{\Omega^1(X)} \cong H^{0,1}(X)$. □

So, we have $\dim H^{1,0}(X) = \dim H^{0,1}(X) = \frac{1}{2} \dim H^1_{dR}(X)$. By relating the de Rham cohomology to a new homology, related to simplices, we will be able to use these results to determine the dimension of $H^{0,1}(X)$.

Definition 2.9 (n-simplex)**.** *Let $(v_0, v_1, \ldots, v_n)$ be an $(n+1)$-tuplet of points in Euclidean space such that $v_1 - v_0, v_2 - v_0, \ldots, v_n - v_0$ is linearly independent. Then, the n-simplex determined by them is $[v_0, v_1, \ldots, v_n]$ and is the smallest convex set containing $v_0, v_1, \ldots, v_n$.*

The condition "convex" means that for any two points p_1, p_2 in the n-simplex, every point on the line segment between p_1 and p_2 is in the n-simplex, and the condition "smallest" means every other set satisfying the other conditions for an n-simplex is a superset of the n-simplex.

The 0-simplices are points, the 1-simplices are line segments, the 2-simplices are triangles, the 3-simplices are tetrahedra, and the other simplices are higher-dimensional analogues of these spaces. The linear independence condition prevents any degeneracy (like, for instance, a "tetrahedron" with four coplanar vertices) in the simplex. All simplices inherit the subspace topology from the euclidean space, and in fact all n-simplices are homeomorphic to the n-ball. For more discussion, see [**?**]. We can use simplices to understand any topological space X by defining simplices in X.

Definition 2.10 (Singular n-simplex)**.** *Let X be a topological space. A singular n-simplex in X is the image $\sigma([v_0, \ldots, v_n])$ of a continuous map*

$$\sigma : [v_0, \ldots, v_n] \to X$$

where $[v_0, \ldots, v_n]$ is an n-simplex.

As with divisors, we define formal sums on singular n-simplices to get a space of interest.

Definition 2.11 (Singular n-chains). *We define $C_n(X, \mathbb{C})$, the space of singular n-chains, to be the space of formal sums*

$$\sum_a g_a a$$

where all but finitely many terms are zero, the coefficients g_a are elements of $\mathbb{C}$, and the sum is over singular n-simplices a in X.

A boundary operator on singular n-chains allows chains to have structure similar to a complex:

Definition 2.12 (Boundary map). *We define the boundary map $\delta_n : C_n(X, \mathbb{C}) \to C_{n-1}(X, \mathbb{C})$ by defining it on singular n-simplices and letting it be linear on $C_n(X, \mathbb{C})$. Let $\sigma([v_0, \ldots, v_n])$ be a singular n-simplex. Then, we define*

$$\delta_n(\sigma([v_0, \ldots, v_n])) = \sum_{i=0}^{n} (-1)^i \sigma([v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n]).$$

We have $\delta_{n-1} \circ \delta_n = 0$ (see [Hat02] for a proof), so we can define a complex moving in the reverse direction:

$$0 \xleftarrow{\delta_0} C_0(X, \mathbb{C}) \xleftarrow{\delta_1} C_1(X, \mathbb{C}) \xleftarrow{\delta_2} C_2(X, \mathbb{C}) \xleftarrow{\delta_3} \cdots$$

This gives a homology, not cohomology:

Definition 2.13 (Singular homology). *We define the nth singular homology space as*

$$H_n(X, \mathbb{C}) := \ker(\delta_n) / \operatorname{Im}(\delta_{n+1})$$

Especially important is the space $H_1(X, \mathbb{C})$, as it is closely related to the genus of X. In Figure 2, the two generators of the fundamental group are also a basis for $H_1(X, \mathbb{C})$, and, in general, the dimension of $H_1(X, \mathbb{C})$ is twice the genus of a space. This is sometimes even taken to be the definition of genus.

As the name suggests, cohomology is dual to homology, and there is a dual singular cohomology.

Definition 2.14. *Consider the complex*

$$0 \xrightarrow{\delta_0^*} C_0(X, \mathbb{C})^* \xrightarrow{\delta_1^*} C_1(X, \mathbb{C})^* \xrightarrow{\delta_2^*} C_2(X, \mathbb{C})^* \xrightarrow{\delta_3^*} \cdots$$

made up of dual spaces and dual maps to the "reverse complex" for singular homology. The nth singular cohomology space is defined as

$$H^n(X, \mathbb{C}) := \ker(\delta_n^*) / \operatorname{Im}(\delta_{n-1}^*)$$

In the case where X is a compact Riemann surface of genus g, $H_1(X, \mathbb{C})$ has dimension $2g$, and $H_1(X, \mathbb{C})$ is isomorphic to its dual space $\operatorname{Hom}(H_1(X, \mathbb{C}), \mathbb{C})$, which therefore also has dimension $2g$. The map from $H^1_{dR}(X)$ to $\operatorname{Hom}(H_1(X, \mathbb{C}), \mathbb{C})$ is constructed as follows. Let $[\omega] \in H^1_{dR}(X)$, then we map $[\omega]$ to the element of $\operatorname{Hom}(H_1(X, \mathbb{C}), \mathbb{C})$ that takes, a 1-chain $[C_1]$, to $\int_{C_1} \omega$. This is well-defined by Stokes' theorem and the definitions of de Rham cohomology and singular homology.

Indeed, for every smooth function f and 2-chain C_2 :

$$\begin{aligned}\int_{C_1+\delta C_2} \omega + df &= \int_{C_1+\delta C_2} \omega + \int_{C_1+\delta C_2} df = \int_{C_1+\delta C_2} \omega + \int_{\delta(C_1+\delta C_2)} f = \int_{C_1+\delta C_2} \omega \\ &= \int_{C_1} \omega + \int_{\delta C_2} \omega = \int_{C_1} \omega + \int_{C_2} d\omega \\ &= \int_{C_1} \omega.\end{aligned}$$

So, this map has the same output for different representatives of homology and cohomology classes. De Rham's Theorem says this map is actually an isomorphism.

Theorem 2.14 (De Rham's Theorem)**.** *Let X be a compact Riemann surface. Then,*

$$H^1_{dR}(X) \cong \mathrm{Hom}(H_1(X, \mathbb{C}), \mathbb{C}).$$

See [GH94] for a proof of Theorem 2.14. As a result we now know that the dimension of $H^1_{dR}(X)$ is $2g$, where g is the genus of X, as $\dim H_1(X, \mathbb{C}) = \dim \mathrm{Hom}(H_1(X, \mathbb{C}), \mathbb{C}) = 2g$. Furthermore, from Theorem 2.11, $H^1_{dR}(X) \cong H^{1,0}(X) \oplus H^{0,1}(X)$ and by Theorem 2.13, $H^{1,0}(X) \cong H^{0,1}(X)$ therefore, we get that $\dim H^{0,1}(X) = g$.

3. Riemann's Existence Theorem and Final Proofs

We can now show another theorem, which will be instrumental in our proof of Riemann-Roch. Along the way, we will show the Uniformization Theorem in the genus 0 case.

Theorem 3.1 (Riemann's Existence Theorem)**.** *Let X be a compact Riemann surface. There is a nonconstant meromorphic function on X.*

Recall, for a Riemann surface X of genus g, we had constructed a smooth function $B \cdot \frac{1}{z_p}$ on $X \setminus \{p\}$ and with a pole at p, and we were looking for a smooth f on X such that $f + B \cdot \frac{1}{z_p}$ is holomorphic. This came down to finding smooth f such that $\bar{\partial} f = -A$ for a globally defined $(0, 1)$-form A.

Theorem 3.2. *Let X be a Riemann surface of genus 0. X is biholomorphic to the Riemann sphere $\hat{\mathbb{C}}$.*

Proof. If $g = 0$, then $\dim H^{0,1}(X) = 0$, so

$$H^{0,1}(X) = \Omega^{0,1}(X)/\,\mathrm{Im}(\bar{\partial} : \Omega^0(X) \to \Omega^{0,1}(X)) = \{0\}$$

so every globally defined $(0, 1)$-form is in the image of the $\bar{\partial}$ operator on the smooth functions on X. Therefore in the genus 0 case, there is a smooth function f such that $\bar{\partial} f = -A$.

This means $f + B \cdot \frac{1}{z_p}$ is holomorphic on $X \setminus \{p\}$ (as $\bar{\partial}(f + B \cdot \frac{1}{z_p}) = 0$), and since $B \cdot \frac{1}{z_p}$ increases without bound near p, $f + B \cdot \frac{1}{z_p}$ is a meromorphic function on $X \setminus \{p\}$ with one simple pole at p. Therefore

$$f + B \cdot \frac{1}{z_p} : X \to \hat{\mathbb{C}}$$

is a nonconstant holomorphic function, so by Theorem 1.1 $f + B \cdot \frac{1}{z_p}$ is a ramified cover of $\hat{\mathbb{C}}$. Since $f + B \cdot \frac{1}{z_p}$ has just one pole of order 1, ∞ has one preimage of multiplicity 1, and therefore all points in $\hat{\mathbb{C}}$ have exactly one preimage under this map. Therefore $f + B \cdot \frac{1}{z_p}$ is a biholomorphism between X and $\hat{\mathbb{C}}$. $\square$

In particular, X is a quotient of $\hat{\mathbb{C}}$ by the trivial group, so X follows form 2 of the Uniformization Theorem.

For surfaces X of genus $g > 0$, it takes a little more work to guarantee a nonconstant meromorphic function on X, as there are now $(0,1)$-forms on X that are not images of smooth functions under the $\bar{\partial}$ operator. We get around this problem by adding more poles to our meromorphic function.

Let $p_1, \ldots, p_{g+1}$ be distinct points in X. For each p_i, let U_i be a neighborhood of p_i on which z_i is a coordinate chart such that $z_i(p_i) = 0$. Then, as when we were constructing $B \cdot \frac{1}{z_p}$, for each $1 \leq i \leq g+1$, let B_i be a smooth bump function on X such that $B_i \equiv 0$ outside of U and $B_i = 1$ in a neighborhood V_i of p_i. Then, $B_i \cdot \frac{1}{z_i}$ is smooth on U_i and can be extended outside of U_i by zero to be smooth on X. Also, $B_i \cdot \frac{1}{z_i}$ is holomorphic on $V_i \setminus \{p\}$, so $\bar{\partial}(B_i \cdot \frac{1}{z_i})$ is zero on $V_i \setminus \{p\}$, so we can extend $\bar{\partial}(B_i \cdot \frac{1}{z_i})$ to a smooth global $(0,1)$-form on X, which we call A_i.

Now, since $[A_i] \in H^{0,1}(X)$ for each $1 \leq i \leq g+1$, and $\dim H^{0,1}(X) = g$, there exist $\lambda_1, \ldots, \lambda_{g+1} \in \mathbb{C}$, not all zero, such that

$$-\lambda_1[A_1] - \cdots - \lambda_{g+1}[A_{g+1}] = 0.$$

By the definition of $H^{0,1}(X)$, there exists some smooth f on X such that

$$\bar{\partial} f = -\lambda_1 A_1 - \cdots - \lambda_{g+1} A_{g+1}.$$

Then,

$$M := f + \lambda_1 B_1 \frac{1}{z_1} + \cdots + \lambda_{g+1} B_{g+1} \frac{1}{z_{g+1}}$$

is an meromorphic function on X with poles only in $\{p_1, \ldots, p_{g+1}\}$, and with all of these poles simple. Indeed, $\bar{\partial}(M) = 0$ everywhere away from the p_i for which $\lambda_i \neq 0$, so M is holomorphic away from finitely many points, which are its poles, so M is in fact meromorphic. Since M has some poles (as there are nonzero λ_i), M is nonconstant. This concludes the proof of Riemann's Existence Theorem.

3.1. **Proof of Riemann-Roch.** We now are ready for our proof of Riemann-Roch, as stated in Theorem 2.5. We will use ideas from [Don11].

Let X be a compact Riemann surface of genus g, and let $D = \{p_1, \ldots, p_d\}$ be a divisor of X, such that the p_i are all distinct elements of X. We will first investigate $H^0(D)$, the space of meromorphic functions on X with all poles simple and in D, which we will eventually relate to $H^0(K - D)$, the space of holomorphic one-forms on X that vanish at every point in D.

It would be nice to be classify elements of $H^0(D)$ by their behavior in D. For motivation, let $D = \{p_1, \ldots, p_d\}$ be a divisor of $\hat{\mathbb{C}}$ such that $D \subset \mathbb{C}$ and consider $H^0(D)$. By the definition of $\hat{\mathbb{C}}$ (the complex plane with a point at infinity) there are two canonical coordinates $z : \hat{\mathbb{C}} \setminus \{\infty\} = \mathbb{C} \to \mathbb{C}$ where z is the identity map on $\mathbb{C}$, and $w : \hat{\mathbb{C}} \setminus \{0\} \to \mathbb{C}$ where $w = \frac{1}{z}$ on the overlap. Then, a meromorphic function is determined, up to addition by a constant, by its residues in these local coordinates. Indeed, from the proof of Theorem 2.6, if $c_1, \ldots, c_d$ are the residues of some meromorphic F, when expressed in z, at $p_1, \ldots, p_d$, then the function

$$f(z) = F(z) - \frac{c_1}{z - p_1} - \frac{c_2}{z - p_2} - \cdots - \frac{c_d}{z - p_d}$$

is constant on $\hat{\mathbb{C}}$. So, the residues, when expressed in z, do indeed determine F on the entire surface.

However, suppose we defined a different local coordinate $z' : \mathbb{C} \to \mathbb{C}$ such that $z' = 2z$. Then, a function $g : \mathbb{C} \to \mathbb{C}$ such that $g(z) = \frac{1}{z}$ would now be expressed $g(z') = \frac{2}{z'}$, and the residues of g with respect to z and z' are different. This is an even greater problem with surfaces that are not $\hat{\mathbb{C}}$, where there is no canonical coordinate and we might not even know the nature of the coordinate charts. We circumvent this problem by defining a new notion of residue.

Definition 3.1 (Tangent Residue)**.** *Let z be a local coordinate on a compact Riemann surface X, let $z(p) = 0$, and let f be a meromorphic function on X with Laurent expansion at p*

$$f(z) = \sum_{i=-1} a_i z^i.$$

Then, we define the tangent residue of f at p to be $\operatorname{Res}_p f = a_{-1}\frac{\partial}{\partial z} \in T_pX$, where T_pX is the tangent space of X at p.

In this and what follows, we consider the tangent space as a complex vector space, which has complex dimension 1 and basis $\frac{\partial}{\partial z}$, where z is a local coordinate.

The notion of "residue" is now well-defined, regardless of local coordinate, and meromorphic functions with only simple poles are still determined, up to addition by a constant, by these residues. Let $D = \{p_1, \dots, p_d\}$ be a divisor on a compact Riemann Surface X, and let $f_1, f_2 \in H^0(D)$ such that $\operatorname{Res}_{p_i} f_1 = \operatorname{Res}_{p_i} f_2$ for all $1 \le i \le d$. Then, f_1 and f_2 differ by a holomorphic function, which is constant by Theorem 1.2.

This gives a map

$$R : H^0(D) \to \bigoplus_{i=1}^{d} T_{p_i}X$$

taking a function f to the d-tuplet of its tangent residues. Now, consider the following sequence:

$$0 \to \mathbb{C} \xrightarrow{I} H^0(D) \xrightarrow{R} \bigoplus_{i=1}^{d} T_{p_i}X$$

where I is an injective inclusion map taking $c \in \mathbb{C}$ to the function $f_c \equiv c$. This is therefore a complex, since $R \circ I = 0$. Furthermore, the associated cohomology group $\ker R/\operatorname{Im} I$ is trivial, as $\ker R$ is exactly the space of holomorphic functions on X, which is just the constants by Theorem 1.2, so $\ker R = \operatorname{Im} I$.

We would like to determine the value $h^0(D) = \dim H^0(D)$. By the Rank-Nullity Theorem, we have

$$\dim H^0(D) = \dim \ker R + \dim \operatorname{Im} R = \dim \operatorname{Im} I + \dim \operatorname{Im} R = 1 + \dim \operatorname{Im} R.$$

So, we now investigate $\operatorname{Im} R$.

We define a map from $\bigoplus_{i=1}^{d} T_{p_i}X$ to $H^{0,1}(X)$ such that its kernel is $\dim \operatorname{Im} R$. For each $1 \le i \le d$, let $\mathcal{A}_i : T_{p_i}X \to H^{0,1}(X)$ be a linear map that, for a local coordinate z_i of p_i, takes $\frac{\partial}{\partial z}$ to the cohomology class of the global $(0,1)$-form A_i, the extension of $\bar{\partial}(B_i \cdot \frac{1}{z_i})$, as defined in the proof of Riemann's Existence Theorem. $\mathcal{A}_i$ is then extended over the rest of $T_{p_i}X$ to be linear. The image $\mathcal{A}_i(\lambda_i \frac{\partial}{\partial z_i}) = \lambda_i[A_i]$, where A_i is defined in terms of the local coordinate z_i, is independent of choice of local z_i.

Next, we define $\mathcal{A} : \bigoplus_{i=1}^{d} T_{p_i}X \to H^{0,1}(X)$ such that, for any $(t_1, \dots, t_d) \in \bigoplus_{i=1}^{d} T_{p_i}X$,

$$\mathcal{A}(t_1, \dots, t_d) = \sum_{i=1}^{d} \mathcal{A}_i(t_i)$$

Now, we have $\mathcal{A} \circ R = 0$, since given meromorphic $f \in H^0(D)$,

$$\mathcal{A} \circ R(f) = \sum_{i=1}^{d} \lambda_i [A_i],$$

such that $\lambda_i \frac{\partial}{\partial z_i}$ is the tangent residue of f at p_i for each $1 \leq i \leq d$. Then, consider the function

$$F = f - \sum_{i=1}^{d} \lambda_i B_i \cdot \frac{1}{z_i}.$$

F has no poles on X by the definition of tangent residue and since f only has simple poles, all in D, so F is smooth on X. Therefore,

$$[\bar{\partial} F] = [\bar{\partial}(f - \sum_{i=1}^{d} \lambda_i B_i \cdot \frac{1}{z_i})] = 0$$

in $H^{0,1}(X)$.

Note since f is holomorphic away from the p_i, $\bar{\partial} f = 0$ on $X \setminus D$. We can extend $\bar{\partial} f$ to be zero on all of X.

Then,

$$[\bar{\partial} f] - \sum_{i=1}^{d} \lambda_i [A_i] = [\bar{\partial} f] - [\sum_{i=1}^{d} \bar{\partial}(\lambda_i B_i \cdot \frac{1}{z_i})] = 0$$

in $H^{0,1}(X)$, so

$$\mathcal{A} \circ R(f) = \sum_{i=1}^{d} \lambda_i [A_i] = 0$$

and $\operatorname{Im} R$ is a subspace of $\ker \mathcal{A}$.

Furthermore, we actually have $\operatorname{Im} R = \ker \mathcal{A}$. Let $\Lambda = (\lambda_1 \frac{\partial}{\partial z_1}, \ldots, \lambda_d \frac{\partial}{\partial z_d}) \in \ker \mathcal{A}$. Then, we have

$$\sum_{i=1}^{d} \lambda_1 [A_1] = 0,$$

so by the definition of $H^{0,1}(X)$ there exists some smooth g on X such that $\bar{\partial} g = -\lambda_1 A_1 - \cdots - \lambda_d A_d$. Then, $\bar{\partial}(g + \lambda_1 B_1 \cdot \frac{1}{z_1} + \cdots + \lambda_d B_d \cdot \frac{1}{z_d}) = 0$, so

$$G := g + \lambda_1 B_1 \cdot \frac{1}{z_1} + \cdots + \lambda_d B_d \cdot \frac{1}{z_d}$$

has all of its poles simple and in D and is holomorphic away from its poles, so $G \in H^0(D)$. Also, $R(G) = \Lambda$. So, $\ker \mathcal{A}$ is a subspace if $\operatorname{Im} R$. So, $\operatorname{Im} R = \ker \mathcal{A}$. Since, from our above rank-nullity claim, we have $\dim H^0(D) = 1 + \dim \operatorname{Im} R$, we have

$$\dim H^0(D) = 1 + \dim \ker \mathcal{A}.$$

We also have $\dim \bigoplus_{i=1}^{d} T_{p_i} X = d$, since each tangent space has complex dimension 1. So, using facts from linear algebra,

$$\begin{aligned} d - \dim \ker \mathcal{A} &= \dim \operatorname{Im} \mathcal{A} = \dim H^{0,1}(X) - \dim(\operatorname{Im} \mathcal{A})^{\perp} = \dim H^{0,1}(X) - \dim \ker \mathcal{A}^* \\ \implies \dim \ker \mathcal{A} &= d - \dim H^{0,1}(X) + \dim \ker \mathcal{A}^* \end{aligned}$$

where $\mathcal{A}^*$ is the dual map or transpose of $\mathcal{A}$.

By the way dual spaces interact with the direct sum, we have $(\bigoplus_{i=1}^{d} T_{p_i}X)^*$ with $\bigoplus_{i=1}^{d} T^*_{p_i}X$, where $T^*_{p_i}X$ is the cotangent space attached to p_i. We also want to associate a space to $(H^{0,1}(X))^*$ to better understand $(H^{0,1}(X))^*$. Consider the following construction, which identifies $H^{1,0}(X)$ with $(H^{0,1}(X))^*$:

Given a one-form $\alpha \in H^{1,0}(X)$, we let α correspond to the element of $(H^{0,1}(X))^*$ that takes the class of forms $[\theta] \in H^{0,1}(X)$ to $\int_X \alpha \wedge \theta$. We abuse notation and consider α as a function in $(H^{0,1}(X))^*$, writing $\alpha([\theta]) = \int_X \alpha \wedge \theta$. This expression is well-defined, regardless of representative for the class $[\theta]$, since if $[\theta'] = [\theta]$, we have that there exists smooth f such that $\theta' = \theta + \bar{\partial} f$. Then, we have

$$\begin{aligned}\alpha([\theta']) &= \int_X \alpha \wedge \theta' = \int_X \alpha \wedge (\theta + \bar{\partial} f) = \int_X \alpha \wedge \theta + \int_X \alpha \wedge \bar{\partial} f \\ &= \alpha([\theta]) + \int_X d(\alpha \wedge f) \\ &= \alpha([\theta]) + \int_{\delta X = \emptyset} \alpha \wedge f = \alpha([\theta])\end{aligned}$$

(We have $\alpha \wedge \bar{\partial} f = d(\alpha \wedge f)$ since, because α is a holomorphic 1-form, $\partial\alpha \wedge f$, $\bar{\partial}\alpha \wedge f$, and $\alpha \wedge \partial f$ all vanish. The next line follows from this by Stokes' Theorem.) This shows the correspondence between $H^{1,0}(X)$ and $(H^{0,1}(X))^*$ is well-defined. It is shown in [Don11] that this correspondence defines an isomorphism.

So, we can consider $\mathcal{A}^*$ as a map from $H^{1,0}(X)$ to $\bigoplus_{i=1}^{d} T^*_{p_i}X$. We are now able to relate $\ker \mathcal{A}^*$ to one of the key terms in Riemann-Roch.

Theorem 3.3. *We have* $\ker \mathcal{A}^* = H^0(K - D)$.

Proof. Let $\alpha \in H^{1,0}(X)$ and consider α as an element of $(H^{0,1}(X))^*$ as above. We consider the conditions for $\mathcal{A}^*(\alpha) = 0$. $\mathcal{A}^*(\alpha)$ is the functional that takes an element

$$(t_1, \ldots, t_d) \in \bigoplus_{i=1}^{d} T_{p_i}X$$

to

$$\alpha(\mathcal{A}(t_1, \ldots, t_d)).$$

Since each tangent space has complex dimension 1, each t_i must actually be $\lambda_i \frac{\partial}{\partial z_i}$ for some $\lambda_i \in \mathbb{C}$. Since $\mathcal{A}^*(\alpha)$ is linear, we have that $\mathcal{A}^*(\alpha)$ vanishes on all vectors in $\bigoplus_{i=1}^{d} T^*_{p_i}X$ if and only it vanishes on all vectors $\frac{\partial}{\partial z_i}$. So, we consider

$$\mathcal{A}^*(\alpha)(\frac{\partial}{\partial z_i}).$$

This is equal to

$$\alpha \circ \mathcal{A}(\frac{\partial}{\partial z_i}) = \alpha([A_i]) = \int_X \alpha \wedge A_i.$$

In the local coordinate z_i, let α be represented as $g(z_i)dz_i$. Recall that A_i is the extension of $\bar{\partial}(B_i \frac{1}{z_i})$ for some smooth bump function B_i, and that z_i is defined such that z_i is 0 at p_i. Let U be a neighborhood of p_i such that B_i is 1 on U. Then, let γ be a path in U around p. Then, γ partitions X into two regions, that is, $X \setminus \gamma$ has two connected components. Let C be the connected component of $X \setminus \gamma$ containing p_i, so $p_i \in C \subset U$. We have that γ is the boundary for both C and $X \setminus C$. Then, we have, using Stokes' theorem,

$$\begin{aligned}\alpha([A_i]) &= \int_X \alpha \wedge A_i \\ &= \int_{X\setminus C} \alpha \wedge A_i + \int_C \alpha \wedge A_i \\ &= \int_{X\setminus C} \alpha \wedge \bar{\partial}(B_i \cdot \frac{1}{z_i}) + \int_C \alpha \wedge \bar{\partial}(B_i \cdot \frac{1}{z_i}) \\ &= \int_{\delta(X\setminus C)=\gamma} B_i \cdot \frac{1}{z_i}\alpha + \int_{\delta C=\gamma} B_i \cdot \frac{1}{z_i}\alpha \\ &= 2\int_\gamma B_i \cdot \frac{1}{z_i}\alpha \\ &= 2\int_\gamma \frac{1}{z_i}\alpha \\ &= 2\int_\gamma \frac{g(z_i)}{z_i} dz_i.\end{aligned}$$

So we have

$$\begin{aligned}d(B_i\frac{1}{z_i}\alpha) &= \alpha \wedge d(B_i\frac{1}{z_i}) - B_i\frac{1}{z_i}d\alpha \\ &= \alpha \wedge \partial(B_i\frac{1}{z_i}) + \alpha \wedge \bar{\partial}(B_i\frac{1}{z_i}) - B_i\frac{1}{z_i}\partial\alpha - B_i\frac{1}{z_i}\bar{\partial}\alpha \\ &= \alpha \wedge \bar{\partial}(B_i\frac{1}{z_i}) \\ &= \alpha \wedge A_i,\end{aligned}$$

since α is a holomorphic (1, 0)-form, allowing us to apply Stokes' Theorem.

By the Residue Theorem, the integral

$$2\int_\gamma \frac{g(z_i)}{z_i} dz_i$$

evaluates to $2(2\pi i g(0))$. So, $\alpha([A_i])$ vanishes if and only if $g(0) = 0$, which occurs exactly when α is 0 at p_i. So, $\mathcal{A}^*(\alpha) = 0$ if and only α vanishes on all $[A_i]$'s, equivalent to the condition that α vanishes at every p_i. So, $\alpha \in \ker \mathcal{A}^*$ if and only if $\alpha \in H^0(K - D)$. □

Now, combining the two dimension relations

$$\dim H^0(D) = 1 + \dim \ker \mathcal{A}$$

$$\dim \ker \mathcal{A} = d - \dim H^{0,1}(X) + \dim \ker \mathcal{A}^*$$

gives

$$\begin{aligned}\dim H^0(D) &= 1 + d - \dim H^{0,1}(X) + \dim \ker \mathcal{A}^* \\ &= d - g + 1 + \dim H^0(K - D)\end{aligned}$$

So,

$$h^0(D) - h^0(K - D) = 1 + d - g.$$

The Riemann-Roch Theorem is proved.

3.2. **Proof of Uniformization.** We conclude with a proof of the Uniformization Theorem, following [Don11]. Note we have already dealt with the case where X is compact and simply connected. In this case, X is of genus 0, so X is biholomorphic to $\hat{\mathbb{C}}$ by Theorem 3.2.

Theorem 3.4 (The Uniformization Theorem, version 3)**.** *Let X be a non-compact simply connected Riemann Surface. Then, X is biholomorphic to either $\mathbb{C}$ or $\mathbb{D}$.*

Proof. We look to construct a map $f : X \to \hat{\mathbb{C}}$ such that f is injective and proper. That is, f is injective and inverse images under f of compact sets are compact. We use a similar theorem to Theorem 2.12 to show the following result.

Lemma 3.1. *Let X be a simply connected, non-compact Riemann surface. If ρ is a form on X with compact support and such that $\int_X \rho = 0$, then there exists a smooth function ϕ on X such that $\partial\bar{\partial}\phi = \rho$ and, for any $\epsilon > 0$, there is a compact subset $K \subset X$ such that $|\phi(x)| < \epsilon$ on $X \setminus K$. (That is,"ϕ tends to 0 at infinity in X").*

For a proof of Lemma 3.1 see [Don11]. We also will also need the Riemann Mapping Theorem.

Lemma 3.2 (Riemann Mapping Theorem)**.** *Let U be a simply connected proper subset of $\mathbb{C}$. Then, there is a biholomorphism $b : U \to \mathbb{D}$.*

For a proof of Lemma 3.2 see [Fil]. — Now we put all of our results together to demonstrate the existence of the desired injective, proper map $f : X \to \hat{\mathbb{C}}$.

Let $p \in X$, and let z be a local coordinate on a neighborhood U of p such that $z(p) = 0$. Then, as in the proof of Riemann's Existence Theorem, let B be a bump function such that B is identically 1 on a smaller neighborhood of p, and B is compactly supported with support entirely within U. Then, $B \cdot \frac{1}{z}$ has a pole at p but is not meromorphic. Again, let $A = \bar{\partial}(B \cdot \frac{1}{z})$ on $X \setminus \{p\}$, and extend A to be a global $(0, 1)$-form since A is 0 in a neighborhood of p.

Then, $\rho = \partial A$ is a $(1, 1)$-form with compact support, as ∂A vanishes outside the support of B. Then,

$$\int_X \rho = \int_X \partial A = \int_X dA = 0,$$

by Stokes' Theorem. By Theorem 3.1, there is a smooth g on X such that $\partial\bar{\partial}g = \rho$ and g tends to zero at infinity in X. Now, consider the 1-form

$$a = (A - \bar{\partial}g) + \overline{A - \bar{\partial}g}.$$

a is twice the real part of $A - \bar{\partial}g$, so a is a real one-form. Then, we have $d(A - \bar{\partial}g) = 0$, since $\partial(A - \bar{\partial}g) = \partial A - \partial\bar{\partial}g = 0$ and $A - \bar{\partial}g$ is a $(0, 1)$-form, so $\bar{\partial}(A - \bar{\partial}g)$ also vanishes. Similarly, we have $d(\overline{A - \bar{\partial}g}) = 0$. So, $da = 0$.

Because X is simply connected, its 1st singular homology group is trivial, so by Theorem 2.14, we have $H^1_{dR}(X, \mathbb{R}) = 0$. This theorem was stated with both the singular homology and the de Rham cohomology defined over the complex numbers, but it still holds true when they are defined over the reals. The de Rham cohomology over the reals is defined where complex-valued smooth functions are replaced with real-valued smooth functions, complex one-forms are replaced with real one-forms, and so on.

Then, since $da = 0$ and a is a real one-form, we have $[a] = 0$ in $H^1_{dR}(X, \mathbb{R})$. By the definition of $H^1_{dR}(X, \mathbb{R})$, there is a real-valued smooth function ψ on X such that $a = d\psi$. Then, $A - \bar{\partial}g = \bar{\partial}\psi$, because the left is the $(0,1)$-form component of a and the right is the $(0,1)$-form component of $d\psi$. So,

$$\bar{\partial}(B \cdot \frac{1}{z} - (g + \psi)) = 0.$$

Let $f = B \cdot \frac{1}{z} - (g + \psi)$. Now, f is meromorphic with one simple pole at p, since it is holomorphic away from p by Theorem 2.9, and it has simple pole behavior at p. Furthermore, the imaginary part of f tends to 0 at infinity in X. This is because $B \cdot \frac{1}{z}$ and g tend to zero at infinity in X and ψ is real-valued.

Now, consider the spaces $\mathbb{H}_+ = \{z \in \mathbb{C} \mid \operatorname{im} z > 0\}$ and $\mathbb{H}_- = \{z \in \mathbb{C} \mid \operatorname{im} z < 0\}$. We also define $X_+ = f^{-1}(\mathbb{H}_+)$ and $X_- = f^{-1}(\mathbb{H}_-)$, along with $f_+ : X_+ \to \mathbb{H}_+$ and $f_- : X_- \to \mathbb{H}_-$ the (respective) restriction maps. Since f is continuous, X_+ and X_- are open. Also, f_+ and f_- are holomorphic maps.

Now, f_+ and f_- are proper maps. Suppose B is a compact subset of $\mathbb{H}_+$. Then, there is $\epsilon > 0$ such that $\operatorname{im} z > \epsilon$ for all $z \in K$. Using an argument from [Fil], suppose $f_+^{-1}(B)$ did not have compact closure. Then, we would have a sequence a_n in $f_+^{-1}(B)$ such that $\operatorname{im} f(a_n)$ approaches 0 as n approaches ∞, since the imaginary part of f tends to 0 at infinity in X. This violates the condition of a positive lower bound on $\operatorname{im} z$, so $f_+^{-1}(B)$ must have compact closure. Since $\mathbb{H}_+$ is Hausdorff, B must be closed, so $f_+^{-1}(B)$ is closed and therefore equals its closure, which is compact. So f_+ is proper. A similar argument shows f_- is also proper. Because f_+ and f_- are proper maps, they are ramified covers of their images [Don11], Proposition 7.

Since f has a simple pole at p, there is a neighborhood of ∞ in the image of f. This neighborhood must intersect both $\mathbb{H}_+$ and $\mathbb{H}_-$, so both X_+ and X_- are nonempty. So, both f_+ and f_- are ramified coverings with degree at least 1, possibly infinite. We show the degree in fact must be 1.

Suppose to the contrary f_+ has degree at least 2. Then, using the fact that the imaginary part of f tends to 0 at infinity, let K be a compact subset of X such that $\operatorname{im} f < 1$ on $X \setminus K$. Then, for each positive integer n, there is x_n, y_n such that $f_+(x_n) = f_+(y_n) = ni$ and $x_n = y_n$ only if $f_+'(x_n) = 0$ (that is, x_n and y_n are distinct unless they are a branch point for f_+). The sequences (x_n) and (y_n) are contained within K, so there is a convergent subsequence of both (x_n) and (y_n), converging to limits x, y. Since f_+ is continuous, we must have $f_+(x)$ and $f_+(y)$ equal ∞, since $(f_+(x_n))$ and $(f_+(y_n))$ approach ∞. This means, since f_+ has one pole at p, $x = y = p$ and there are sequence elements of both (x_n) and (y_n) arbitrarily close to p. This violates the simple pole behavior of f at p, as it either violates injectivity or the condition that the derivative of f cannot vanish in a neighborhood of p.

So, f_+, and f_- by a similar argument, are ramified coverings of degree 1 and therefore bijections. We now show f is injective on its entire domain. Suppose there were two points $x_1 \neq x_2$ in X such that $f(x_1) = f(x_2)$. Then, we must have $f(x_1) \in \mathbb{R}$, since f only has one preimage for points in $\mathbb{H}_+ \cup \mathbb{H}_- \cup \{\infty\}$. But, then there is a neighborhood $U_{f(x_1)}$ of $f(x_1) \in \hat{\mathbb{C}}$ such that there are disjoint neighborhoods U_1 of x_1 and U_2 of x_2 such that $f(U_1)$ and $f(U_2)$ both contain $U_{f(x_1)}$. Then, $U_{f(x_1)}$ contains elements of $\mathbb{H}^+$, so U_1 and U_2 both contain preimages of $\mathbb{H}^+$, contradicting the fact that f is injective on $\mathbb{H}^+$. So, f must be injective.

So, f is a biholomorphism from X to $f(X)$. We also have $\mathbb{H}^+ \cup \mathbb{H}^- \cup \{\infty\} \subset f(X)$. So,

$$\hat{\mathbb{C}} \setminus f(X) \subset \mathbb{R}.$$

Since X is simply connected, $f(X)$ is simply connected, so $\hat{\mathbb{C}} \setminus f(X)$ is connected and therefore is some interval $I \subset \mathbb{R}$. Furthermore, f attains the value ∞ in $\hat{\mathbb{C}}$, so f attains a neighborhood of ∞ and therefore there is an upper and lower bound on points in I. So, I is an interval of finite length.

Because $f(X)$ must be open, I must be closed, so we have $I = [a, b]$ for some $a \leq b \in \mathbb{R}$. — Notice I must be nonempty since X is not compact. Therefore we conclude by considering two cases.

Case 1: If $a = b$, then X is biholomorphic to $f(X)$, which is the Riemann sphere with one point removed and therefore biholomorphic to the complex plane, where the biholomorphism is given by an automorphism of the Riemann Sphere that shifts the point removed from the Riemann Sphere to ∞.

Case 2: If $a < b$, then $f(X)$ is biholomorphic to a proper subset of the complex plane, given by an automorphism that shifts one of the points removed to ∞. In this case the Riemann Mapping theorem, Theorem 3.2, guarantees that $f(X)$ is biholomorphic to $\mathbb{D}$, so X is biholomorphic to $\mathbb{D}$.

In one case, X is biholomorphic to $\mathbb{C}$. In the other case, X is biholomorphic to $\mathbb{D}$. The Uniformization theorem is proved. □

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