Fast Phase Retrieval from Masked Fourier Measurements

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Abstract

The phase retrieval problem is a fascinating but challenging inverse problem that arises in molecular imaging methods such as X-ray crystallography, ptychography, and other diffraction imaging techniques. The physics of the diffraction process dictates that measurements observed are squared magnitude of the Fourier transform of the diffracted signal's amplitudes. Because distinct signals can have the same magnitude, different phase factors can generate multiple solutions for the same problem. Many phase recovery algorithms used in practice are heuristic in nature, and so have no mathematical assurances of obtaining a correct solution. We use discrete Fourier analysis in conjunction with spectral analysis of strategically constructed, circulant-like matrices to recover the phase accurately and efficiently. We explore the 1-dimensional phase retrieval problem – recovering a signal $\mathbf{x} \in \mathbb{C}^N$ – to build the 2-dimensional analog: reconstructing a signal $\mathcal{X} \in \mathbb{C}^{N_1 \times N_2}$. We present our new theoretical results as well as empirical simulations verifying the accuracy and efficiency of the proposed framework.

1 Introduction

1.1 Motivation

The research presented in this paper concerns methods of mathematically recovering a signal – a quantity encoding amplitude and phase information – from indirect measurements. For instance, we want to recover a signal $\mathbf{x} \in \mathbb{C}^N$, but due to some underlying physical process, represented by a transformation $T : \mathbb{C}^N \to \mathbb{C}^M$, we can only quantitatively observe a vector $\mathbf{y}$ such that

$$|T\mathbf{x}| = \mathbf{y} \in \mathbb{R}^M,$$

where $|\cdot|$ is considered component-wise. Thus, solely real-valued, magnitude-only measurements are physically detectable, and hence, phase information is lost. Phase encodes critical information

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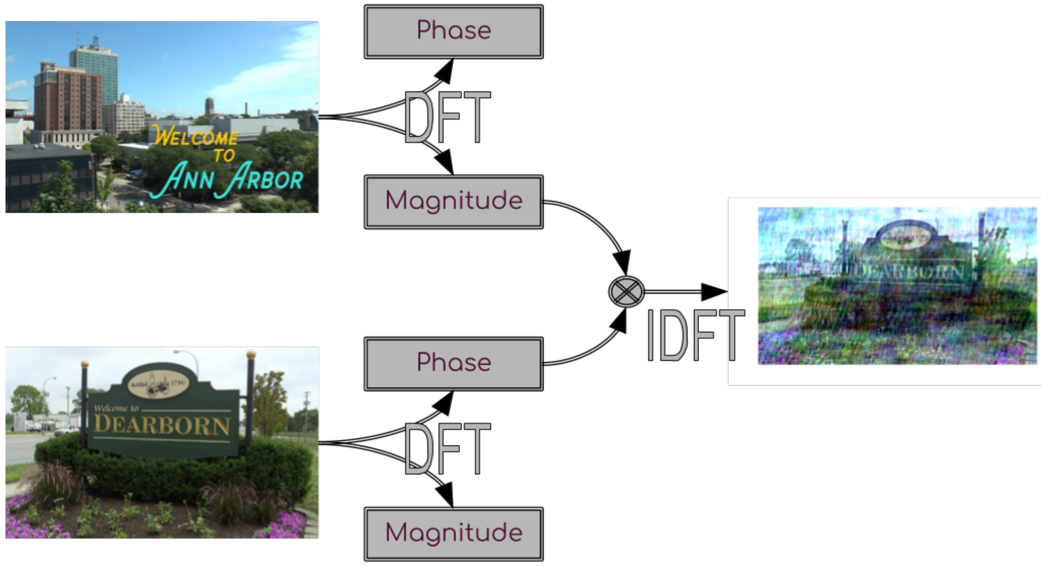

Figure 1: Merging phase from Dearborn image and magnitude from Ann Arbor image

regarding the structure of an object, as Figure 1 demonstrates, when reconstructing an accurate, detailed image. We use the following definition for phase.

Definition 1.1.1. *Let $z \in \mathbb{C}$. The* **phase** *of z is the complex exponential $e^{i\theta}$ such that $z = |z|e^{i\theta}$, where $|z|$ is the modulus or amplitude of z and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. For a vector $\mathbf{x} \in \mathbb{C}^N$, the* **vector of phases** *is*

$$\frac{\mathbf{x}}{|\mathbf{x}|} = \begin{bmatrix} |x_0|e^{i\theta_0}/|x_0| \\ |x_1|e^{i\theta_1}/|x_1| \\ \vdots \\ |x_{N-1}|e^{i\theta_{N-1}}/|x_{N-1}| \end{bmatrix} = \begin{bmatrix} e^{i\theta_0} \\ e^{i\theta_1} \\ \vdots \\ e^{i\theta_{N-1}} \end{bmatrix},$$

where $\div$ and $|\cdot|$ are defined component-wise.

The motivating application for phase recovery methods presented in this paper stems from ptychography, a coherent diffraction imaging (CDI) technique widely used in molecular imaging to reveal the underlying structure of a specimen of interest, like cells, viruses, or nanocrystals. An electromagnetic wave is said to be *diffracted* if any perturbation of the propagating wave from a straight-linear path cannot be described as reflection or refraction [6]. In the ptychographic setting, the measurement apparatus utilizes a transmitter of illumination, such as light, X-rays, or electron beams, to radiate the sample and produce several diffraction patterns which appear as light and dark spots within a detection field. Processing the measurements of the signals' diffracted amplitudes, imaging scientists then use a mathematical algorithm to transform the data into an image of the microscopic structure [9].

Measuring multiple (at least 2) interference patterns differentiates ptychography from many other imaging techniques, and the strategy is known to bolster mathematically the performance of phase retrieval algorithms to obtain a unique solution. The measurement apparatus generates these collections of diffraction patterns by the use of a moving probe or window function often called the *mask*. The mask's physical and mathematical characteristics are known or can be calculated; in particular, the mask is often designed to be support restricted. To have local support means mathematically that the nonzero components of the mask's signal will be significantly smaller than the size N of the signal diffracted from the sample. This ensures that the radiation source, guided by the mask, will illuminate small sections of the sample at time. So, mathematically speaking, each measurement, a sum of diffracted amplitudes from the sample and mask, will also be locally supported. Thus, to reconstruct an image, the mask is moved or *shifted* to gather several measurements encompassing the whole sample [9, 13]. Such locally supported masks are labeled *bandlimited*.

The mathematics of diffracted amplitudes of a propagating wave $u(x, y, z; t)$ in space $(x, y, z) \in \mathbb{R}^3$ and time $t \in \mathbb{R}$ derives from a solution to the wave equation

$$\Delta u(x, y, z; t) = k^2 \frac{\partial^2}{\partial t^2} u(x, y, z; t),$$

where Δ is the spatial Laplacian and k^2 is a parameter determined by the medium of propagation and the speed of light in a vacuum. The physics of the diffraction process dictates that only intensity I, squared-magnitude measurements, of the signal is physically observable. Based on setup parameters of the ratio of the signal's wavelength to the distance from the illumination source to the detection field, measurements can be calculated by the Fraunhofer approximation,

$$I \propto \left| \iint_{\mathbb{R}^2} u(x, y, 0; t) e^{-\frac{2\pi i}{z\lambda}(x\xi_1 + y\xi_2)} \, dx \, dy \right|^2 = |\mathcal{F}[u(x, y, 0; t)]|^2 ,$$

where $\mathcal{F}$ is the 2-dimensional Fourier transform, λ is the wavelength of the electromagnetic ray, and ξ_1, ξ_2 represent the Fourier or frequency domain variables in the detection field, sometimes referred to abstractly as the *Fraunhofer plane* [6]. Thus, in the ptychographic setting, intensity observables are squared-magnitude Fourier measurements and contain no phase information. For our purposes, we will assume the proportional constant is 1.

1.2 Background and Statement of 1D and 2D Problem

This REU project examined the prospects of developing a phase retrieval algorithm for 2-dimensional signals that is computationally efficient, robust, and that allows analysis to produce mathematical assurances of obtaining a correct solution − still an open question for 2-dimensional phase retrieval. The strategy is to extend the 1-dimensional results of Perlmutter et al. in [8] to the 2-dimensional case, where we represent discretized signals as matrices $\mathcal{X} \in \mathbb{C}^{N_1 \times N_2}$. In the one-dimensional setting, where there is much traction for mathematical phase recovery, the specimen under study or the mask is considered to have only one axis of movement for shifts, whereas in the two-dimensional setting, shifts can be either horizontal or vertical. (See Figures 2 and 3.) In practice, however, physicists generally require two- and/or three-dimensional frameworks. Therefore, with some exceptions[1], practitioners in the field view 1-dimensional phase retrieval as a mathematical toy problem.

The origins of the phase retrieval problem trace back to X-ray crystallography in the 1940s; yet in 1953, phase reconstruction methods effected a watershed moment in biology and medical science when Watson and Crick employed such algorithms to uncover the double helix structure of DNA [9]. Nevertheless, even today critical obstacles remain in obtaining flawless phase retrieval, such as computational efficiency and robustness to noise or measurement error, which exacerbate the

[1]Audio speech processing, for example, is a 1D phase problem [1]

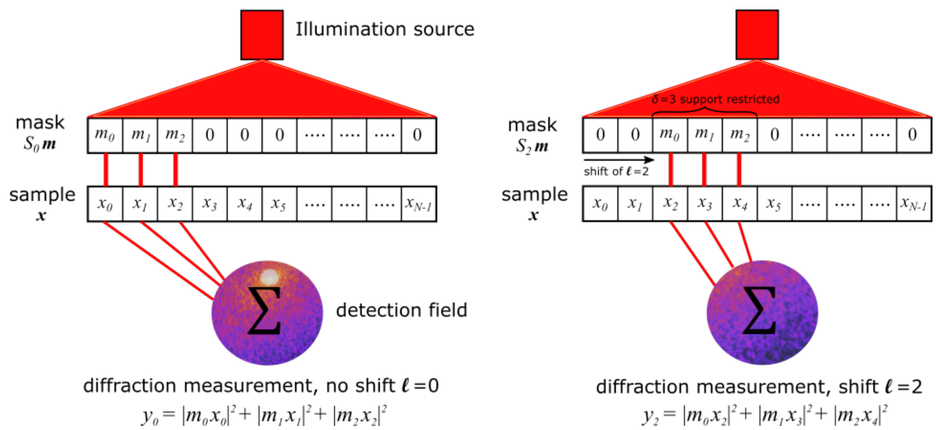

Figure 2: 1D Diffraction measurement setup

indeterminable qualities inherent in the *ill-posed* phase retrieval problem [10]. Take for example a simple system of linear equations over $\mathbb{C}$.

> **Motivating Example.**
>
> $$\begin{aligned} 3x_1 - (2i)x_2 &= 3 - 3i \\ x_1 + (1-i)x_2 &= 4 - 2i \end{aligned} \tag{1.1}$$

It can easily be shown that the unique solution is $\mathbf{x} = [x_1 \;\; x_2]^T = [1+i \;\; 3]^T$. On the other hand, taking the modulus or magnitude of these equations shows that the system

$$\begin{aligned} |3x_1 - (2i)x_2|^2 &= |3 - 3i|^2 \\ |x_1 + (1-i)x_2|^2 &= |4 - 2i|^2 \end{aligned}$$

is now nonlinear, and plus, no longer has a unique solution. Because the magnitude $\|\mathbf{x}\| = \|\mathbf{x}e^{i\theta}\|$ for any $\theta \in \mathbb{R}\backslash 2\pi\mathbb{Z}$, a solution is unique only up to a global phase factor of $e^{i\theta}$. In other words, the nonlinear system

$$\begin{aligned} |3(x_1e^{i\theta}) - 2i(x_2e^{i\theta})|^2 &= |3 - 3i|^2 \\ |(x_1e^{i\theta}) + (1-i)(x_2e^{i\theta})|^2 &= |4 - 2i|^2 \end{aligned}$$

has an infinite number of solutions. Moreover, in the signal processing setting, signals which have been circularly shifted, time-reversed, or have the same auto-correlation function also have the same Fourier magnitude [1].

Since the inception of the phase retrieval problem, the imaging science community has discovered many mathematical methods to overcome these ill-posed complications. Nonetheless, state of the art, 2-dimensional phase retrieval algorithms used in practice tend to be heuristic in nature, thus lacking the needed mathematical framework to provide assurances of recovering the precise phase and induce analysis for the robustness of solutions obtained. Two main classes of these algorithms used for phase retrieval are *alternating projections* (AP) and *semi-definite programming* (SDP). Introduced in 1972 by Gerchberg and Saxton in the pioneering paper [4], the AP method entails projecting the phase retrieval problem and its constraints onto a convex set and using well-established convex optimization techniques to solve the problem. But because the constraints of the problem are not actually convex, the iterative GS algorithm has no theoretical guarantee of obtaining a correct solution, let alone a unique one, and more problematically, it is known to converge to local optima. In 1982, Fienup modified the GS algorithm for 2-dimensional phase retrieval with the *hybrid input-output* (HIO) method in [3]. HIO showed marked improvement as the algorithm showed empirically to avoid local optima. However, there is still no proof that HIO converges, and it also fails to perform in the presence of high noise or measurement error.

SDP realizes that Fourier intensity measurements establish a system of quadratic equations that can be linearized or *lifted* into a higher dimension. Then for an observed signal $\mathbf{x} \in \mathbb{C}^N$, the method minimizes the rank of the matrix $\mathbf{X} = \mathbf{x}\mathbf{x}^*$ generated by the diffraction measurements with a projection onto the set of positive semi-definite, rank 1 matrices. Rank minimization is an extremely difficult combinatorial problem despite its convex constraints, but one convex adjacent strategy, *PhaseLift*, tries to minimize the trace norm $\|\mathbf{X}\|_\Sigma$ instead of the rank of the matrices generated from the measurements. In general, the number of measurements needed for phase recovery by this method must be proportional to the square of the signal size, and thus, lifting the dimension significantly increases the computation time. There do exist robustness and recovery guarantees for PhaseLift, and there are many interesting mathematical results from it, including a novel graph-theoretic approach by Singer in [11]. However, the physical constraints of the diffraction apparatus setup needed to satisfy the algebraic structure of the PhaseLift algorithm are severely infeasible in practice [1].

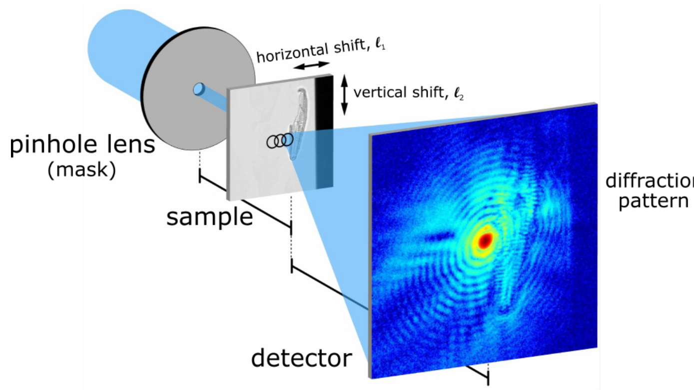

Figure 3: 2D Diffraction measurement setup; Adapted[2] with permission from [5] © The Optical Society

Unlike the many algorithmic variations of AP and SDP used in practice, the 1-dimensional method by Perlmutter et al. [8] avoids costly iterations altogether, instead relying on the mathematical properties of Fourier analysis. Thus, this paper explores the 1-dimensional case and the mathematical foundation of phase retrieval method Algorithm 1 (p. 37) adapted from [8]. We then build the 2-dimensional analog by presenting the underlying mathematics and theory corroborating Algorithm 2 (pp. 38, 39), phase retrieval in 2 dimensions.

2 1D Phase Retrieval

2.1 Preliminaries and Notation

Given the finite nature of measurement data collected, the natural setting for the mathematics for signal recovery is discrete Fourier analysis. So, first we introduce suitable notation and conventions used for the 1-dimensional case. Discretized, 1-dimensional signals are represented as vectors $\mathbf{x} \in \mathbb{C}^N$ with circular, *i.e.* modular, indexing: for $n \in \mathbb{Z}_N$, the complex component x_n is the element of $\mathbf{x}$

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evaluated at an n^{th} spatial position. So, with slight abuse of notation, we let the integer n imply the equivalence class $[n]$ modulo N. We can also think of $\mathbf{x}$ as a function $f_{\mathbf{x}}\colon \mathbb{Z}_N \to \mathbb{C}$ that assigns a point $n \in \mathbb{Z}_N$ in the admitting electromagnetic field a complex sinusoid $x_n \in \mathbb{C}$. If the signal is periodic, we say

Definition 2.1.1. *An **N-periodic signal** $\mathbf{x} \in \mathbb{C}^N$ is a sequence of complex numbers $\{x_n\}_{n\in\mathbb{Z}}$ such that $x_{n+N} = x_n$ for all $n \in \mathbb{Z}_N$.*

Including definition 1.1.1, these additional conventions are used: the symbol $i = \sqrt{-1}$, the absolute value or modulus operation $|\cdot|$ is considered component-wise for a vector or matrix quantity, and the notation $(\mathbf{y})_k$ refers to the k^{th} element of vector $\mathbf{y}$. Two critical transformations, the discrete Fourier transform (DFT) and inverse discrete Fourier transform (IDFT), allow us to interchangeably maneuver between discrete spatial (or time) domain n and discrete frequency domain j, also known as the Fourier mode.

Definition 2.1.2. *Let $\mathbf{x} \in \mathbb{C}^N$ and $n, j \in \mathbb{Z}_N$. The Discrete Fourier Transform $F_N\colon \mathbb{C}^N \to \mathbb{C}^N$ and Inverse Discrete Fourier Transform $F_N^{-1}\colon \mathbb{C}^N \to \mathbb{C}^N$ are defined as*

$$\widehat{x}_j = (F_N\mathbf{x})_j := \sum_{n=0}^{N-1} x_n e^{\frac{-2\pi i n j}{N}}$$

$$x_n = (F_N^{-1}\mathbf{x})_n := \frac{1}{N}\sum_{j=0}^{N-1} \widehat{x}_j e^{\frac{2\pi i n j}{N}}.$$

Table 1 contains elementary operations on signals used throughout the 1-dimensional work. Unless otherwise noted, all indexing of signals $\mathbf{x}, \mathbf{h} \in \mathbb{C}^N$ in the table are implied to be modulo N.

Operation	**Definition**
Complex Conjugate	$\overline{\mathbf{x}}$
Time Reversal	$\widetilde{x}_n := x_{-n}$
Circular Shift in Time Operator $S_\ell : \mathbb{C}^N \to \mathbb{C}^N$	$(S_\ell\mathbf{x})_n := x_{n-\ell}$
Modulation Operator $W_k\colon \mathbb{C}^N \to \mathbb{C}^N$	$(W_k\mathbf{x})_n := x_n e^{-\frac{2\pi i k n}{N}}$
Hadamard (Element-wise) Product	$(\mathbf{x} \circ \mathbf{h})_n := x_n h_n$
Circular Convolution $\circledast_N : \mathbb{C}^N \times \mathbb{C}^N \to \mathbb{C}^N$	$(\mathbf{x} \circledast_N \mathbf{h})_n := \sum_{k=0}^{N-1} x_k h_{n-k}$

Table 1: 1D Elementary Operations

We now look at the governing measurements equation in the ptychographic setting. Let $\mathbf{x} \in \mathbb{C}^N$ be the signal of the unknown sample and the mask $\mathbf{m} \in \mathbb{C}^N$, and let $Y \in \mathbb{R}^{N\times L}$ contain the $N \cdot L$ real-valued, phaseless measurements, where L is the total number of shifts of the mask. Therefore, our objective for 1-dimensional phase retrieval is to recover $\mathbf{x}$ from the intensity measurements of the diffracted amplitudes of $\mathbf{x}$ and $\mathbf{m}$, which are the squared-magnitude Fourier measurements

$$Y_{j,\ell} = \left| \sum_{n=0}^{N-1} x_n m_{n-\ell} e^{\frac{-2\pi i n j}{N}} \right|^2 + \eta_{j,\ell}, \quad \text{for } n, j, \ell \in \mathbb{Z}_N, \tag{2.1}$$

where $\eta_{j,\ell} \in \mathbb{R}$ represents random noise or measurement error for each entry of Y and shift ℓ of the mask. Observe that we strategically populate the measurements matrix Y so that the ℓ^{th} column is the ℓ^{th} shift of the mask and the k^{th} row is the k^{th} Fourier mode of column vector $\mathbf{y}_\ell \in \mathbb{C}^N$. Furthermore, using precise notation, we can also state that

$$Y_{j,\ell} = \left| (F_N(\mathbf{x} \circ S_\ell \mathbf{m}))_j \right|^2 + \eta_{j,\ell}. \tag{2.2}$$

For mathematical convenience, we ignore the noise term η in proofs and, if needed, assume each entry $Y_{j,\ell}$ is a noisy measurement.

2.2 Mathematical Foundation for Algorithm 1

The goal of this section is to provide the theoretical foundation for the 1-dimensional phase retrieval method of Algorithm 1, provided by mentor Dr. Aditya Viswanathan and co-author of the 1-dimensional case work in [8]. The 3 lemmas and consequent theorem presented here are also featured in [8], but we provide our own proofs as well as consequences of the results. First, we state without proof well-known properties, largely taken from [12] (except Properties 1(iv.) and 1(viii.) whose proofs are left for Appendix A), of the 1-dimensional DFT and other elementary operations on signals.

Properties 1. *Let $\mathbf{x}, \mathbf{h} \in \mathbb{C}^N$ be arbitrary signals and $\alpha, \beta \in \mathbb{C}$ scalars. Then for all $n, j, k, \ell \in \mathbb{Z}_N$, the following properties hold:*

i. $(F_N(\alpha\mathbf{x} + \beta\mathbf{h}))_j = \alpha\widehat{x}_j + \beta\widehat{h}_j$

ii. $(F_N(S_\ell\mathbf{x}))_j = (W_\ell\widehat{\mathbf{x}})_j$

iii. $(F_N\widetilde{\mathbf{x}})_j = \widehat{\widetilde{x}}_j = \widetilde{\widehat{x}}_j$

iv. $(\widetilde{S_\ell\mathbf{x}})_n = (S_{-\ell}\widetilde{\mathbf{x}})_n$

v. $(F_N(\mathbf{h} \circledast_N \mathbf{x}))_j = (\widehat{\mathbf{h}} \circ \widehat{\mathbf{x}})_j$

vi. $(F_N(\mathbf{h} \circ \mathbf{x}))_j = \dfrac{1}{N}(\widehat{\mathbf{h}} \circledast_N \widehat{\mathbf{x}})_j$

vii. $(F_N\overline{\mathbf{x}})_j = \widehat{\overline{x}}_j = \overline{\widetilde{\widehat{x}}}_j$

viii. $(F_N(F_N\mathbf{x}))_j = (F_N\widehat{\mathbf{x}})_j = N\widetilde{x}_j$.

We show the following lemmas and definitions to aid in the proof of main theoretical results, Theorem 1 and Corollary 1.

Lemma 2.2.1. *Let $\mathbf{x} \in \mathbb{C}^N$ and $j, \ell \in \mathbb{Z}_N$, then*

$$(F_N(\mathbf{x} \circ S_\ell\overline{\mathbf{x}}))_j = \frac{1}{N} e^{\frac{-2\pi i j\ell}{N}} (F_N(\widehat{\mathbf{x}} \circ S_j\overline{\widehat{\mathbf{x}}}))_{-\ell}.$$

Proof.

$$
\begin{aligned}
(F_N(\mathbf{x} \circ S_\ell \overline{\mathbf{x}}))_j &= \frac{1}{N} \left[\widehat{\mathbf{x}} \circledast_N (F_N(S_\ell \overline{\mathbf{x}}))\right]_j && \text{(by Property 1(vi.))} \\
&= \frac{1}{N} \left(\widehat{\mathbf{x}} \circledast_N (W_\ell \widehat{\overline{\mathbf{x}}})\right)_j && \text{(by Property 1(ii.))} \\
&= \frac{1}{N} \sum_{k=0}^{N-1} \widehat{x}_k (W_\ell \widehat{\overline{\mathbf{x}}})_{j-k} \\
&= \frac{1}{N} \sum_{k=0}^{N-1} \widehat{x}_k \widehat{\overline{x}}_{j-k} e^{\frac{-2\pi i (j-k)\ell}{N}} \\
&= \frac{1}{N} e^{\frac{-2\pi i j \ell}{N}} \sum_{k=0}^{N-1} \widehat{x}_k \widehat{\overline{x}}_{j-k} e^{\frac{2\pi i k \ell}{N}} \\
&= \frac{1}{N} e^{\frac{-2\pi i j \ell}{N}} \sum_{k=0}^{N-1} \widehat{x}_k \widetilde{\widehat{\overline{x}}}_{k-j} e^{\frac{2\pi i k \ell}{N}}. && \text{(by Property 1(vii.))}
\end{aligned}
$$

Noting that the time reversal of the time reversal of a signal is the original signal, *i.e.*

$$
\widetilde{\widetilde{x}}_n = x_{-(-n)} = x_n,
$$

we can use Property 1(vii.) to rewrite the term $\widetilde{\widehat{\overline{x}}}_{k-j}$:

$$
\widetilde{\widehat{\overline{\mathbf{x}}}} = \widetilde{F_N \overline{\mathbf{x}}} = \widetilde{\widetilde{\overline{\widehat{\mathbf{x}}}}} = \overline{\widehat{\mathbf{x}}}.
$$

So, we have

$$
\begin{aligned}
(F_N(\mathbf{x} \circ S_\ell \overline{\mathbf{x}}))_j &= \frac{1}{N} e^{\frac{-2\pi i j \ell}{N}} \sum_{k=0}^{N-1} \widehat{x}_k \overline{\widehat{x}}_{k-j} e^{\frac{2\pi i k \ell}{N}} \\
&= \frac{1}{N} e^{\frac{-2\pi i j \ell}{N}} \sum_{k=0}^{N-1} \widehat{x}_k \overline{\widehat{x}}_{k-j} e^{\frac{-2\pi i k(-\ell)}{N}} \\
&= \frac{1}{N} e^{\frac{-2\pi i j \ell}{N}} \sum_{k=0}^{N-1} \left(\widehat{\mathbf{x}} \circ S_j \overline{\widehat{\mathbf{x}}}\right)_k e^{\frac{-2\pi i k(-\ell)}{N}} \\
&= \frac{1}{N} e^{\frac{-2\pi i j \ell}{N}} (F_N(\widehat{\mathbf{x}} \circ S_j \overline{\widehat{\mathbf{x}}}))_{-\ell}. && \text{(by def. of DFT)}
\end{aligned}
$$

□

Lemma 2.2.2. *Let $\mathbf{x} \in \mathbb{C}^N$ be the unknown signal to be recovered and the mask $\mathbf{m} \in \mathbb{C}^N$ known. Let $\mathbf{y}_\ell \in \mathbb{R}^N$ be the ℓ^{th} column vector of measurements matrix Y, as defined by equation (2.1), corresponding to the ℓ^{th} shift of $\mathbf{m}$. Then, for any $k \in \mathbb{Z}_N$,*

$$
(F_N(F_N \mathbf{y}_\ell))_k = N \left[F_N(\mathbf{x} \circ S_k \overline{\mathbf{x}}) \circ F_N(\widetilde{\mathbf{m}} \circ S_{-k} \overline{\widetilde{\mathbf{m}}})\right]_\ell.
$$

Proof. The measurements equation (2.2) tells us that

$$\mathbf{y}_\ell = |F_N(\mathbf{x} \circ S_\ell \mathbf{m})|^2 = [F_N(\mathbf{x} \circ S_\ell \mathbf{m})] \circ \overline{[F_N(\mathbf{x} \circ S_\ell \mathbf{m})]}.$$

First, we show that we can rewrite the conjugate term as $NF_N^{-1}(\overline{\mathbf{x}} \circ S_\ell \overline{\mathbf{m}})$. For $j \in \mathbb{Z}_N$, we can write

$$\begin{aligned}
\overline{(F_N(\mathbf{x} \circ S_\ell \mathbf{m}))_j} &= \overline{\sum_{n=0}^{N-1} x_n m_{n-\ell} e^{\frac{-2\pi inj}{N}}} \\
&= \sum_{n=0}^{N-1} \overline{x}_n \overline{m}_{n-\ell} e^{\frac{2\pi inj}{N}} \\
&= N \left(\frac{1}{N} \sum_{n=0}^{N-1} \overline{x}_n \overline{m}_{n-\ell} e^{\frac{2\pi inj}{N}} \right) \\
&= N \left(F_N^{-1}(\overline{\mathbf{x}} \circ \overline{S_\ell \mathbf{m}}) \right)_j \\
&= N \left(F_N^{-1}(\overline{\mathbf{x}} \circ S_\ell \overline{\mathbf{m}}) \right)_j .
\end{aligned}$$

So, with this result, we see that

$$(\mathbf{y}_\ell)_j = \left[(F_N(\mathbf{x} \circ S_\ell \mathbf{m})) \circ N(F_N^{-1}(\overline{\mathbf{x}} \circ S_\ell \overline{\mathbf{m}})) \right]_j .$$

Using Properties 1(vi.) and 1(viii.), we take the DFT,

$$\begin{aligned}
F_N \mathbf{y}_\ell &= \frac{1}{N} [F_N F_N(\mathbf{x} \circ S_\ell \mathbf{m})] \circledast_N [N(\overline{\mathbf{x}} \circ S_\ell \overline{\mathbf{m}})] && \text{(by Property 1(vi.))} \\
&= \frac{1}{N} [N\widetilde{(\mathbf{x} \circ S_\ell \mathbf{m})}] \circledast_N [N(\overline{\mathbf{x}} \circ S_\ell \overline{\mathbf{m}})] && \text{(by Property 1(viii.))} \\
&= N[(\widetilde{\mathbf{x}} \circ \widetilde{S_\ell \mathbf{m}}) \circledast_N (\overline{\mathbf{x}} \circ S_\ell \overline{\mathbf{m}})] \\
&= N[(\widetilde{\mathbf{x}} \circ S_{-\ell} \widetilde{\mathbf{m}}) \circledast_N (\overline{\mathbf{x}} \circ S_\ell \overline{\mathbf{m}})]. && \text{(by Property 1(iv.))}
\end{aligned}$$

By the definition of convolution,

$$\begin{aligned}
(F_N \mathbf{y}_\ell)_k &= N \sum_{n=0}^{N-1} \widetilde{x}_n (S_{-\ell} \widetilde{\mathbf{m}})_n \overline{x}_{k-n} (S_\ell \overline{\mathbf{m}})_{k-n} \\
&= N \sum_{n=0}^{N-1} \widetilde{x}_n \overline{x}_{k-n} \widetilde{m}_{n+\ell} (S_\ell \overline{\mathbf{m}})_{k-n} \\
&= N \sum_{n=0}^{N-1} \widetilde{x}_n \overline{x}_{k-n} \widetilde{m}_{n+\ell} (\widetilde{S_\ell \overline{\mathbf{m}}})_{n-k} \\
&= N \sum_{n=0}^{N-1} x_{-n} \overline{x}_{k-n} \widetilde{m}_{n+\ell} (S_{-\ell} \overline{\widetilde{\mathbf{m}}})_{n-k}. && \text{(by Property 1(iv.))}
\end{aligned}$$

By substitution, let $p = -n$. The indexing set for $p \in \mathbb{Z}_N$ in the summation becomes $\{-(N-1), -(N-2), \ldots, -1, 0\}$ which is equivalent to $\{0, 1, \ldots, N-1\}$ because each element is considered modulo N. So, we have

$$
\begin{aligned}
(F_N \mathbf{y}_\ell)_k &= N \sum_{p=0}^{N-1} x_p \overline{x}_{k+p} \widetilde{m}_{-p+\ell} (S_{-\ell} \overline{\widetilde{\mathbf{m}}})_{-p-k} \\
&= N \sum_{p=0}^{N-1} [x_p (S_{-k} \overline{\mathbf{x}})_p][\widetilde{m}_{-p+\ell} (S_k \overline{\widetilde{\mathbf{m}}})_{-p+\ell}] \\
&= N \sum_{p=0}^{N-1} (\mathbf{x} \circ S_{-k} \overline{\mathbf{x}})_p (\widetilde{\mathbf{m}} \circ S_k \overline{\widetilde{\mathbf{m}}})_{\ell-p} \\
&= N \left[(\mathbf{x} \circ S_{-k} \overline{\mathbf{x}}) \circledast_N (\widetilde{\mathbf{m}} \circ S_k \overline{\widetilde{\mathbf{m}}}) \right]_\ell . \qquad \text{(by def. of convolution)}
\end{aligned}
$$

So, given that

$$
(F_N \mathbf{y}_\ell)_k = N \left[(\mathbf{x} \circ S_{-k} \overline{\mathbf{x}}) \circledast_N (\widetilde{\mathbf{m}} \circ S_k \overline{\widetilde{\mathbf{m}}}) \right]_\ell , \tag{2.3}
$$

we use Property 1(v.) and take one final DFT to yield

$$
F_N (F_N \mathbf{y}_\ell) = N \left[F_N (\mathbf{x} \circ S_{-k} \overline{\mathbf{x}}) \circ F_N (\widetilde{\mathbf{m}} \circ S_k \overline{\widetilde{\mathbf{m}}}) \right] .
$$

□

Note that for measurements equations (2.1), (2.2) and the previous lemmas, there were no assumptions on the total number L of shifts of the mask $\mathbf{m}$. But for our main result, Theorem 1, we want to look at a sub-sample of equally spaced shifts such that

$$
\ell \in \left\{ 0, \frac{N}{L}, \frac{2N}{L}, \ldots, \frac{(L-1)N}{L} \right\} .
$$

First, we define a uniform, sub-sampling operator and prove a lemma concerning *undersampled* signals.

Definition 2.2.3. *Let $\mathbf{x} \in \mathbb{C}^N$ be a 1-dimensional signal and $L \in \mathbb{N}$ such that L divides N. Then, the* **uniform sub-sampling operator** *$Z_L \colon \mathbb{C}^N \to \mathbb{C}^{N/L}$ is defined as*

$$
(Z_L \mathbf{x})_n := x_{nL}, \quad \forall n \in \mathbb{Z}_{N/L} .
$$

Lemma 2.2.4 (Aliasing)**.** *Let $\mathbf{x} \in \mathbb{C}^N$ be arbitrary and suppose $L \in \mathbb{N}$ divides N. Then, for any $j \in \mathbb{Z}_{N/L}$,*

$$
\left(F_{N/L} (Z_L \mathbf{x}) \right)_j = \frac{1}{L} \sum_{p=0}^{L-1} \widehat{x}_{j-pN/L} .
$$

Proof. By definitions 2.2.3 and of the DFT,

$$
\left(F_{N/L} (Z_L \mathbf{x}) \right)_j = \sum_{n=0}^{\frac{N}{L}-1} x_{nL} e^{\frac{-2\pi i n j}{N/L}}
$$

$$
\begin{aligned}
&= \sum_{n=0}^{\frac{N}{L}-1} \left(\frac{1}{N} \sum_{k=0}^{N-1} \widehat{x}_k e^{\frac{2\pi i (nL)k}{N}} \right) e^{\frac{-2\pi i n j}{N/L}} \qquad \text{(by def. of IDFT)} \\
&= \frac{1}{N} \sum_{k=0}^{N-1} \widehat{x}_k \sum_{n=0}^{\frac{N}{L}-1} e^{\frac{2\pi i n (k-j)}{N/L}} \\
&= \frac{1}{N} \sum_{k=0}^{N-1} \widehat{x}_k \frac{N}{L} \delta_{k-j \bmod N/L} \\
&= \frac{1}{L} \sum_{k=0}^{N-1} \widehat{x}_k \delta_{k-j \bmod N/L}.
\end{aligned}
$$

The Kronecker delta function here is defined as

$$
\delta_{k-j \bmod N/L} = \begin{cases} 1, & \text{if } k = j \bmod N/L \\ 0, & \text{otherwise.} \end{cases}
$$

Because the indexing set for the above summation is for $k \in \{0, 1, \ldots, N-1\}$ and thus has cardinality N, and we have that L divides N, then there must be exactly L values that satisfy $\delta_{k-j \bmod N/L} = 1$. So, let $p \in \mathbb{Z}_N$. We can rewrite the Kronecker delta function as

$$
\delta_{k-j \bmod N/L} = \begin{cases} 1, & k = j - pN/L, \\ 0, & \text{otherwise.} \end{cases}
$$

Therefore, we have

$$
\left(F_{N/L}(Z_L \mathbf{x}) \right)_j = \frac{1}{L} \sum_{p=0}^{L-1} \widehat{x}_{j-pN/L}.
$$

□

This interesting result tells us that if a subset of some (periodic) signal $\mathbf{x} \in \mathbb{C}^N$ is uniformly sampled in the spatial domain, then the DFT of the uniformly sub-sampled signal $Z_L\mathbf{x} \in \mathbb{C}^{N/L}$ yields an overlapping of $\widehat{\mathbf{x}}$ Fourier modes. Thus, in the frequency domain, we can no longer distinguish between the high-frequency sinusoids from the low-frequency sinusoids of the original signal $\mathbf{x}$. To see this clearly, we let $\mathbf{w} = Z_L\mathbf{x}$ and take the DFT:

$$
\widehat{w}_j = \frac{1}{L} \left(\widehat{x}_j + \widehat{x}_{j-N/L} + \widehat{x}_{j-2N/L} + \cdots + \widehat{x}_{j-(L-1)N/L} \right).
$$

This overlapping, a sum of the high and low Fourier modes $j \in Z_{N/L}$ of $\widehat{\mathbf{x}}$, caused by undersampling is known as *aliasing*. Aliased signals can present mathematical difficulties for phase retrieval, particularly when data measurements recorded are non-bandlimited and thus can prevent successful reconstruction of the original signal [6]. For observing masked measurements, knowledge of aliasing can be pertinent, because it is not ideal, or perhaps impossible, to take all possible N shifts of the mask to illuminate the specimen. Thus, the bandlimited parameter of the mask aids in minimizing the number of shifts and, more importantly, eases numerical calculations for phase retrieval. So, we now formalize definitions for the *support* and *bandlimited* parameter δ of the mask.

Definition 2.2.5. *Let $\mathbf{x} \in \mathbb{C}^N$ with components x_n for all $n \in \mathbb{Z}_N$. Then, the* **support** *of* $\mathbf{x}$, *denoted* $\operatorname{supp}(\mathbf{x})$*, is defined as the set*

$$\operatorname{supp}(\mathbf{x}) := \{n \in \mathbb{Z}_N : x_n \neq 0\}.$$

If the mask $\mathbf{m}$ *is* $\boldsymbol{\delta}$**-bandlimited** *for* $\delta \in \mathbb{Z}_N$*, then*

$$\operatorname{supp}(\mathbf{m}) = \{0, 1, \ldots, \delta - 1\}.$$

We now come to the main theoretical results from [8] which provide the basis of Algorithm 1.

Theorem 1 (Perlmutter et al. '19)**.** *Let* $\mathbf{x} \in \mathbb{C}^N$ *be the signal of the unknown sample and the bandlimited mask* $\mathbf{m} \in \mathbb{C}^N$ *a known quantity. And for some* L *which divides* N*, let* $Y \in \mathbb{R}^{N \times L}$ *contain* $N \cdot L$ *noisy measurements of the form* (2.1)*. Then, for any* $\omega \in \mathbb{Z}_L$ *and* $k \in \mathbb{Z}_N$*,*

$$\left(F_L(F_N Y)^T\right)_{\omega,k} = \frac{L}{N^2} \sum_{p=0}^{\frac{N}{L}-1} \left(F_N\left(\widehat{\mathbf{x}} \circ S_{\omega - pL}\overline{\widehat{\mathbf{x}}}\right)\right)_k \left(F_N\left(\widehat{\mathbf{m}} \circ S_{pL-\omega}\overline{\widehat{\mathbf{m}}}\right)\right)_k.$$

Proof. In the proof of Lemma 7, we showed equation (2.3):

$$(F_N \mathbf{y}_\ell)_k = N\left[(\mathbf{x} \circ S_{-k}\overline{\mathbf{x}}) \circledast_N (\widetilde{\mathbf{m}} \circ S_k\overline{\widetilde{\mathbf{m}}})\right]_\ell.$$

With slight abuse of notation, we will also use the symbol F_N as the matrix representation of the N-period DFT. So, by $F_N Y \in \mathbb{C}^{N \times L}$, we imply the matrix multiplication of F_N on each ℓ^{th} column of the measurements matrix Y. Thus, we can write the (k, ℓ) entry of $F_N Y$ as

$$(F_N Y)_{k,\ell} = N[(\mathbf{x} \circ S_{-k}\overline{\mathbf{x}}) \circledast_N (\widetilde{\mathbf{m}} \circ S_k\overline{\widetilde{\mathbf{m}}})]_\ell,$$

or, equivalently, taking the transpose yields

$$((F_N Y)^T)_{\ell,k} = N[(\mathbf{x} \circ S_{-k}\overline{\mathbf{x}}) \circledast_N (\widetilde{\mathbf{m}} \circ S_k\overline{\widetilde{\mathbf{m}}})]_\ell.$$

Define $\mathbf{u} \in \mathbb{C}^N$ as $\mathbf{u} = N[(\mathbf{x} \circ S_{-k}\overline{\mathbf{x}}) \circledast_N (\widetilde{\mathbf{m}} \circ S_k\overline{\widetilde{\mathbf{m}}})]$. If we take ℓ at L equally spaced shifts such that L divides N, then u_ℓ corresponds to the ℓ^{th} sub-sampled element for

$$\ell \in \left\{0, \frac{N}{L}, \frac{2N}{L}, \ldots, \frac{(L-1)N}{L}\right\}.$$

Now, we want to take the L-period DFT of the column vectors of the matrix $(F_N Y)^T \in \mathbb{C}^{L \times N}$. Let $(F_L(F_N Y)^T)_{\omega,k}$ denote the the ω^{th} mode of the k^{th} column vector of $F_L(F_N Y)^T \in \mathbb{C}^{L \times N}$ for $\omega \in \mathbb{Z}_L$ and $k \in \mathbb{Z}_N$. Then, by definition 2.2.3, we can now write

$$(F_L(F_N Y)^T)_{\omega,k} = (F_L(Z_{N/L}\mathbf{u}))_\omega.$$

Then by Lemma 2.2.4, with N/L replacing L,

$$
\begin{aligned}
(F_L(Z_{N/L}\mathbf{u}))_\omega &= \frac{1}{N/L}\sum_{p=0}^{\frac{N}{L}-1}\widehat{u}_{\omega-p\frac{N}{N/L}} \\
&= \frac{L}{N}\sum_{p=0}^{\frac{N}{L}-1}\widehat{u}_{\omega-pL} \\
&= \frac{L}{N}\sum_{p=0}^{\frac{N}{L}-1} N[F_N(\mathbf{x}\circ S_{-k}\overline{\mathbf{x}})]_{\omega-pL}[F_N(\widetilde{\mathbf{m}}\circ S_k\overline{\widetilde{\mathbf{m}}})]_{\omega-pL} && \text{(by Lemma 2.2.2)} \\
&= L\sum_{p=0}^{\frac{N}{L}-1}[F_N(\mathbf{x}\circ S_{-k}\overline{\mathbf{x}})]_{\omega-pL}[F_N(\widetilde{\mathbf{m}}\circ S_k\overline{\widetilde{\mathbf{m}}})]_{\omega-pL}.
\end{aligned}
$$

By the definition of time reversal, we then have

$$
\begin{aligned}
(F_L(F_NY)^T)_{\omega,k} &= L\sum_{p=0}^{\frac{N}{L}-1}[F_N(\mathbf{x}\circ S_{-k}\overline{\mathbf{x}})]_{\omega-pL}\widetilde{[F_N(\widetilde{\mathbf{m}}\circ S_k\overline{\widetilde{\mathbf{m}}})]}_{pL-\omega} \\
&= L\sum_{p=0}^{\frac{N}{L}-1}[F_N(\mathbf{x}\circ S_{-k}\overline{\mathbf{x}})]_{\omega-pL}[F_N(\widetilde{\widetilde{\mathbf{m}}}\circ \widetilde{S_k\overline{\widetilde{\mathbf{m}}}})]_{pL-\omega} && \text{(by Property 1(iii.))} \\
&= L\sum_{p=0}^{\frac{N}{L}-1}[F_N(\mathbf{x}\circ S_{-k}\overline{\mathbf{x}})]_{\omega-pL}[F_N(\widetilde{\widetilde{\mathbf{m}}}\circ S_{-k}\widetilde{\overline{\widetilde{\mathbf{m}}}})]_{pL-\omega} && \text{(by Property 1(iv.))} \\
&= L\sum_{p=0}^{\frac{N}{L}-1}[F_N(\mathbf{x}\circ S_{-k}\overline{\mathbf{x}})]_{\omega-pL}[F_N(\mathbf{m}\circ S_{-k}\overline{\mathbf{m}})]_{pL-\omega}.
\end{aligned}
$$

Therefore, by Lemma 2.2.1, we have

$$
\begin{aligned}
\left(F_L(F_NY)^T\right)_{\omega,k} &= L\sum_{p=0}^{\frac{N}{L}-1}\frac{1}{N}e^{\frac{-2\pi i(\omega-pL)(-k)}{N}}\left[F_N\left(\widehat{\mathbf{x}}\circ S_{\omega-pL}\overline{\widehat{\mathbf{x}}}\right)\right]_k \cdot \\
&\qquad\qquad \frac{1}{N}e^{\frac{-2\pi i(pL-\omega)(-k)}{N}}\left[F_N\left(\widehat{\mathbf{m}}\circ S_{pL-\omega}\overline{\widehat{\mathbf{m}}}\right)\right]_k \\
&= \frac{L}{N^2}\sum_{p=0}^{\frac{N}{L}-1}e^{\frac{2\pi i(\omega-pL)k}{N}}e^{\frac{-2\pi i(\omega-pL)k}{N}}\left[F_N\left(\widehat{\mathbf{x}}\circ S_{\omega-pL}\overline{\widehat{\mathbf{x}}}\right)\right]_k\left[F_N\left(\widehat{\mathbf{m}}\circ S_{pL-\omega}\overline{\widehat{\mathbf{m}}}\right)\right]_k \\
&= \frac{L}{N^2}\sum_{p=0}^{\frac{N}{L}-1}\left(F_N\left(\widehat{\mathbf{x}}\circ S_{\omega-pL}\overline{\widehat{\mathbf{x}}}\right)\right)_k\left(F_N\left(\widehat{\mathbf{m}}\circ S_{pL-\omega}\overline{\widehat{\mathbf{m}}}\right)\right)_k.
\end{aligned}
$$

□

We began with measurements as defined by equation (2.2) where the Fourier intensity of the diffracted amplitudes from the signal $\mathbf{x}$ and the mask $\mathbf{m}$ are intertwined as the Hadamard product of $\mathbf{x}$ and shifts of $\mathbf{m}$. So, observe the 2 critical consequences of Theorem 1: (1) we can decouple quantities of the unknown signal $\mathbf{x}$ from $\mathbf{m}$; and (2) we can write 2 Fourier transforms of the measurements in matrix Y as a linear combination of the known quantity $F_N(\widehat{\mathbf{m}} \circ S_{pL-\omega}\overline{\widehat{\mathbf{m}}})$, which leaves us to simply solve for $F_N(\widehat{\mathbf{x}} \circ S_{\omega-pL}\overline{\widehat{\mathbf{x}}})$, an altered quantity of the signal we want to recover. The following corollary shows we can further simplify the new linear system and, as discussed in the next section, will allow us to strategically construct special matrices for Algorithm 1 to recover the desired phase information.

Corollary 1. *Assume* $\mathbf{x}, \mathbf{m} \in \mathbb{C}^N$ *and* $L \in \mathbb{N}$ *as in Theorem 1. If* $\widehat{\mathbf{m}}$ *is* δ*-bandlimited, i.e.* $\operatorname{supp}(\widehat{\mathbf{m}}) = \{0, 1, \ldots, \delta - 1\}$, *and* $L = 2\delta - 1$, *then for any* $\omega \in \mathbb{Z}_L$ *and* $k \in \mathbb{Z}_N$, *the summation in Theorem 1 collapses to exactly one of two terms:*

$$\frac{N^2}{L}\left(F_L(F_N Y)^T\right)_{\omega,k} = \begin{cases} \left(F_N(\widehat{\mathbf{x}} \circ S_\omega \overline{\widehat{\mathbf{x}}})\right)_k \left(F_N(\widehat{\mathbf{m}} \circ S_{-\omega} \overline{\widehat{\mathbf{m}}})\right)_k, & \text{if } \omega \in \mathbb{Z}_\delta \\ \left(F_N(\widehat{\mathbf{x}} \circ S_{\omega-L} \overline{\widehat{\mathbf{x}}})\right)_k \left(F_N(\widehat{\mathbf{m}} \circ S_{L-\omega} \overline{\widehat{\mathbf{m}}})\right)_k, & \text{if } \omega \in \mathbb{Z}_L \backslash \mathbb{Z}_\delta. \end{cases} \tag{2.4}$$

Proof. We want to find the conditions under which the intersection $\operatorname{supp}(\widehat{\mathbf{m}}) \cap \operatorname{supp}(S_{pL-\omega}\overline{\widehat{\mathbf{m}}}) \neq \emptyset$ or equivalently when $\widehat{\mathbf{m}} \circ S_{pL-\omega}\overline{\widehat{\mathbf{m}}} \neq \mathbf{0}$. To satisfy this criterion, note that this intersection is nonempty if and only if $|pL - \omega| \leq \delta - 1$, the maximal shift within support restrictions of $\widehat{\mathbf{m}}$. Because $|\operatorname{supp}(\widehat{\mathbf{m}})| = |\operatorname{supp}(S_{pL-\omega}\overline{\widehat{\mathbf{m}}})| = \delta$, a shift greater than $\delta - 1$ will cause the support sets to be disjoint, and hence the element-wise product $\widehat{\mathbf{m}} \circ S_{pL-\omega}\overline{\widehat{\mathbf{m}}} = \mathbf{0}$. So, for the nonempty intersection, we have

$$\begin{gathered} -(\delta - 1) \leq pL - \omega \leq \delta - 1 \\ \omega - (\delta - 1) \leq pL \leq \omega + \delta - 1 \\ \frac{\omega - (\delta - 1)}{L} \leq p \leq \frac{\omega + \delta - 1}{L} \\ \frac{\omega - (\delta - 1)}{2\delta - 1} \leq p \leq \frac{\omega + \delta - 1}{2\delta - 1}. \end{gathered}$$

So, we look at when $\omega = 0$ and $\omega = \delta - 1$. For $\omega = 0$,

$$\begin{gathered} -\frac{\delta - 1}{2\delta - 1} \leq p \leq \frac{\delta - 1}{2\delta - 1} \\ -1 < -\frac{\delta - 1}{2\delta - 1} \leq p \leq \frac{\delta - 1}{2\delta - 1} < 1. \end{gathered}$$

And if $\omega = \delta - 1$,

$$\begin{aligned} \frac{\delta - 1 - (\delta - 1)}{2\delta - 1} \leq p &\leq \frac{\delta - 1 + \delta - 1}{2\delta - 1} \\ 0 \leq p &\leq \frac{2\delta - 2}{2\delta - 1} \\ 0 \leq p &\leq \frac{2\delta - 2}{2\delta - 1} < \frac{2\delta - 2}{2\delta - 2} = 1. \end{aligned}$$

Because $p \in \mathbb{Z}_{N/L}$ is an integer, the bounds $-1 < p < 1$ imply $p = 0$ for all $\omega \in \mathbb{Z}_\delta$. Now, we look at $\omega = \delta$,

$$\begin{aligned}
\frac{\delta-(\delta-1)}{2\delta-1} \le p &\le \frac{\delta+\delta-1}{2\delta-1}\\
\frac{1}{2\delta-1} \le p &\le 1\\
0 < \frac{1}{2\delta-1} \le p &\le 1.
\end{aligned}$$

And for $\omega = L-1 = 2\delta-2$,

$$\begin{aligned}
\frac{2\delta-2-(\delta-1)}{2\delta-1} \le p &\le \frac{2\delta-2+\delta-1}{2\delta-1}\\
\frac{2\delta-1}{2\delta-1} \le p &\le \frac{3\delta-3}{2\delta-1}\\
1 \le p &\le \frac{3\delta-3}{2\delta-1} \le \frac{3\delta-3}{2\delta-2}\\
1 \le p &\le \frac{3}{2}.
\end{aligned}$$

Thus, the bounds $0 < p \le \frac{3}{2}$ imply $p = 1$ for all $\omega \in \mathbb{Z}_L \backslash \mathbb{Z}_\delta$. Therefore, the summation in Theorem 1 collapses to the desired terms. $\square$

2.3 Angular Synchronization and Algorithm 1

Recall the linear system (1.1) from the motivating example:

$$\begin{aligned}
3x_1 - (2i)x_2 &= 3-3i\\
x_1 + (1-i)x_2 &= 4-2i.
\end{aligned}$$

When we take the modulus of each equation, the linear system expands to the nonlinear one,

$$\begin{aligned}
9|x_1|^2 + 6ix_1\overline{x}_2 - 6i\overline{x}_1x_2 + 4|x_2|^2 &= 18\\
|x_1|^2 + (1+i)x_1\overline{x}_2 + (1-i)\overline{x}_1x_2 + 2|x_2|^2 &= 20.
\end{aligned}$$

In addition, there are now several unknown values but only 2 equations; hence, the difficulty in finding a unique solution still remains. Suppose we are able to obtain 2 more equations which, in practice, would correspond to obtaining more measurements or gleaning more information from numerical data. We then can linearize, or *lift*, the system by treating $|x_1|^2, x_1\overline{x}_2, \overline{x}_1x_2, |x_2|^2$ as our unknowns. Thus, such a linear system is solvable and has a unique solution (assuming invertibility of the coefficients matrix):

$$\begin{bmatrix} 9 & 6i & -6i & 4\\ 1 & 1+i & 1-i & 2\\ 4 & 2 & 2 & 1\\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} |x_1|^2\\ x_1\overline{x}_2\\ \overline{x}_1x_2\\ |x_2|^2 \end{bmatrix} = \begin{bmatrix} 18\\ 20\\ 29\\ 5 \end{bmatrix} \implies \begin{bmatrix} |x_1|^2\\ x_1\overline{x}_2\\ \overline{x}_1x_2\\ |x_2|^2 \end{bmatrix} = \begin{bmatrix} 2\\ 3+3i\\ 3-3i\\ 9 \end{bmatrix}.$$

To recover $\mathbf{x} = [x_1 \;\; x_2]^T$, we rearrange the vector of new linear variables into a square matrix $\mathbf{X}$:

$$\begin{bmatrix} |x_1|^2 \\ x_1\overline{x}_2 \\ \overline{x}_1 x_2 \\ |x_2|^2 \end{bmatrix} \xrightarrow{\text{rearrange}} \mathbf{X} = \begin{bmatrix} |x_1|^2 & x_1\overline{x}_2 \\ \overline{x}_1 x_2 & |x_2|^2 \end{bmatrix} = \mathbf{x}\mathbf{x}^*.$$

Note that our new matrix $\mathbf{X}$ is rank 1, because it is the outer product of $\mathbf{x}$ and $\mathbf{x}^*$. Therefore, recovering the vector (or signal) $\mathbf{x} = [x_1 \ \ x_2]^T$ is a matter of finding the leading eigenvector of $\mathbf{X}$ (other eigenvectors will have eigenvalues equal to 0):

$$\begin{bmatrix} |x_1|^2 & x_1\overline{x}_2 \\ \overline{x}_1 x_2 & |x_2|^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \mathbf{x}\mathbf{x}^*\mathbf{x} = \|\mathbf{x}\|^2\mathbf{x} \quad \xrightarrow[\lambda=\|\mathbf{x}\|^2]{\text{eigenvector}} \quad \mathbf{x} = \begin{bmatrix} 1+i \\ 3 \end{bmatrix}.$$

This is an *eigenvector-based synchronization* approach to recovering phase information which mirrors steps in SDP algorithms like PhaseLift. Now, because we look at masked measurements, many of the entries of the rank 1 matrix $\mathbf{X} = \mathbf{x}\mathbf{x}^*$ will be replaced by zeros, prompting us to use *phase* or *angular synchronization* to recover the signal. Indeed, angular synchronization is the critical step in Algorithm 1 (as well as Algorithm 2 for 2D phase retrieval), and its implementation is based on the bandlimited parameter δ of the mask $\mathbf{m}$ and a sufficient number of shifts L of the mask as required by Corollary 1.

So, recall from Corollary 1 that $\operatorname{supp}(\widehat{\mathbf{m}}) = \{0, 1, \ldots, \delta - 1\}$, which implies $|\operatorname{supp}(\widehat{\mathbf{m}})| = \delta$, and that $L = 2\delta - 1$. Additionally, for all $k \in \mathbb{Z}_N$, and after one IDFT, we can recover the vector quantity $\widehat{\mathbf{x}} \circ S_{\omega - pL}\overline{\widehat{\mathbf{x}}}$ from (2.4):

$$\widehat{\mathbf{x}} \circ S_{\omega}\overline{\widehat{\mathbf{x}}} = \frac{N^2}{L} F_N^{-1} \left[\frac{(F_L(F_N Y)^T)_\omega}{F_N\left(\widehat{\mathbf{m}} \circ S_{-\omega}\overline{\widehat{\mathbf{m}}}\right)} \right], \qquad \forall \omega \in \mathbb{Z}_\delta$$

$$\widehat{\mathbf{x}} \circ S_{\omega - L}\overline{\widehat{\mathbf{x}}} = \frac{N^2}{L} F_N^{-1} \left[\frac{(F_L(F_N Y)^T)_\omega}{F_N\left(\widehat{\mathbf{m}} \circ S_{L-\omega}\overline{\widehat{\mathbf{m}}}\right)} \right], \quad \forall \omega \in \mathbb{Z}_L \backslash \mathbb{Z}_\delta$$

where division is component-wise and $(F_L(F_N Y)^T)_\omega \in \mathbb{C}^N$ denotes the ω^{th} row vector of the matrix. The Hadamard product of $\widehat{\mathbf{x}}$ and shifts of $\overline{\widehat{\mathbf{x}}}$ is analogous to the solution of the lifted system in the opening example generated by $\mathbf{X}$. So, we want to strategically rearrange the terms $\widehat{\mathbf{x}} \circ S_{\omega - pL}\overline{\widehat{\mathbf{x}}}$ into a *circular banded system* that resembles the rank 1 matrix $\widehat{\mathbf{x}}\widehat{\mathbf{x}}^*$. However, some off-diagonal bands of this matrix will be all zeros due to support parameters of $\widehat{\mathbf{m}}$. So, we define a banded matrix operator T_δ.

Definition 2.3.1. *Let $A \in \mathbb{C}^{N \times N}$ with row and column indices $i, j \in \mathbb{Z}_N$, respectively. Then, we define the banded matrix operator $T_\delta \colon \mathbb{C}^{N\times N} \to \mathbb{C}^{N \times N}$ with respect to parameter $\delta \in \mathbb{Z}_N$ as*

$$(T_\delta A)_{ij} := \begin{cases} A_{ij}, & \text{if } i - j \ (\text{mod } N) < \delta \\ 0, & \text{otherwise} \end{cases}.$$

To better visualize the matrix $T_\delta(\widehat{\mathbf{x}}\widehat{\mathbf{x}}^*)$, we let $N = 8$, $\delta = 3$, and $L = 2\delta - 1 = 5$. By Corollary 1, these parameters imply shifts $\omega - pL = -2, -1, 0, 1, 2$. Now, with $L = 5$ recovered vectors, we construct the matrix using a circular banded structure:

$$\begin{matrix} \widehat{\mathbf{x}} \circ S_{-2}\overline{\widehat{\mathbf{x}}} \\ \widehat{\mathbf{x}} \circ S_{-1}\overline{\widehat{\mathbf{x}}} \\ \widehat{\mathbf{x}} \circ \boldsymbol{S_0}\overline{\widehat{\mathbf{x}}} \\ \widehat{\mathbf{x}} \circ S_{1}\overline{\widehat{\mathbf{x}}} \\ \widehat{\mathbf{x}} \circ S_{2}\overline{\widehat{\mathbf{x}}} \end{matrix} \xrightarrow{arrange} \begin{bmatrix} |\widehat{\boldsymbol{x}}_\mathbf{0}|^\mathbf{2} & \widehat{x}_0\overline{\widehat{x}}_1 & \widehat{x}_0\overline{\widehat{x}}_2 & 0 & 0 & 0 & \widehat{x}_0\overline{\widehat{x}}_6 & \widehat{x}_0\overline{\widehat{x}}_7 \\ \widehat{x}_1\overline{\widehat{x}}_0 & |\widehat{\boldsymbol{x}}_\mathbf{1}|^\mathbf{2} & \widehat{x}_1\overline{\widehat{x}}_2 & \widehat{x}_1\overline{\widehat{x}}_3 & 0 & 0 & 0 & \widehat{x}_1\overline{\widehat{x}}_7 \\ \widehat{x}_2\overline{\widehat{x}}_0 & \widehat{x}_2\overline{\widehat{x}}_1 & |\widehat{\boldsymbol{x}}_\mathbf{2}|^\mathbf{2} & \widehat{x}_2\overline{\widehat{x}}_3 & \widehat{x}_2\overline{\widehat{x}}_4 & 0 & 0 & 0 \\ 0 & \widehat{x}_3\overline{\widehat{x}}_1 & \widehat{x}_3\overline{\widehat{x}}_2 & |\widehat{\boldsymbol{x}}_\mathbf{3}|^\mathbf{2} & \widehat{x}_3\overline{\widehat{x}}_4 & \widehat{x}_3\overline{\widehat{x}}_5 & 0 & 0 \\ 0 & 0 & \widehat{x}_4\overline{\widehat{x}}_2 & \widehat{x}_4\overline{\widehat{x}}_3 & |\widehat{\boldsymbol{x}}_\mathbf{4}|^\mathbf{2} & \widehat{x}_4\overline{\widehat{x}}_5 & \widehat{x}_4\overline{\widehat{x}}_6 & 0 \\ 0 & 0 & 0 & \widehat{x}_5\overline{\widehat{x}}_3 & \widehat{x}_5\overline{\widehat{x}}_4 & |\widehat{\boldsymbol{x}}_\mathbf{5}|^\mathbf{2} & \widehat{x}_5\overline{\widehat{x}}_6 & \widehat{x}_5\overline{\widehat{x}}_7 \\ \widehat{x}_6\overline{\widehat{x}}_0 & 0 & 0 & 0 & \widehat{x}_6\overline{\widehat{x}}_4 & \widehat{x}_6\overline{\widehat{x}}_5 & |\widehat{\boldsymbol{x}}_\mathbf{6}|^\mathbf{2} & \widehat{x}_6\overline{\widehat{x}}_7 \\ \widehat{x}_7\overline{\widehat{x}}_0 & \widehat{x}_7\overline{\widehat{x}}_1 & 0 & 0 & 0 & \widehat{x}_7\overline{\widehat{x}}_5 & \widehat{x}_7\overline{\widehat{x}}_6 & |\widehat{\boldsymbol{x}}_\mathbf{7}|^\mathbf{2} \end{bmatrix} = T_\delta(\widehat{\mathbf{x}}\widehat{\mathbf{x}}^*).$$

Observe that the squared magnitude components of $\widehat{\mathbf{x}}$ are along the main diagonal; the Hadamard product of $\widehat{\mathbf{x}}$ with positive shifts of $\widehat{\mathbf{x}}$ begin under the main diagonal; the product with negative shifts above the main diagonal; and each row has $N - L = 3$ zeros and $L = 5$ nonzero values.

Now, the banded matrix $T_\delta(\widehat{\mathbf{x}}\widehat{\mathbf{x}}^*)$ is not rank 1 nor is the vector $\widehat{\mathbf{x}}$, the Fourier coefficients of the signal we want to recover, an eigenvector of the matrix. So, we normalize the nonzero entries of the banded matrix to obtain the *relative phases* of the recovered Fourier quantities, which are of the form $e^{i(\widehat{\theta}_n - \widehat{\theta}_m)}$ for $n, m \in \mathbb{Z}_N$. Let $\mathbf{X} = \widehat{\mathbf{x}}\widehat{\mathbf{x}}^*$ and define

$$(\mathring{\mathbf{X}})_{ij} := \begin{cases} \dfrac{X_{ij}}{|X_{ij}|}, & \text{if } X_{ij} \neq 0 \\ 0, & \text{if } X_{ij} = 0 \end{cases}$$

as the normalized matrix desired. Then, the matrix of relative phases in this example is

$$T_\delta\mathring{\mathbf{X}} = \begin{bmatrix} \mathbf{1} & e^{i(\widehat{\theta}_0-\widehat{\theta}_1)} & e^{i(\widehat{\theta}_0-\widehat{\theta}_2)} & 0 & 0 & 0 & e^{i(\widehat{\theta}_0-\widehat{\theta}_6)} & e^{i(\widehat{\theta}_0-\widehat{\theta}_7)} \\ e^{i(\widehat{\theta}_1-\widehat{\theta}_0)} & \mathbf{1} & e^{i(\widehat{\theta}_1-\widehat{\theta}_2)} & e^{i(\widehat{\theta}_1-\widehat{\theta}_3)} & 0 & 0 & 0 & e^{i(\widehat{\theta}_1-\widehat{\theta}_7)} \\ e^{i(\widehat{\theta}_2-\widehat{\theta}_0)} & e^{i(\widehat{\theta}_2-\widehat{\theta}_1)} & \mathbf{1} & e^{i(\widehat{\theta}_2-\widehat{\theta}_3)} & e^{i(\widehat{\theta}_2-\widehat{\theta}_4)} & 0 & 0 & 0 \\ 0 & e^{i(\widehat{\theta}_3-\widehat{\theta}_1)} & e^{i(\widehat{\theta}_3-\widehat{\theta}_2)} & \mathbf{1} & e^{i(\widehat{\theta}_3-\widehat{\theta}_4)} & e^{i(\widehat{\theta}_3-\widehat{\theta}_5)} & 0 & 0 \\ 0 & 0 & e^{i(\widehat{\theta}_4-\widehat{\theta}_2)} & e^{i(\widehat{\theta}_4-\widehat{\theta}_3)} & \mathbf{1} & e^{i(\widehat{\theta}_4-\widehat{\theta}_5)} & e^{i(\widehat{\theta}_4-\widehat{\theta}_6)} & 0 \\ 0 & 0 & 0 & e^{i(\widehat{\theta}_5-\widehat{\theta}_3)} & e^{i(\widehat{\theta}_5-\widehat{\theta}_4)} & \mathbf{1} & e^{i(\widehat{\theta}_5-\widehat{\theta}_6)} & e^{i(\widehat{\theta}_5-\widehat{\theta}_7)} \\ e^{i(\widehat{\theta}_6-\widehat{\theta}_0)} & 0 & 0 & 0 & e^{i(\widehat{\theta}_6-\widehat{\theta}_4)} & e^{i(\widehat{\theta}_6-\widehat{\theta}_5)} & \mathbf{1} & e^{i(\widehat{\theta}_6-\widehat{\theta}_7)} \\ e^{i(\widehat{\theta}_7-\widehat{\theta}_0)} & e^{i(\widehat{\theta}_7-\widehat{\theta}_1)} & 0 & 0 & 0 & e^{i(\widehat{\theta}_7-\widehat{\theta}_5)} & e^{i(\widehat{\theta}_7-\widehat{\theta}_6)} & \mathbf{1} \end{bmatrix}.$$

As per strategy, the matrix $T_\delta\mathring{\mathbf{X}}$ conveniently decomposes into 3 matrices: 2 diagonal matrices of relative phases and a 0-1 circulant matrix,

$$
T_\delta \mathring{\mathbf{X}} = \begin{bmatrix} e^{i\widehat{\theta}_0} & 0 & \cdots & 0 \\ 0 & e^{i\widehat{\theta}_1} & 0 & \vdots \\ \vdots & \cdots & \ddots & \vdots \\ 0 & \cdots & \cdots & e^{i\widehat{\theta}_7} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-i\widehat{\theta}_0} & 0 & \cdots & 0 \\ 0 & e^{-i\widehat{\theta}_1} & 0 & \vdots \\ \vdots & \cdots & \ddots & \vdots \\ 0 & \cdots & \cdots & e^{-i\widehat{\theta}_7} \end{bmatrix}.
$$

Many spectral properties of these 0-1 circulant matrices are well-known. This strategic construction of the circularly banded matrix exploits the fact that the leading eigenvector of the circulant matrix is $[1\,1\ \cdots\ 1]^T$. So, then note that the vector of phases of the Fourier coefficients of $\mathbf{x}$, $\widehat{\mathbf{x}}/|\widehat{\mathbf{x}}| = [e^{i\widehat{\theta}_0}\ e^{i\widehat{\theta}_1}\ \cdots\ e^{i\widehat{\theta}_7}]^T$, is indeed an eigenvector of $T_\delta \mathring{\mathbf{X}}$ and its associated eigenvalue is $L = 5$:

$$
\begin{aligned}
T_\delta \mathring{\mathbf{X}} \begin{bmatrix} e^{i\widehat{\theta}_0} \\ e^{i\widehat{\theta}_1} \\ \vdots \\ e^{i\widehat{\theta}_7} \end{bmatrix} &= \begin{bmatrix} e^{i\widehat{\theta}_0} & 0 & \cdots & 0 \\ 0 & e^{i\widehat{\theta}_1} & 0 & \vdots \\ 0 & \cdots & \ddots & \vdots \\ 0 & \cdots & \cdots & e^{i\widehat{\theta}_7} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-i\widehat{\theta}_0} & 0 & \cdots & 0 \\ 0 & e^{-i\widehat{\theta}_1} & 0 & \vdots \\ \vdots & \cdots & \ddots & \vdots \\ 0 & \cdots & \cdots & e^{-i\widehat{\theta}_7} \end{bmatrix} \begin{bmatrix} e^{i\widehat{\theta}_0} \\ e^{i\widehat{\theta}_1} \\ \vdots \\ e^{i\widehat{\theta}_7} \end{bmatrix} \\
&= \begin{bmatrix} e^{i\widehat{\theta}_0} & 0 & \cdots & 0 \\ 0 & e^{i\widehat{\theta}_1} & 0 & \vdots \\ \vdots & \cdots & \ddots & \vdots \\ 0 & \cdots & \cdots & e^{i\widehat{\theta}_7} \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \\
&= 5 \begin{bmatrix} e^{i\widehat{\theta}_0} & 0 & \cdots & 0 \\ 0 & e^{i\widehat{\theta}_1} & 0 & \vdots \\ 0 & \cdots & \ddots & \vdots \\ 0 & \cdots & \cdots & e^{i\widehat{\theta}_7} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \\
&= 5 \begin{bmatrix} e^{i\widehat{\theta}_0} \\ e^{i\widehat{\theta}_1} \\ \vdots \\ e^{i\widehat{\theta}_7} \end{bmatrix} = L \frac{\widehat{\mathbf{x}}}{|\widehat{\mathbf{x}}|}.
\end{aligned}
$$

Thus, it can be shown for the general case that L is the dominant eigenvalue and $\widehat{\mathbf{x}}/|\widehat{\mathbf{x}}|$ is the unique, leading eigenvector. In the presence of noise, however, the desired eigenvector is not unique which makes the task of finding the correct phase is nontrivial; the signal can only be recovered up to certain global phase factors. The guiding article [8] presents robustness bounds for the estimated signal $\mathbf{x}_{\text{rec}}$ from noisy measurements, a distinguishing factor resulting from the underlying mathematics of Theorem and Corollary 1. The 1-dimensional phase retrieval method, Algorithm 1, is presented in Appendix B on page 37.

2.4 1D Numerical Testing

To test the performance of Algorithm 1, we implemented two numerical experiments in MATLAB: (1) a test for accuracy and robustness to random noise added to the phaseless Fourier measurements $Y_{j,\ell}$ as defined by equation (2.1); and (2) a test of execution time based on the length N of the 1-dimensional signal. For the first test, Gaussian noise $\eta_{j,\ell}$ was added using signal-to-noise ratios (SNR) 10, 20, 30, 40, 50, 60, 70, and 80. Per each SNR value, the code was looped 50 times where each time a different true signal $\mathbf{x} \in \mathbb{C}^N$ was randomly generated by a Gaussian distribution. Then, finally, the relative error values, computed in decibels by the formula

$$\text{rel err} = 10 \log_{10} \left(\frac{\|\mathbf{x} - \mathbf{x}_{\text{rec}}\|_2^2}{\|\mathbf{x}\|_2^2} \right),$$

where $\mathbf{x} \in \mathbb{C}^N$ is the true (randomly generated) signal and $\mathbf{x}_{\text{rec}}$ the signal recovered by Algorithm 1, were averaged.

Recall that the bandlimited parameter $\delta = |\text{supp}(\mathbf{m})| = |\text{supp}(\widehat{\mathbf{m}})|$. Figure 4 demonstrates that as we increase the support parameter δ closer to the length N of the signal, we obtain a recovered signal $\mathbf{x}_{\text{rec}}$ closer to the true signal up to the level of noise added to each measurmenet $Y_{j,\ell}$ as expected.

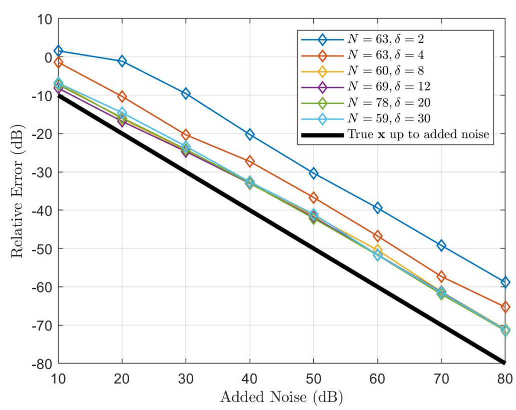


Figure 4: Relative Error vs. Added Noise

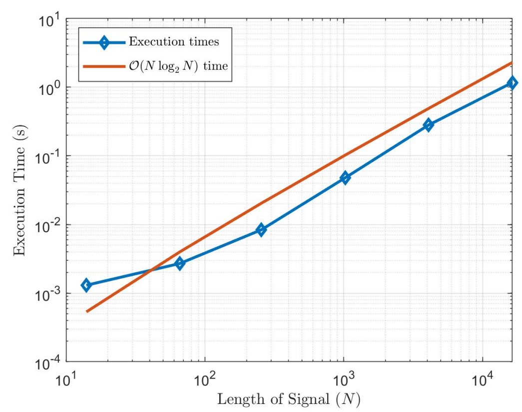


Figure 5: Log-Log of Execution Time vs. Signal Length

In the execution time testing, we similarly looped the code 50 times for each 6 differently sized and randomly generated (Gaussian) signals: $N = 2^4, 2^6, 2^8, 2^{10}, 2^{12}, 2^{14}$. Time was measured in seconds from when the phaseless measurements matrix, with added SNR $= 30$, was created to when $\mathbf{x}_{\text{rec}}$ was recovered. One difference in this numerical test is that a different bandlimited mask was generated for each of the 50 loops as well as the true signal $\mathbf{x}$. Figure 5 is a log-log plot comparison of the execution times of the code measured against $\mathcal{O}(N \log_2 N)$ time, the time associated with the number of computations needed to compute the fast Fourier transform (FFT). The $\mathcal{O}(N \log_2 N)$ time graph was scaled to judge the rate of change of the 2 graphs. Note that Algorithm 1 performs faster than FFT time for signal sizes up to approximately $N = 300$ and performs comparably for higher values of N.

As these metrics show much promise for use in the field, therefore the goal is to achieve similar results for the 2-dimensional phase retrieval code of Algorithm 2.

3 2D Phase Retrieval

3.1 2D Preliminaries and Notation

As stated before, the objective of this research is to extend the results of [8] for the 1-dimensional case to the 2-dimensional case. First, we introduce notation used throughout this section. Using script notation, we represent discrete, 2-dimensional signals as matrices $\mathcal{X} \in \mathbb{C}^{N_1 \times N_2}$, and following notation from [2], we use lowercase letters with bold vector indexing to denote the components of a 2-dimensional signal. Thus, for $\mathbf{n} = [n_1 \;\; n_2]^T \in \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$, the component $x_{\mathbf{n}}$ denotes the $n_1{}^{\text{th}}$ horizontal row element of the $n_2{}^{\text{th}}$ vertical column of signal $\mathcal{X}$ where $n_1 \in \mathbb{Z}_{N_1}$ and $n_2 \in \mathbb{Z}_{N_2}$.

Operation	**Definition**
(Rectangular) Periodicity Matrix N	$N = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix}$
Region of Summation R_N	$R_N := \{\mathbf{n} = [n_1 \quad n_2]^T : n_1 \in \mathbb{Z}_{N_1},\, n_2 \in \mathbb{Z}_{N_2}\}$
Conjugate, Transpose, & Conjugate Transpose	$\overline{\mathcal{X}}, \mathcal{X}^T, \mathcal{X}^*$
2D Time Reversal	$\widetilde{x}_{\mathbf{n}} := x_{-\mathbf{n}}$
Circular Shift in Time Operator $S_{\boldsymbol{\ell}} : \mathbb{C}^{N_1 \times N_2} \to \mathbb{C}^{N_1 \times N_2}$	$(S_{\boldsymbol{\ell}}\mathcal{X})_{\mathbf{n}} := x_{\mathbf{n}-\boldsymbol{\ell}}$
Modulation Operator $W_{\mathbf{k}} : \mathbb{C}^{N_1 \times N_2} \to \mathbb{C}^{N_1 \times N_2}$	$(W_{\mathbf{k}}\mathcal{X})_{\mathbf{n}} := x_{\mathbf{n}} e^{-2\pi i \mathbf{k}^T N^{-1} \mathbf{n}}$
Hadamard (Element-wise) product	$(\mathcal{X} \circ \mathcal{H})_{\mathbf{n}} := x_{\mathbf{n}} h_{\mathbf{n}}$
Circular Convolution $\circledast_N : \mathbb{C}^{N_1 \times N_2} \times \mathbb{C}^{N_1 \times N_2} \to \mathbb{C}^{N_1 \times N_2}$	$(\mathcal{X} \circledast_N \mathcal{H})_{\mathbf{n}} := \sum_{\mathbf{k} \in R_N} x_{\mathbf{k}} h_{\mathbf{n}-\mathbf{k}}$

Table 2: 2D Elementary Operations

Elementary operations on signals, as shown in Table 2, are defined similarly to the 1-dimensional case but with vector indexing. All indexing of 2-dimensional signals $\mathcal{X}, \mathcal{H} \in \mathbb{C}^{N_1 \times N_2}$ are implied to be modulo N_1 horizontally (by row) and modulo N_2 vertically (by column).

Note that the periodicity of an $N_1 \times N_2$ signal is represented by a 2×2 matrix. We could consider the 1-dimensional signals as having periodicity defined by a 1×1 matrix, so that the 1-dimensional case is a special case of a general periodicity definition. If N is a diagonal matrix, then the signal is rectangularly periodic, that is the periodicity can be seen in horizontal and vertical blocks parallel to the axes of shifts.

Definition 3.1.1. *A 2-dimensional signal $\mathcal{X} \in \mathbb{C}^{N_1 \times N_2}$ is* **rectangularly N-periodic** *if there exists $N_1, N_2 \in \mathbb{N}$ such that $x_{n_1+N_1,n_2} = x_{n_1,n_2} = x_{n_1,n_2+N_2}$ for all $n_1 \in \mathbb{Z}_{N_1}$ and all $n_2 \in \mathbb{Z}_{N_2}$. The periodicity of the signal $\mathcal{X}$ is defined by the diagonal matrix*

$$N = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix}.$$

A more general representation of a 2-dimensional periodic signal has nonzero terms on the skew-diagonal of its periodicity matrix N. In our analysis, as well as in diffraction imaging, assuming rectangular periodicity is valid since most signals investigated are not periodic.

Lastly, we define the 2-dimensional analogs of the DFT and IDFT.

Definition 3.1.2. *Let $\mathcal{X} \in \mathbb{C}^{N_1 \times N_2}$ and $\mathbf{n}, \mathbf{j} \in \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$. The Discrete Fourier Transform $F_N \colon \mathbb{C}^{N_1 \times N_2} \to \mathbb{C}^{N_1 \times N_2}$ and Inverse Discrete Fourier Transform $F_N^{-1} \colon \mathbb{C}^{N_1 \times N_2} \to \mathbb{C}^{N_1 \times N_2}$ are defined as*

$$\widehat{x}_{\mathbf{j}} = (F_N \mathcal{X})_{\mathbf{j}} := \sum_{\mathbf{n} \in R_N} x_{\mathbf{n}} e^{-2\pi i \mathbf{n}^T N^{-1} \mathbf{j}}$$

$$x_{\mathbf{n}} = (F_N^{-1} \widehat{\mathcal{X}})_{\mathbf{n}} := \frac{1}{\det N} \sum_{\mathbf{j} \in R_N} \widehat{x}_{\mathbf{j}} e^{2\pi i \mathbf{j}^T N^{-1} \mathbf{n}}.$$

Traditionally, the 2-dimensional DFT $F_N \colon \mathbb{C}^{N_1 \times N_2} \to \mathbb{C}^{N_1 \times N_2}$ is given with 2 indexing sets for a double summation:

$$(F_N \mathcal{X})_{j_1,j_2} = \widehat{x}_{j_1,j_2} := \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x_{n_1,n_2} e^{\frac{-2\pi i n_1 j_1}{N_1}} e^{\frac{-2\pi i n_2 j_2}{N_2}}.$$

Nonetheless, vector indexing – though a slight abuse of notation since indices $\mathbf{n}, \mathbf{j} \in \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$ are not ordered pairs – provides compactness of notation. Furthermore, we believe this notation can ease the translation of our theoretical results to 3 or even higher dimensions.

3.2 2D Measurements Equation

Let $\mathcal{X} \in \mathbb{C}^{N_1 \times N_2}$ be the 2-dimensional signal of the unknown sample and the bandlimited mask $\mathcal{M} \in \mathbb{C}^{N_1 \times N_2}$. The real-valued, phaseless Fourier measurements now populate a fourth-order tensor $\mathcal{Y} \in \mathbb{R}^{N_1 \times N_2 \times L_1 \times L_2}$, where L_1, L_2 represent the total horizontal and vertical shifts of the mask, respectively. Thus, the notation $\mathcal{Y}_{j_1,j_2,\ell_1,\ell_2}$ denotes the measurement corresponding to the $\ell_1{}^{\text{th}}$ and $\ell_2{}^{\text{th}}$ horizontal and vertical shifts, and the $k_1{}^{\text{th}}$ and $k_2{}^{\text{th}}$ horizontal and vertical Fourier modes, respectively. The physics of diffracted amplitudes in 2 dimensions prescribes the governing measurements equation

$$\mathcal{Y}_{j_1,j_2,\ell_1,\ell_2} = \left| \sum_{n_1=0}^{N_1-1} \sum_{n_2=0}^{N_2-1} x_{n_1,n_2} m_{n_1-\ell_1,n_2-\ell_2} e^{\frac{-2\pi i n_1 j_1}{N_1}} e^{\frac{-2\pi i n_2 j_2}{N_2}} \right|^2 + \eta_{j_1,j_2,\ell_1,\ell_2},$$

for $n_1, j_1, \ell_1 \in \mathbb{Z}_{N_1}$ and $n_2, j_2, \ell_2 \in \mathbb{Z}_{N_2}$. But for compactness, we use vector indexing and bracket notation: Let $\mathbf{j}, \mathbf{n}, \boldsymbol{\ell} \in \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$ and let $\mathcal{Y}_{[\mathbf{j}\ \boldsymbol{\ell}]}$ correspond to the measurement $\mathcal{Y}_{j_1, j_2, \ell_1, \ell_2} \in \mathbb{R}$, then we can rewrite the measurements equation as

$$\begin{aligned} \mathcal{Y}_{[\mathbf{j}\ \boldsymbol{\ell}]} &= \left| \sum_{\mathbf{n} \in R_N} x_{\mathbf{n}} m_{\mathbf{n}-\boldsymbol{\ell}} e^{-2\pi i \mathbf{n}^T N^{-1} \mathbf{j}} \right|^2 + \eta_{[\mathbf{j}\ \boldsymbol{\ell}]} \\ &= \left| (F_N(\mathcal{X} \circ S_{\boldsymbol{\ell}} \mathcal{M}))_{\mathbf{j}} \right|^2 + \eta_{[\mathbf{j}\ \boldsymbol{\ell}]}. \end{aligned} \tag{3.1}$$

3.3 Preliminary Lemmas

In this section, we show that Lemmas 2.2.1, 2.2.2, and 2.2.4 from the 1-dimensional case extend readily to the 2-dimensional case. Similarly, the mathematical properties of the DFT in 2 dimensions, taken from [2] (proofs for Properties 2(iv.) and 2(viii.) are in Appendix A), bear resemblance to the 1-dimensional properties.

Properties 2. *Let $\mathcal{X}, \mathcal{H} \in \mathbb{C}^{N_1 \times N_2}$ be arbitrary, 2-dimensional signals and $\alpha, \beta \in \mathbb{C}$ scalars. Then for all $\mathbf{n}, \mathbf{j}, \mathbf{k}, \boldsymbol{\ell} \in \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$, the following properties hold:*

i. $(F_N(\alpha \mathcal{X} + \beta \mathcal{H}))_{\mathbf{j}} = \alpha \widehat{x}_{\mathbf{j}} + \beta \widehat{h}_{\mathbf{j}}$

ii. $(F_N(S_{\boldsymbol{\ell}} \mathcal{X}))_{\mathbf{j}} = (W_{\boldsymbol{\ell}} \widehat{\mathcal{X}})_{\mathbf{j}}$

iii. $(F_N \widetilde{\mathcal{X}})_{\mathbf{j}} = \widehat{\widetilde{x}}_{\mathbf{j}} = \widetilde{\widehat{x}}_{\mathbf{j}}$

iv. $(\widetilde{S_{\boldsymbol{\ell}} \mathcal{X}})_{\mathbf{n}} = (S_{-\boldsymbol{\ell}} \widetilde{\mathcal{X}})_{\mathbf{n}}$

v. $(F_N(\mathcal{H} \circledast_N \mathcal{X}))_{\mathbf{j}} = (\widehat{\mathcal{H}} \circ \widehat{\mathcal{X}})_{\mathbf{j}}$

vi. $(F_N(\mathcal{H} \circ \mathcal{X}))_{\mathbf{j}} = \dfrac{1}{\det N} (\widehat{\mathcal{H}} \circledast_N \widehat{\mathcal{X}})_{\mathbf{j}}$

vii. $(F_N \overline{\mathcal{X}})_{\mathbf{j}} = \widehat{\overline{x}}_{\mathbf{j}} = \overline{\widetilde{\widehat{x}}}_{\mathbf{j}}$

viii. $(F_N(F_N \mathcal{X}))_{\mathbf{j}} = (F_N \widehat{\mathcal{X}})_{\mathbf{j}} = (\det N)\ \widetilde{x}_{\mathbf{j}}$.

Lemma 3.3.1. *Let $\mathcal{X} \in \mathbb{C}^{N_1 \times N_2}$ and $\mathbf{j}, \boldsymbol{\ell} \in \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$, then*

$$\left(F_N(\mathcal{X} \circ S_{\boldsymbol{\ell}} \overline{\mathcal{X}})\right)_{\mathbf{j}} = \frac{1}{\det N} e^{-2\pi i \mathbf{j}^T N^{-1} \boldsymbol{\ell}} \left(F_N(\widehat{\mathcal{X}} \circ S_{\mathbf{j}} \overline{\widehat{\mathcal{X}}})\right)_{-\boldsymbol{\ell}}.$$

Proof.

$$\begin{aligned} \left(F_N(\mathcal{X} \circ S_{\boldsymbol{\ell}} \overline{\mathcal{X}})\right)_{\mathbf{j}} &= \frac{1}{\det N} (\widehat{\mathcal{X}} \circledast_N F_N(S_{\boldsymbol{\ell}} \overline{\mathcal{X}}))_{\mathbf{j}} && \text{(by Property 2(vi.))} \\ &= \frac{1}{\det N} (\widehat{\mathcal{X}} \circledast_N W_{\boldsymbol{\ell}} \widehat{\overline{\mathcal{X}}})_{\mathbf{j}} && \text{(by Property 2(ii.))} \\ &= \frac{1}{\det N} \sum_{\mathbf{n} \in R_N} \widehat{x}_{\mathbf{n}} (W_{\boldsymbol{\ell}} \widehat{\overline{\mathcal{X}}})_{\mathbf{j}-\mathbf{n}} \\ &= \frac{1}{\det N} \sum_{\mathbf{n} \in R_N} \widehat{x}_{\mathbf{n}} \widehat{\overline{x}}_{\mathbf{j}-\mathbf{n}} e^{-2\pi i \boldsymbol{\ell}^T N^{-1} (\mathbf{j}-\mathbf{n})} \end{aligned}$$

$$= \frac{1}{\det N} e^{-2\pi i \boldsymbol{\ell}^T N^{-1}\mathbf{j}} \sum_{\mathbf{n}\in R_N} \widehat{x}_{\mathbf{n}} \widehat{\overline{\widetilde{x}}}_{\mathbf{j}-\mathbf{n}} e^{2\pi i \boldsymbol{\ell}^T N^{-1}\mathbf{n}}$$

$$= \frac{1}{\det N} e^{-2\pi i \boldsymbol{\ell}^T N^{-1}\mathbf{j}} \sum_{\mathbf{n}\in R_N} \widehat{x}_{\mathbf{n}} \widetilde{\widehat{\overline{\widetilde{x}}}}_{\mathbf{n}-\mathbf{j}} e^{2\pi i \boldsymbol{\ell}^T N^{-1}\mathbf{n}}. \qquad \text{(by Property 2(vii.))}$$

Note that the term $\widetilde{\widehat{\overline{\widetilde{x}}}}_{\mathbf{n}-\mathbf{j}}$ can be written as $\overline{\widehat{x}}_{\mathbf{n}-\mathbf{j}}$, since for any signal $\mathcal{X}$,

$$\widetilde{\widehat{\overline{\widetilde{\mathcal{X}}}}} = \widetilde{F_N \overline{\widetilde{\mathcal{X}}}} = \widetilde{\widetilde{\overline{\widehat{\mathcal{X}}}}} = \overline{\widehat{\mathcal{X}}},$$

by Property 2(vii.) and because the reversal of the reversal of a signal is the original signal. Therefore, we have

$$\begin{aligned}
(F_N(\mathcal{X} \circ S_{\boldsymbol{\ell}}\overline{\mathcal{X}}))_{\mathbf{j}} &= \frac{1}{\det N} e^{-2\pi i \boldsymbol{\ell}^T N^{-1}\mathbf{j}} \sum_{\mathbf{n}\in R_N} \widehat{x}_{\mathbf{n}} \overline{\widehat{x}}_{\mathbf{n}-\mathbf{j}} e^{2\pi i \boldsymbol{\ell}^T N^{-1}\mathbf{n}} \\
&= \frac{1}{\det N} e^{-2\pi i \boldsymbol{\ell}^T N^{-1}\mathbf{j}} \sum_{\mathbf{n}\in R_N} \widehat{x}_{\mathbf{n}} (S_{\mathbf{j}}\overline{\widehat{\mathcal{X}}})_{\mathbf{n}} e^{-2\pi i (-\boldsymbol{\ell})^T N^{-1}\mathbf{n}} \\
&= \frac{1}{\det N} e^{-2\pi i \boldsymbol{\ell}^T N^{-1}\mathbf{j}} \sum_{\mathbf{n}\in R_N} \left(\widehat{\mathcal{X}} \circ S_{\mathbf{j}}\overline{\widehat{\mathcal{X}}}\right)_{\mathbf{n}} e^{-2\pi i (-\boldsymbol{\ell})^T N^{-1}\mathbf{n}} \\
&= \frac{1}{\det N} e^{-2\pi i \boldsymbol{\ell}^T N^{-1}\mathbf{j}} \left(F_N(\widehat{\mathcal{X}} \circ S_{\mathbf{j}}\overline{\widehat{\mathcal{X}}})\right)_{-\boldsymbol{\ell}}. \qquad \text{(by def. of DFT)}
\end{aligned}$$

□

Before the next lemma, we give some notation regarding the manipulation of the 4-tensor $\mathcal{Y}$. Using colon and bracket notation, we define the matrix $\mathcal{Y}_{[:\,\boldsymbol{\ell}]} \in \mathbb{R}^{N_1\times N_2}$ as the 2-dimensional *slice*[3] of 4-dimensional array $\mathcal{Y} \in \mathbb{R}^{N_1\times N_2\times L_1\times L_2}$ for a fixed vector index $\boldsymbol{\ell} = [\ell_1\ \ell_2]^T \in \mathbb{Z}_{L_1} \times \mathbb{Z}_{L_2}$. And thus, the notation $(\mathcal{Y}_{[:\,\boldsymbol{\ell}]})_{\mathbf{k}}$ refers to the (k_1, k_2) entry of $\mathcal{Y}_{[:\,\boldsymbol{\ell}]}$.

Lemma 3.3.2. *Let $\mathcal{X} \in \mathbb{C}^{N_1\times N_2}$ be the unknown 2-dimensional signal to be recovered and the mask $\mathcal{M} \in \mathbb{C}^{N_1\times N_2}$ known. Let $\mathcal{Y}_{[:\,\boldsymbol{\ell}]} \in \mathbb{R}^{N_1\times N_2}$ be the 2-dimensional slice of 4^{th}-order tensor $\mathcal{Y} \in \mathbb{R}^{N_1\times N_2\times L_1\times L_2}$ as defined by measurements equation (3.1) for a fixed $\boldsymbol{\ell} \in \mathbb{Z}_{L_1} \times \mathbb{Z}_{L_2}$, i.e. the $N_1 \times N_2$ matrix of Fourier modes for the ${\ell_1}^{th}$ horizontal shift and ${\ell_2}^{th}$ vertical shift. Then for any $\mathbf{k} \in \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$,*

$$\left(F_N\left(F_N \mathcal{Y}_{[:\,\boldsymbol{\ell}]}\right)\right)_{\mathbf{k}} = (\det N)\left[F_N(\mathcal{X} \circ S_{\mathbf{k}}\overline{\mathcal{X}}) \circ F_N(\widetilde{\mathcal{M}} \circ S_{-\mathbf{k}}\overline{\widetilde{\mathcal{M}}})\right]_{\boldsymbol{\ell}}.$$

Proof. From the 2D measurements equation (3.1), we have that

$$\mathcal{Y}_{[:\,\boldsymbol{\ell}]} = |F_N(\mathcal{X} \circ S_{\boldsymbol{\ell}}\mathcal{M})|^2 = [F_N(\mathcal{X} \circ S_{\boldsymbol{\ell}}\mathcal{M})] \circ \overline{[F_N(\mathcal{X} \circ S_{\boldsymbol{\ell}}\mathcal{M})]}.$$

Note that we can rewrite the conjugate term:

$$\overline{(F_N(\mathcal{X} \circ S_{\boldsymbol{\ell}}\mathcal{M}))_{\mathbf{j}}} = \overline{\sum_{\mathbf{n}\in R_N} x_{\mathbf{n}} (S_{\boldsymbol{\ell}}\mathcal{M})_{\mathbf{n}} e^{-2\pi i \mathbf{n}^T N^{-1}\mathbf{j}}}$$

[3] A term commonly used in computer science and programming for the manipulation of multidimensional arrays

$$
\begin{aligned}
&= \sum_{\mathbf{n}\in R_N} \overline{x}_{\mathbf{n}} \overline{m}_{\mathbf{n}-\boldsymbol{\ell}} e^{2\pi i \mathbf{n}^T N^{-1}\mathbf{j}} \\
&= \det N \left(\frac{1}{\det N} \sum_{\mathbf{n}\in R_N} \overline{x}_{\mathbf{n}} \overline{m}_{\mathbf{n}-\boldsymbol{\ell}} e^{2\pi i \mathbf{n}^T N^{-1}\mathbf{j}} \right) \\
&= (\det N)\ [F_N^{-1}(\overline{\mathcal{X}} \circ S_{\boldsymbol{\ell}} \overline{\mathcal{M}})]_{\mathbf{j}}. && \text{(by def. of IDFT)}
\end{aligned}
$$

So, for $\mathbf{j} \in \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$,

$$
\mathcal{Y}_{[\mathbf{j}\ \boldsymbol{\ell}]} = \left[(F_N(\mathcal{X} \circ S_{\boldsymbol{\ell}} \mathcal{M})) \circ (\det N) \left(F_N^{-1}(\overline{\mathcal{X}} \circ S_{\boldsymbol{\ell}} \overline{\mathcal{M}}) \right) \right]_{\mathbf{j}}.
$$

Taking the 2D DFT of the measurements $\mathcal{Y}_{[:\ \boldsymbol{\ell}]}$,

$$
\begin{aligned}
F_N \mathcal{Y}_{[:\ \boldsymbol{\ell}]} &= F_N [F_N(\mathcal{X} \circ S_{\boldsymbol{\ell}} \mathcal{M}) \circ (\det N) F_N^{-1}(\overline{\mathcal{X}} \circ S_{\boldsymbol{\ell}} \overline{\mathcal{M}})] \\
&= \frac{1}{\det N} \left[F_N(F_N(\mathcal{X} \circ S_{\boldsymbol{\ell}} \mathcal{M})) \circledast_N (\det N)(\overline{\mathcal{X}} \circ S_{\boldsymbol{\ell}} \overline{\mathcal{M}}) \right]_{\mathbf{j}} && \text{(by Property 2(vi.))} \\
&= \frac{1}{\det N} \left[(\det N)(\widetilde{\mathcal{X} \circ S_{\boldsymbol{\ell}} \mathcal{M}}) \circledast_N (\det N)(\overline{\mathcal{X}} \circ S_{\boldsymbol{\ell}} \overline{\mathcal{M}}) \right] && \text{(by Property 2(viii.))} \\
&= (\det N) \ \left[(\widetilde{\mathcal{X}} \circ S_{-\boldsymbol{\ell}} \widetilde{\mathcal{M}}) \circledast_N (\overline{\mathcal{X}} \circ S_{\boldsymbol{\ell}} \overline{\mathcal{M}}) \right]. && \text{(by Property 2(iv.))}
\end{aligned}
$$

By the definition of convolution, we have for $\mathbf{k} \in \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$

$$
\begin{aligned}
\left(F_N \mathcal{Y}_{[:\ \boldsymbol{\ell}]} \right)_{\mathbf{k}} &= (\det N) \sum_{\mathbf{n}\in R_N} \widetilde{x}_{\mathbf{n}} (S_{-\boldsymbol{\ell}} \widetilde{\mathcal{M}})_{\mathbf{n}} \overline{x}_{\mathbf{k}-\mathbf{n}} (S_{\boldsymbol{\ell}} \overline{\mathcal{M}})_{\mathbf{k}-\mathbf{n}} \\
&= (\det N) \sum_{\mathbf{n}\in R_N} x_{-\mathbf{n}} \widetilde{m}_{\mathbf{n}+\boldsymbol{\ell}} \overline{x}_{\mathbf{k}-\mathbf{n}} \overline{m}_{\mathbf{k}-\mathbf{n}-\boldsymbol{\ell}} \\
&= (\det N) \sum_{\mathbf{n}\in R_N} x_{-\mathbf{n}} \overline{x}_{\mathbf{k}-\mathbf{n}} \widetilde{m}_{\mathbf{n}+\boldsymbol{\ell}} \overline{m}_{\mathbf{k}-\mathbf{n}-\boldsymbol{\ell}} \\
&= (\det N) \sum_{\mathbf{n}\in R_N} x_{-\mathbf{n}} \overline{x}_{\mathbf{k}-\mathbf{n}} \widetilde{m}_{\mathbf{n}+\boldsymbol{\ell}} \overline{\widetilde{m}}_{(\mathbf{n}+\boldsymbol{\ell})-\mathbf{k}} && \text{(by def. of reversal)} \\
&= (\det N) \sum_{\mathbf{n}\in R_N} x_{-\mathbf{n}} (S_{-\mathbf{k}} \overline{\mathcal{X}})_{-\mathbf{n}} \widetilde{m}_{\mathbf{n}+\boldsymbol{\ell}} (S_{\mathbf{k}} \overline{\widetilde{\mathcal{M}}})_{\mathbf{n}+\boldsymbol{\ell}}.
\end{aligned}
$$

Let $\mathbf{p} = -\mathbf{n}$. Note that the region of summation for $\mathbf{p}$ is equivalent to that of $\mathbf{n} \in R_N$ due to modular indexing. So, we have

$$
\begin{aligned}
\left(F_N \mathcal{Y}_{[:\ \boldsymbol{\ell}]} \right)_{\mathbf{k}} &= (\det N) \sum_{\mathbf{p}\in R_N} x_{\mathbf{p}} (S_{-\mathbf{k}} \overline{\mathcal{X}})_{\mathbf{p}} \widetilde{m}_{\boldsymbol{\ell}-\mathbf{p}} (S_{\mathbf{k}} \overline{\widetilde{\mathcal{M}}})_{\boldsymbol{\ell}-\mathbf{p}} \\
&= (\det N) \sum_{\mathbf{p}\in R_N} (\mathcal{X} \circ S_{-\mathbf{k}} \overline{\mathcal{X}})_{\mathbf{p}} (\widetilde{\mathcal{M}} \circ S_{\mathbf{k}} \overline{\widetilde{\mathcal{M}}})_{\boldsymbol{\ell}-\mathbf{p}} \\
&= (\det N) \left[\left(\mathcal{X} \circ S_{-\mathbf{k}} \overline{\mathcal{X}} \right) \circledast_N \left(\widetilde{\mathcal{M}} \circ S_{\mathbf{k}} \overline{\widetilde{\mathcal{M}}} \right) \right]_{\boldsymbol{\ell}}. && \text{(by def. of convolution)}
\end{aligned}
$$

So, given that

$$\left(F_N \mathcal{Y}_{[:\,\boldsymbol{\ell}]}\right)_{\mathbf{k}} = (\det N)\left[\left(\mathcal{X} \circ S_{-\mathbf{k}}\overline{\mathcal{X}}\right) \circledast_N \left(\widetilde{\mathcal{M}} \circ S_{\mathbf{k}}\overline{\widetilde{\mathcal{M}}}\right)\right]_{\boldsymbol{\ell}}, \tag{3.2}$$

we take one more DFT and utilize Property 2(v.) to yield

$$\left(F_N\left(F_N \mathcal{Y}_{[:\,\boldsymbol{\ell}]}\right)\right)_{\mathbf{k}} = (\det N)\left[F_N\left(\mathcal{X} \circ S_{-\mathbf{k}}\overline{\mathcal{X}}\right) \circ F_N\left(\widetilde{\mathcal{M}} \circ S_{\mathbf{k}}\overline{\widetilde{\mathcal{M}}}\right)\right]_{\boldsymbol{\ell}}.$$

□

Definition 3.3.3. *Let $\mathcal{X} \in \mathbb{C}^{N_1 \times N_2}$ be a 2-dimensional signal and $L_1, L_2 \in \mathbb{N}$ be such that L_1 and L_2 divide N_1 and N_2, respectively. Then, the* **uniform sub-sampling operator** $Z_L\colon \mathbb{C}^{N_1 \times N_2} \to \mathbb{C}^{N_1/L_1 \times N_2/L_2}$ *is defined as*

$$(Z_L \mathcal{X})_{\mathbf{n}} := x_{L\mathbf{n}}, \quad \forall \mathbf{n} \in \mathbb{Z}_{N_1/L_1} \times \mathbb{Z}_{N_2/L_2}.$$

where the sub-sampling periodicity matrix

$$L = \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix}.$$

Lemma 3.3.4 (Aliasing in 2D)**.** *Let $\mathcal{X} \in \mathbb{C}^{N_1 \times N_2}$ be arbitrary and suppose L_1 and L_2 divide N_1 and N_2, respectively. Then, for any $\mathbf{j} \in \mathbb{Z}_{N_1/L_1} \times \mathbb{Z}_{N_2/L_2}$,*

$$(F_{NL^{-1}}(Z_L \mathcal{X}))_{\mathbf{j}} = \frac{1}{\det L} \sum_{\mathbf{p} \in R_L} \widehat{x}_{\mathbf{j} - NL^{-1}\mathbf{p}},$$

where (rectangular) periodicity matrices $L = \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix}$ and $NL^{-1} = \begin{bmatrix} N_1/L_1 & 0 \\ 0 & N_2/L_2 \end{bmatrix}$, and region of summation $R_L = \{\mathbf{p} : p_1 \in \mathbb{Z}_{L_1}, p_2 \in \mathbb{Z}_{L_2}\}$.

Proof. By the definition of the DFT of signals with periodicity NL^{-1}, $F_{NL^{-1}}\colon \mathbb{C}^{N_1/L_1 \times N_2/L_2} \to \mathbb{C}^{N_1/L_1 \times N_2/L_2}$, we have

$$(F_{NL^{-1}}(Z_L \mathcal{X}))_{\mathbf{j}} = \sum_{\mathbf{n} \in R_{NL^{-1}}} x_{L\mathbf{n}} e^{-2\pi i \mathbf{n}^T N^{-1} L \mathbf{j}},$$

where the region of summation $R_{NL^{-1}} = \{\mathbf{n} : n_1 \in \mathbb{Z}_{N_1/L_1}, n_2 \in \mathbb{Z}_{N_2/L_2}\}$. To show that our exponent agrees with the definition of this transform, we see that

$$\begin{aligned} -2\pi i \mathbf{n}^T N^{-1} L \mathbf{j} &= -2\pi i \begin{bmatrix} n_1 & n_2 \end{bmatrix} \begin{bmatrix} \frac{1}{N_1} & 0 \\ 0 & \frac{1}{N_2} \end{bmatrix} \begin{bmatrix} L_1 & 0 \\ 0 & L_2 \end{bmatrix} \begin{bmatrix} j_1 \\ j_2 \end{bmatrix} \\ &= -2\pi i \begin{bmatrix} \frac{n_1}{N_1} & \frac{n_2}{N_2} \end{bmatrix} \begin{bmatrix} L_1 j_1 \\ L_2 j_2 \end{bmatrix} \\ &= -2\pi i \left(\frac{n_1 L_1 j_1}{N_1} + \frac{n_2 L_2 j_2}{N_2} \right) \end{aligned}$$

$$= -2\pi i \left(\frac{n_1 j_1}{N_1/L_1} + \frac{n_2 j_2}{N_2/L_2} \right).$$

By definition of the IDFT,

$$\begin{aligned}
(F_{NL^{-1}}(Z_L\mathcal{X}))_{\mathbf{j}} &= \sum_{\mathbf{n}\in R_{NL^{-1}}} \left(\frac{1}{\det N} \sum_{\mathbf{k}\in R_N} \widehat{x}_{\mathbf{k}} e^{2\pi i \mathbf{k}^T N^{-1} L\mathbf{n}} \right) e^{-2\pi i \mathbf{n}^T N^{-1} L\mathbf{j}} \\
&= \frac{1}{\det N} \sum_{\mathbf{k}\in R_N} \widehat{x}_{\mathbf{k}} \sum_{\mathbf{n}\in R_{NL^{-1}}} e^{2\pi i \mathbf{k}^T N^{-1} L\mathbf{n}} e^{-2\pi i \mathbf{n}^T N^{-1} L\mathbf{j}} \\
&= \frac{1}{\det N} \sum_{\mathbf{k}\in R_N} \widehat{x}_{\mathbf{k}} \sum_{\mathbf{n}\in R_{NL^{-1}}} e^{2\pi i \mathbf{n}^T N^{-1} L\mathbf{k}} e^{-2\pi i \mathbf{n}^T N^{-1} L\mathbf{j}} \\
&= \frac{1}{\det N} \sum_{\mathbf{k}\in R_N} \widehat{x}_{\mathbf{k}} \sum_{\mathbf{n}\in R_{NL^{-1}}} e^{2\pi i \mathbf{n}^T N^{-1} L(\mathbf{k}-\mathbf{j})} \\
&= \frac{1}{\det N} \sum_{\mathbf{k}\in R_N} \widehat{x}_{\mathbf{k}} \left(\det(NL^{-1})\right) \delta_{\mathbf{k}-\mathbf{j} \bmod NL^{-1}} \\
&= \frac{\det(NL^{-1})}{\det N} \sum_{\mathbf{k}\in R_N} \widehat{x}_{\mathbf{k}} \delta_{\mathbf{k}-\mathbf{j} \bmod NL^{-1}} \\
&= \frac{(\det N)(\det L^{-1})}{\det N} \sum_{\mathbf{k}\in R_N} \widehat{x}_{\mathbf{k}} \delta_{\mathbf{k}-\mathbf{j} \bmod NL^{-1}} \\
&= \frac{1}{\det L} \sum_{\mathbf{k}\in R_N} \widehat{x}_{\mathbf{k}} \delta_{\mathbf{k}-\mathbf{j} \bmod NL^{-1}},
\end{aligned}$$

where mod NL^{-1} denotes modulo N_1/L_1 horizontally (first index component) and modulo N_2/L_2 vertically (second index component). Let $\mathbf{p} \in \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$. Here, the Kronecker delta is defined as

$$\delta_{\mathbf{k}-\mathbf{j} \bmod NL^{-1}} = \begin{cases} 1, & \mathbf{k} = \mathbf{j} - NL^{-1}\mathbf{p} \\ 0, & \text{otherwise} \end{cases}.$$

We can see more explicitly that this Kronecker delta is 1 when

$$\begin{aligned}
\begin{bmatrix} k_1 \\ k_2 \end{bmatrix} &= \begin{bmatrix} j_1 \\ j_2 \end{bmatrix} - \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix} \begin{bmatrix} \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \\
&= \begin{bmatrix} j_1 \\ j_2 \end{bmatrix} - \begin{bmatrix} p_1 \frac{N_1}{L_1} \\ p_2 \frac{N_2}{L_2} \end{bmatrix} \\
&= \begin{bmatrix} j_1 - p_1 \frac{N_1}{L_1} \\ j_2 - p_2 \frac{N_2}{L_2} \end{bmatrix}
\end{aligned}$$

for $p_1 \in \mathbb{Z}_{N_1}$ and $p_2 \in \mathbb{Z}_{N_2}$. Then, similarly as in Lemma 2.2.4, there are L_1 terms of p_1 and L_2 terms of p_2 which satisfy $\delta_{\mathbf{k}-\mathbf{j} \bmod NL^{-1}} = 1$. Therefore, we have

$$(F_{NL^{-1}}(Z_L\mathcal{X}))_{\mathbf{j}} = \frac{1}{\det L}\sum_{\mathbf{p}\in R_L} \widehat{x}_{\mathbf{j}-NL^{-1}\mathbf{p}}.$$

□

3.4 Main Result for 2D

Before providing the main result, we again discuss some preliminary notation. Recall in the 1-dimensional case for Lemma 2.2.2 and Theorem 1, we held the shift $\ell \in \mathbb{Z}_L$ fixed and took the DFT of the measurements for varying Fourier modes of $\mathbf{y}_\ell \in \mathbb{R}^N$, which are the column vectors of $Y \in \mathbb{R}^{N\times L}$. Then, we transposed Y and fixed the Fourier modes $k \in \mathbb{Z}_N$ to take the DFT for variable shifts.

With higher dimensional arrays, transposing n^{th}-order tensors requires considering the permutation symmetry group S_n of its $n! - 1$ transpositions [7]. Upon investigating the 2-dimensional variant of Theorem 1, we discovered there is only one transposition permutation $T\sigma$ that is needed: we want to fix the shift $\boldsymbol{\ell} \in \mathbb{Z}_{L_1}\times\mathbb{Z}_{L_2}$ and take the 2D DFT of $\mathcal{Y}_{[:\ \boldsymbol{\ell}]} \in \mathbb{R}^{N_1\times N_2}$ for varying modes $\mathbf{j}\in\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2}$, transpose the tensor[4] $F_N\mathcal{Y} \in \mathbb{C}^{N_1\times N_2\times L_1\times L_2}$ and then take the L-period 2D DFT of the matrices $((F_N\mathcal{Y})^{T\sigma})_{[:\ \mathbf{k}]} \in \mathbb{C}^{L_1\times L_2}$ with fixed Fourier modes $\mathbf{k}\in\mathbb{Z}_{L_1}\times\mathbb{Z}_{L_2}$ and varying shifts $\boldsymbol{\ell}$. Thus, for any 4-tensor $M \in \mathbb{C}^{N_1\times N_2\times L_1\times L_2}$ and indices i,j,k,l we define

$$(M^{T\sigma})_{ijkl} := M_{klij}.$$

Thus, in bracket notation, the desired transpose permutation for the measurements tensor $\mathcal{Y}$ is

$$(\mathcal{Y}^{T\sigma})_{[\boldsymbol{\ell}\ \mathbf{j}]} = \mathcal{Y}_{[\mathbf{j}\ \boldsymbol{\ell}]}.$$

Theorem 2. *Let the 2-dimensional signal $\mathcal{X}\in\mathbb{C}^{N_1\times N_2}$ be arbitrary and the bandlimited mask $\mathcal{M}\in\mathbb{C}^{N_1\times N_2}$ known. And for $L_1, L_2 \in \mathbb{N}$ such that L_1 divides N_1 and L_2 divides N_2, let $\mathcal{Y}\in\mathbb{R}^{N_1\times N_2\times L_1\times L_2}$ contain $N_1\cdot N_2\cdot L_1\cdot L_2$ measurements of the form* (3.1)*. Then, for any $\boldsymbol{\omega}\in\mathbb{Z}_{L_1}\times\mathbb{Z}_{L_2}$ and any $\mathbf{k}\in\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2}$,*

$$(F_L(F_N\mathcal{Y})^{T\sigma})_{[\boldsymbol{\omega}\ \mathbf{k}]} = \frac{\det L}{(\det N)^2}\sum_{\mathbf{p}\in R_{NL^{-1}}}\left(F_N\left(\widehat{\mathcal{X}}\circ S_{\boldsymbol{\omega}-L\mathbf{p}}\overline{\widehat{\mathcal{X}}}\right)\right)_{\mathbf{k}}\left(F_N\left(\widehat{\mathcal{M}}\circ S_{L\mathbf{p}-\boldsymbol{\omega}}\overline{\widehat{\mathcal{M}}}\right)\right)_{\mathbf{k}}.$$

Moreover, if $\operatorname{supp}(\widehat{\mathcal{M}}) = \{\mathbf{n}\in\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2} : 0\le n_1\le\delta_1-1, 0\le n_2\le\delta_2-1\}$ and $L_1 = 2\delta_1-1, L_2 = 2\delta_2-1$, then the sum above collapses to exactly one of the four terms:

$$\frac{(\det N)^2}{\det L}(F_L(F_N\mathcal{Y})^{T\sigma})_{[\boldsymbol{\omega}\ \mathbf{k}]} = \begin{cases} (i.)\ \left(F_N(\widehat{\mathcal{X}}\circ S_{\boldsymbol{\omega}}\overline{\widehat{\mathcal{X}}})\right)_{\mathbf{k}}\left(F_N(\widehat{\mathcal{M}}\circ S_{-\boldsymbol{\omega}}\overline{\widehat{\mathcal{M}}})\right)_{\mathbf{k}} \\ (ii.)\ \left(F_N(\widehat{\mathcal{X}}\circ S_{\boldsymbol{\omega}-[0\ L_2]^T}\overline{\widehat{\mathcal{X}}})\right)_{\mathbf{k}}\left(F_N(\widehat{\mathcal{M}}\circ S_{[0\ L_2]^T-\boldsymbol{\omega}}\overline{\widehat{\mathcal{M}}})\right)_{\mathbf{k}} \\ (iii.)\ \left(F_N(\widehat{\mathcal{X}}\circ S_{\boldsymbol{\omega}-[L_1\ 0]^T}\overline{\widehat{\mathcal{X}}})\right)_{\mathbf{k}}\left(F_N(\widehat{\mathcal{M}}\circ S_{[L_1\ 0]^T-\boldsymbol{\omega}}\overline{\widehat{\mathcal{M}}})\right)_{\mathbf{k}} \\ (iv.)\ \left(F_N(\widehat{\mathcal{X}}\circ S_{\boldsymbol{\omega}-[L_1\ L_2]^T}\overline{\widehat{\mathcal{X}}})\right)_{\mathbf{k}}\left(F_N(\widehat{\mathcal{M}}\circ S_{[L_1\ L_2]^T-\boldsymbol{\omega}}\overline{\widehat{\mathcal{M}}})\right)_{\mathbf{k}} \end{cases} \quad (3.3)$$

(i.) if $\boldsymbol{\omega}\in\mathbb{Z}_{\delta_1}\times\mathbb{Z}_{\delta_2}$, (ii.) if $\boldsymbol{\omega}\in\mathbb{Z}_{\delta_1}\times\mathbb{Z}_{L_2}\backslash\mathbb{Z}_{\delta_2}$, (iii.) if $\boldsymbol{\omega}\in\mathbb{Z}_{L_1}\backslash\mathbb{Z}_{\delta_1}\times\mathbb{Z}_{\delta_2}$, and (iv.) if $\boldsymbol{\omega}\in\mathbb{Z}_{L_1}\backslash\mathbb{Z}_{\delta_1}\times\mathbb{Z}_{L_2}\backslash\mathbb{Z}_{\delta_2}$.

[4]This is abuse of notation as in Theorem 1; here, we assume F_N operates on 2-dimensional slices of $\mathcal{Y}$ for fixed $\boldsymbol{\ell}$.

Proof. In Lemma 3.3.2, we showed in equation (3.2) that we can write the $(k_1, k_2, \ell_1, \ell_2)$ entry of $F_N\mathcal{Y}$ as

$$(F_N\mathcal{Y})_{k_1,k_2,\ell_1,\ell_2} = (\det N)\left[\left(\mathcal{X} \circ S_{-[k_1\ k_2]^T}\overline{\mathcal{X}}\right) \circledast_N \left(\widetilde{\mathcal{M}} \circ S_{[k_1\ k_2]^T}\overline{\widetilde{\mathcal{M}}}\right)\right]_{[\ell_1\ \ell_2]^T}.$$

Using bracket notation and the previously defined transpose $T\sigma$, we equivalently have

$$\left((F_N\mathcal{Y})^{T\sigma}\right)_{[\boldsymbol{\ell}\ \mathbf{k}]} = (\det N)\left[\left(\mathcal{X} \circ S_{-\mathbf{k}}\overline{\mathcal{X}}\right) \circledast_N \left(\widetilde{\mathcal{M}} \circ S_{\mathbf{k}}\overline{\widetilde{\mathcal{M}}}\right)\right]_{\boldsymbol{\ell}}.$$

for $\boldsymbol{\ell} \in \mathbb{Z}_{L_1} \times \mathbb{Z}_{L_2}$. Let $\mathcal{U} \in \mathbb{C}^{N_1 \times N_2}$, and define $\mathcal{U} = (\det N)[(\mathcal{X} \circ S_{-\mathbf{k}}\overline{\mathcal{X}}) \circledast_N (\widetilde{\mathcal{M}} \circ S_{\mathbf{k}}\overline{\widetilde{\mathcal{M}}})]$. If we take $\ell_1 \in \mathbb{Z}_{N_1}$ and $\ell_2 \in \mathbb{Z}_{N_2}$ at equally spaced L_1 horizontal and L_2 vertical shifts, respectively, such that L_1 divides N_1 and L_2 divides N_2, then $u_{\boldsymbol{\ell}}$ corresponds to sub-sampled elements of $\mathcal{U}$ for shifts

$$\boldsymbol{\ell} \in \left\{0, \frac{N_1}{L_1}, \frac{2N_1}{L_1}, \dots, \frac{(L_1-1)N_1}{L_1}\right\} \times \left\{0, \frac{N_2}{L_2}, \frac{2N_2}{L_2}, \dots, \frac{(L_2-1)N_2}{L_2}\right\}.$$

Now, we take the 2D L-period DFT of 2-dimensional slices of $(F_N\mathcal{Y})^{T\sigma} \in \mathbb{C}^{L_1 \times L_2 \times N_1 \times N_2}$ for fixed modes of $\mathbf{k} \in \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$. Then for $\boldsymbol{\omega} \in \mathbb{Z}_{L_1} \times \mathbb{Z}_{L_2}$, we have by definition 3.3.3,

$$\left(F_L(F_N\mathcal{Y})^{T\sigma}\right)_{[\boldsymbol{\omega}\ \mathbf{k}]} = (F_L(Z_{NL^{-1}}\mathcal{U}))_{\boldsymbol{\omega}}.$$

Thus, by Lemma 3.3.4, with NL^{-1} replacing L,

$$\begin{aligned}
\left(F_L(F_N\mathcal{Y})^{T\sigma}\right)_{[\boldsymbol{\omega}\ \mathbf{k}]} &= \frac{1}{\det(NL^{-1})} \sum_{\mathbf{p} \in R_{NL^{-1}}} \widehat{u}_{\boldsymbol{\omega} - N(NL^{-1})^{-1}\mathbf{p}} \\
&= \frac{1}{\det(NL^{-1})} \sum_{\mathbf{p} \in R_{NL^{-1}}} \widehat{u}_{\boldsymbol{\omega} - NLN^{-1}\mathbf{p}} \\
&= \frac{1}{\det(NL^{-1})} \sum_{\mathbf{p} \in R_{NL^{-1}}} \widehat{u}_{\boldsymbol{\omega} - L\mathbf{p}} \qquad \text{(diagonal matrices commute)} \\
&= \frac{\det L}{\det N} \sum_{\mathbf{p} \in R_{NL^{-1}}} \widehat{u}_{\boldsymbol{\omega} - L\mathbf{p}}.
\end{aligned}$$

Applying Lemma 3.3.2,

$$\begin{aligned}
\left(F_L(F_N\mathcal{Y})^{T\sigma}\right)_{[\boldsymbol{\omega}\ \mathbf{k}]} &= \frac{\det L}{\det N} \sum_{\mathbf{p} \in R_N} (\det N)\ [F_N(\mathcal{X} \circ S_{-\mathbf{k}}\overline{\mathcal{X}})]_{\boldsymbol{\omega} - L\mathbf{p}} [F_N(\widetilde{\mathcal{M}} \circ S_{\mathbf{k}}\overline{\widetilde{\mathcal{M}}})]_{\boldsymbol{\omega} - L\mathbf{p}} \\
&= (\det L) \sum_{\mathbf{p} \in R_N} [F_N(\mathcal{X} \circ S_{-\mathbf{k}}\overline{\mathcal{X}})]_{\boldsymbol{\omega} - L\mathbf{p}} [F_N(\widetilde{\mathcal{M}} \circ S_{\mathbf{k}}\overline{\widetilde{\mathcal{M}}})]_{\boldsymbol{\omega} - L\mathbf{p}}.
\end{aligned}$$

Now, we utilize that the time reversal of the time reversal of a signal is the original signal and negate the indexing in the second term of the summation, *i.e.* $-(\boldsymbol{\omega} - L\mathbf{p}) = L\mathbf{p} - \boldsymbol{\omega}$.

$$(F_L(F_N\mathcal{Y})^{T\sigma})_{[\boldsymbol{\omega}\ \mathbf{k}]} = (\det L) \sum_{\mathbf{p} \in R_N} [F_N(\mathcal{X} \circ S_{-\mathbf{k}}\overline{\mathcal{X}})]_{\boldsymbol{\omega} - L\mathbf{p}} \widetilde{[F_N(\widetilde{\mathcal{M}} \circ S_{\mathbf{k}}\overline{\widetilde{\mathcal{M}}})]_{L\mathbf{p} - \boldsymbol{\omega}}}.$$

Then, by Properties 2(iii.) and 2(iv.),

$$\begin{aligned}(F_L(F_N\mathcal{Y})^{T\sigma})_{[\boldsymbol{\omega}\ \mathbf{k}]} &= (\det L)\sum_{\mathbf{p}\in R_N}[F_N(\mathcal{X}\circ S_{-\mathbf{k}}\overline{\mathcal{X}})]_{\boldsymbol{\omega}-L\mathbf{p}}[F_N(\widetilde{\widetilde{\mathcal{M}}}\circ \widetilde{S_{\mathbf{k}}\overline{\widetilde{\widetilde{\mathcal{M}}}}})]_{L\mathbf{p}-\boldsymbol{\omega}}\\ &= (\det L)\sum_{\mathbf{p}\in R_N}[F_N(\mathcal{X}\circ S_{-\mathbf{k}}\overline{\mathcal{X}})]_{\boldsymbol{\omega}-L\mathbf{p}}[F_N(\mathcal{M}\circ S_{-\mathbf{k}}\overline{\mathcal{M}})]_{L\mathbf{p}-\boldsymbol{\omega}}.\end{aligned}$$

Therefore, by Lemma 3.3.1, we have

$$\begin{aligned}\left(F_L(F_N\mathcal{Y})^{T\sigma}\right)_{[\boldsymbol{\omega}\ \mathbf{k}]} = (\det L)\sum_{\mathbf{p}\in R_N}&\frac{1}{\det N}e^{-2\pi i(\boldsymbol{\omega}-L\mathbf{p})^T N^{-1}(-\mathbf{k})}\left[F_N\left(\widehat{\mathcal{X}}\circ S_{\boldsymbol{\omega}-L\mathbf{p}}\overline{\widehat{\mathcal{X}}}\right)\right]_{\mathbf{k}}\cdot\\ &\frac{1}{\det N}e^{-2\pi i(L\mathbf{p}-\boldsymbol{\omega})^T N^{-1}(-\mathbf{k})}\left[F_N\left(\widehat{\mathcal{M}}\circ S_{L\mathbf{p}-\boldsymbol{\omega}}\overline{\widehat{\mathcal{M}}}\right)\right]_{\mathbf{k}}\\ = \frac{\det L}{(\det N)^2}\sum_{\mathbf{p}\in R_N}&\left(F_N\left(\widehat{\mathcal{X}}\circ S_{\boldsymbol{\omega}-L\mathbf{p}}\overline{\widehat{\mathcal{X}}}\right)\right)_{\mathbf{k}}\left(F_N\left(\widehat{\mathcal{M}}\circ S_{L\mathbf{p}-\boldsymbol{\omega}}\overline{\widehat{\mathcal{M}}}\right)\right)_{\mathbf{k}}\end{aligned}$$

as desired. To show the second part of Theorem 2, we want to find conditions where

$$\operatorname{supp}\left(\widehat{\mathcal{M}}\right)\cap\operatorname{supp}\left(S_{L\mathbf{p}-\boldsymbol{\omega}}\overline{\widehat{\mathcal{M}}}\right)\neq\emptyset.$$

Recall definition 2.2.5 and note in 2 dimensions we can similarly define the bandlimited parameter $\delta_1\times\delta_2$ of $\widehat{\mathcal{M}}$ as having

$$\operatorname{supp}\left(\widehat{\mathcal{M}}\right)=\{\mathbf{n}\in\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2}:0\le n_1\le\delta_1-1,0\le n_2\le\delta_2-1\}.$$

Thus, we must consider both the horizontal and vertical shifts of $\overline{\widehat{\mathcal{M}}}$ to verify quantities where the intersection is nonempty. Since

$$L\mathbf{p}-\boldsymbol{\omega}=\begin{bmatrix}L_1 & 0\\ 0 & L_2\end{bmatrix}\begin{bmatrix}p_1\\ p_2\end{bmatrix}-\begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix}=\begin{bmatrix}p_1L_1-\omega_1\\ p_2L_2-\omega_2\end{bmatrix},$$

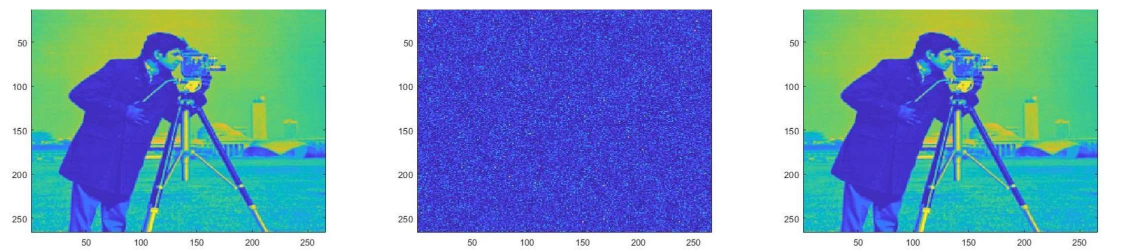
then the set $\operatorname{supp}(\widehat{\mathcal{M}})\cap\operatorname{supp}(S_{L\mathbf{p}-\boldsymbol{\omega}}\overline{\widehat{\mathcal{M}}})$ is nonempty if and only if

$$|p_1L_1-\omega_1|\le\delta_1-1\quad\text{ and }\quad|p_2L_2-\omega_2|\le\delta_2-1.$$

So, by Corollary 1, $p_1=0$ for $\omega_1\in\mathbb{Z}_{\delta_1}$ and $p_1=1$ for $\omega_1\in\mathbb{Z}_{L_1}\backslash\mathbb{Z}_{\delta_1}$; and also $p_2=0$ for $\omega_2\in\mathbb{Z}_{\delta_2}$ and $p_2=1$ for $\omega_2\in\mathbb{Z}_{L_2}\backslash\mathbb{Z}_{\delta_2}$. Thus, the 2 bounds imply the following values for vector $\mathbf{p}$:

$$\mathbf{p}=\begin{cases}[0\ \ 0]^T, & \text{if }\boldsymbol{\omega}\in\mathbb{Z}_{\delta_1}\times\mathbb{Z}_{\delta_2}\\ [0\ \ 1]^T, & \text{if }\boldsymbol{\omega}\in\mathbb{Z}_{\delta_1}\times\mathbb{Z}_{L_2}\backslash\mathbb{Z}_{\delta_2}\\ [1\ \ 0]^T, & \text{if }\boldsymbol{\omega}\in\mathbb{Z}_{L_1}\backslash\mathbb{Z}_{\delta_1}\times\mathbb{Z}_{\delta_2}\\ [1\ \ 1]^T, & \text{if }\boldsymbol{\omega}\in\mathbb{Z}_{L_1}\backslash\mathbb{Z}_{\delta_1}\times\mathbb{Z}_{L_2}\backslash\mathbb{Z}_{\delta_2}.\end{cases}$$

Therefore, we can collapse the summation in Theorem 2 to the four exact terms as desired. □



(a) True Signal (b) Diffraction pattern of noiseless measurements (c) Image recovered by Algorithm 2

Figure 6: Cameraman image as unknown signal $\mathbf{x}$

4 Empirical Results for 2D

4.1 Phase Synchronization in 2D and Algorithm 2

Empirical results after implementing MATLAB code for 2-dimensional phase retrieval seem to confirm the results of Theorem 2. Indeed, without added noise, randomly generated signals could be recovered exactly up to computer error. Figure 6 shows the "Cameraman" image encoded in RGB pixels, which are all positive integer values. The picture has size $N_1 = N_2 = 256$ and the code used $L_1 = L_2 = 19$ number of shifts. Part (b.) shows the diffraction pattern from Fourier measurements of the image and (c.) is the image recovered from (b.) by Algorithm 2.

By Theorem 2 and taking an IDFT, we are able to extract from the 4$^{\text{th}}$-order tensor $\mathcal{Y} \in \mathbb{R}^{N_1 \times N_2 \times L_1 \times L_2}$ the altered quantities of the unknown signal $\mathcal{X} \in \mathbb{C}^{N_1 \times N_2}$ as

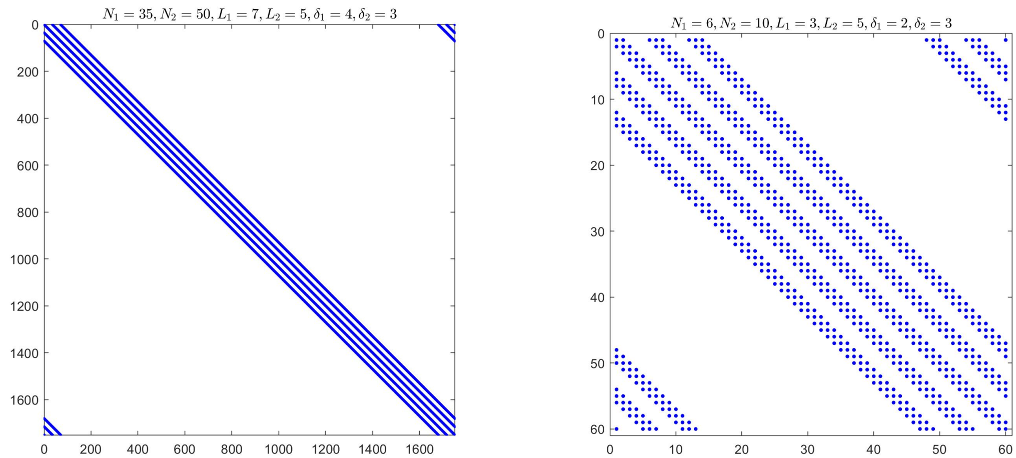
$$\widehat{\mathcal{X}} \circ S_{\boldsymbol{\omega}} \overline{\widehat{\mathcal{X}}} = \frac{(\det N)^2}{\det L} F_N^{-1} \left[\frac{(F_L(F_N \mathcal{Y})^{T\sigma})_{[\boldsymbol{\omega}\ :]}}{F_N\left(\widehat{\mathcal{M}} \circ S_{-\boldsymbol{\omega}} \overline{\widehat{\mathcal{M}}}\right)} \right], \qquad \forall \boldsymbol{\omega} \in \mathbb{Z}_{\delta_1} \times \mathbb{Z}_{\delta_2}$$

$$\widehat{\mathcal{X}} \circ S_{\boldsymbol{\omega} - [0\ L_2]^T} \overline{\widehat{\mathcal{X}}} = \frac{(\det N)^2}{\det L} F_N^{-1} \left[\frac{(F_L(F_N \mathcal{Y})^{T\sigma})_{[\boldsymbol{\omega}\ :]}}{F_N\left(\widehat{\mathcal{M}} \circ S_{[0\ L_2]^T - \boldsymbol{\omega}} \overline{\widehat{\mathcal{M}}}\right)} \right], \qquad \forall \boldsymbol{\omega} \in \mathbb{Z}_{\delta_1} \times \mathbb{Z}_{L_2} \backslash \mathbb{Z}_{\delta_2}$$

$$\widehat{\mathcal{X}} \circ S_{\boldsymbol{\omega} - [L_1\ 0]^T} \overline{\widehat{\mathcal{X}}} = \frac{(\det N)^2}{\det L} F_N^{-1} \left[\frac{(F_L(F_N \mathcal{Y})^{T\sigma})_{[\boldsymbol{\omega}\ :]}}{F_N\left(\widehat{\mathcal{M}} \circ S_{[L_1\ 0]^T - \boldsymbol{\omega}} \overline{\widehat{\mathcal{M}}}\right)} \right], \qquad \forall \boldsymbol{\omega} \in \mathbb{Z}_{L_1} \backslash \mathbb{Z}_{\delta_1} \times \mathbb{Z}_{\delta_2}$$

$$\widehat{\mathcal{X}} \circ S_{\boldsymbol{\omega} - [L_1\ L_2]^T} \overline{\widehat{\mathcal{X}}} = \frac{(\det N)^2}{\det L} F_N^{-1} \left[\frac{(F_L(F_N \mathcal{Y})^{T\sigma})_{[\boldsymbol{\omega}\ :]}}{F_N\left(\widehat{\mathcal{M}} \circ S_{[L_1\ L_2]^T - \boldsymbol{\omega}} \overline{\widehat{\mathcal{M}}}\right)} \right], \qquad \forall \boldsymbol{\omega} \in \mathbb{Z}_{L_1} \backslash \mathbb{Z}_{\delta_1} \times \mathbb{Z}_{L_2} \backslash \mathbb{Z}_{\delta_2}$$

where division is component-wise and $(F_L(F_N\mathcal{Y})^{T\sigma})_{[\boldsymbol{\omega}\ :]} \in \mathbb{C}^{N_1 \times N_2}$ denotes the matrix for fixed $\omega_1 \in \mathbb{Z}_{L_1}, \omega_2 \in \mathbb{Z}_{L_2}$. So, to implement the phase synchronization portion of the algorithm, we vectorize these recovered quantities and construct the matrix with the circular banded structure as in the 1-dimensional case; mathematically, that is, we use a vector transformation, $\text{vec}\colon \mathbb{C}^{N_1 \times N_2} \to \mathbb{C}^{N_1 N_2}$, and lay the recovered quantities $\text{vec}(\widehat{\mathcal{X}} \circ S_{\boldsymbol{\omega} - L\mathbf{p}} \overline{\widehat{\mathcal{X}}})$ along the diagonals of an $N_1N_2 \times N_1N_2$ matrix. Because the shifts have 2 degrees of freedom for movement, the support conditions of the mask, $\text{supp}(\widehat{\mathcal{M}}) = \{\mathbf{n} \in \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} : 0 \leq n_1 \leq \delta_1 - 1, 0 \leq n_2 \leq \delta_2 - 1\}$ generate a slightly different sparsity structure for the banded matrix. Thus, we define $T_{\delta_1 \times \delta_2}\colon \mathbb{C}^{N_1N_2 \times N_1N_2} \to \mathbb{C}^{N_1N_2 \times N_1N_2}$ as the banded matrix operator in 2D with respect to bandlimited parameters $\delta_1 \in \mathbb{Z}_{L_1}$ and $\delta_2 \in \mathbb{Z}_{L_2}$.



(a) 35 shifts of size $N_1 N_2 = 1750$ signal

(b) 15 shifts of size $N_1 N_2 = 60$ signal

Figure 7: Sparsity structure for banded matrices $T_{\delta_1 \times \delta_2}$

Generated in MATLAB, Figure 7 demonstrates the sparsity structure of $T_{\delta_1 \times \delta_2}(\text{vec}(\widehat{\mathcal{X}})\text{vec}(\widehat{\mathcal{X}})^*)$. The blue diagonals are nonzero entries whereas white space signifies zero entries of the matrix. With smaller signal size $N_1 \times N_2 = 6 \times 10$, Figure 7(b.) shows more clearly the structure where the individual nonzero entries are blue dots. As can be seen, there are additional bands of zeros between the quantities placed near the main diagonal unlike in the 1-dimensional case with $T_\delta(\widehat{\mathbf{x}}\widehat{\mathbf{x}}^*)$. The matrix $T_{\delta_1 \times \delta_2}(\text{vec}(\widehat{\mathcal{X}})\text{vec}(\widehat{\mathcal{X}})^*)$ still has a similar "circulant deconstruction." For example, a 10×10 0-1 matrix such as

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \end{bmatrix}$$

is also a circulant matrix with leading eigenvector $[1 \ 1 \ 1 \ \cdots \ 1]^T \in \mathbb{R}^{N_1 N_2}$, and thus, has the desired spectral properties for Algorithm 2 to recover the phase of $\mathcal{X}$. The 2-dimensional phase retrieval method, Algorithm 2, is presented in Appendices C and D on pages 38 and 39, respectively.

4.2 2D Numerical Testing

The same numerical tests were implemented to judge the accuracy and efficiency of phase recovery for 2 dimensions. To test robustness, the 2D code was looped 30 times to compute averaged relative error (in decibels) with the formula

$$\text{rel err} = 10\log_{10}\left(\frac{\|\mathcal{X}-\mathcal{X}_{\text{rec}}\|_F^2}{\|\mathcal{X}\|_F^2}\right),$$

where $\mathcal{X} \in \mathbb{C}^{N_1 \times N_2}$ is the true signal, $\mathcal{X}_{\text{rec}}$ the recovered signal, and $\|\cdot\|_F$ the Frobenius norm. Figure 8 demonstrates the same pattern as in Figure 4 for 1 dimension, a reduction in error when the support parameters δ_1, δ_2 of the mask $\mathcal{M}$ increase. Because of the similar mathematical behavior, consequently we conjecture the possibility of estimating bounds on the robustness to noise as done in [8] in 1 dimension.

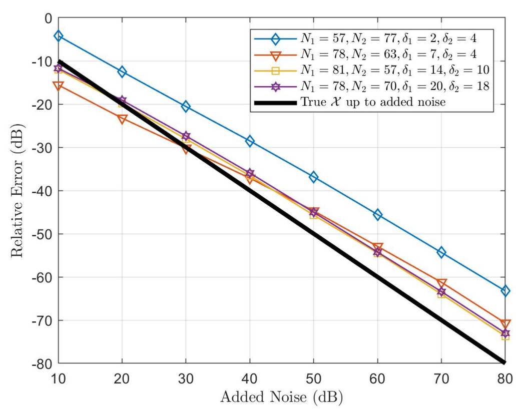


Figure 8: 2D Relative Error vs. Added Noise

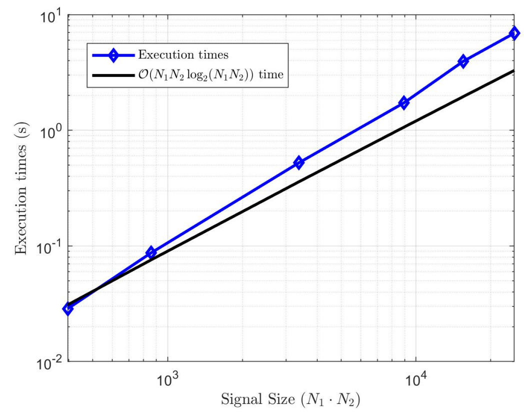


Figure 9: Log-Log of Execution Time vs. 2D Signal Size

Figure 9 compares execution time to recover the signal $\mathcal{X}$ from noisy (SNR = 30) measurements $\mathcal{Y}_{j_1,j_2,\ell_1,\ell_2}$ with FFT, or $\mathcal{O}(N_1 N_2 \log_2(N_1 N_2))$, time. The code was looped 30 times for different size $N_1 \times N_2$ signals $\mathcal{X}$, and the execution times were averaged. The 2D code seems to perform slower than FFT time though not significantly so. In fact, the code recovered an $N_1 N_2 = 109 \cdot 119 = 24871$ size signal from 8,033,333 noisy measurements, *i.e.* entries within $\mathcal{Y} \in \mathbb{R}^{N_1 \times N_2 \times L_1 \times L_2}$, at an average of about 6.9 seconds. Notably, solving for the eigenvectors of the banded matrix $T_{\delta_1 \times \delta_2}(\text{vec}(\widehat{\mathcal{X}})\text{vec}(\widehat{\mathcal{X}})^*)$ constitutes the heaviest computational cost according to MATLAB's run time Profiler, which logs the execution time of functions within the code. The Profiler indicates that at least 90% of the execution time is dedicated to this portion of the algorithm.

5 Concluding Remarks

The end goal of this REU project was to contribute real-world results that will improve upon existing practices of the phase retrieval problem. We developed working software code in MATLAB that can recover phase information for 2-dimensional signals efficiently. And in particular, the algorithm presented in this paper is firmly grounded in the properties of Fourier and spectral analysis and avoids costly iterative methods such as those used by alternating projections and PhaseLift.

Nevertheless, there are some questions that still need attention and further research. Top priority should be given to analyzing rigorously the robustness of Algorithm 2 in the presence of measurement error. The theoretical foundation laid by Theorem 2 greatly facilitates the task of developing similar bounds and inequalities, as in [8] for the 1-dimensional case, between true and recovered signals. Secondly, more investigation of the circular banded structure for the matrix $T_{\delta_1 \times \delta_2}(\text{vec}(\widehat{\mathcal{X}})\text{vec}(\widehat{\mathcal{X}})^*)$

is needed to specify precisely how angular synchronization functions in 2 dimensions so that we may understand better the desired spectral properties the algorithm hinges upon. This could perhaps be crucial in making the 2D MATLAB code more efficient given that solving for the eigenvectors of the banded matrix had the highest numerical cost. Lastly, we would like to extend framework for Algorithm 2 to more generalized masked setups and hence to broader phase retrieval applications. Although we gave particular attention to the ptychographic setting, the mathematical methods presented here potentially have a wide-range of usage for other disciplines, including audio speech processing, optical imaging, astronomy, quantum mechanics, and more.

6 Acknowledgements

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Appendices

A Proofs of Selected 1D and 2D Properties

Property 1(iv.). *Time Reversal of the Shift:*

$$(\widetilde{S_\ell \mathbf{x}})_n = (S_{-\ell}\widetilde{\mathbf{x}})_n.$$

Proof. Starting with the definition of time reversal before applying the shift, we have

$$(\widetilde{S_\ell \mathbf{x}})_n = (S_\ell \mathbf{x})_{-n} = x_{-n-\ell} = x_{-(n+\ell)} = \widetilde{x}_{n+\ell} = (S_{-\ell}\widetilde{\mathbf{x}})_n.$$

$\square$

Property 1(viii.). *The DFT of the DFT:*

$$(F_N(F_N\mathbf{x}))_j = N\widetilde{x}_j.$$

Proof.

$$\begin{aligned}
(F_N(F_N\mathbf{x}))_j &= \left(F_N\left(\sum_{n=0}^{N-1} x_n e^{\frac{-2\pi i n k}{N}}\right)\right)_j \\
&= \sum_{k=0}^{N-1}\sum_{n=0}^{N-1} x_n e^{\frac{-2\pi i n k}{N}} e^{\frac{-2\pi i k j}{N}} \\
&= \sum_{n=0}^{N-1} x_n \sum_{k=0}^{N-1} e^{\frac{-2\pi i k(n+j)}{N}} \\
&= \sum_{n=0}^{N-1} x_n \sum_{k=0}^{N-1} \left(e^{\frac{-2\pi i (n+j)}{N}}\right)^k.
\end{aligned}$$

For the second summation term, we can use the formula for the sum of a finite geometric series. When $n + j \neq 0$,

$$\sum_{k=0}^{N-1}\left(e^{\frac{-2\pi i(n+j)}{N}}\right)^k = \frac{1-e^{-2\pi i(n+j)N/N}}{1-e^{-2\pi i(n+j)/N}} = \frac{1-e^{-2\pi i(n+j)}}{1-e^{-2\pi i(n+j)/N}} = 0.$$

The last equality follows from noting that $e^{-2\pi i n} = 1$ for any integer $n \in \mathbb{Z}$ and the denominator $e^{-2\pi i(n+j)/N} \neq 0$ for all $n, j \in \mathbb{Z}_N$. When $n + j = 0$, or equivalently $n = -j$,

$$\sum_{k=0}^{N-1}\left(e^{\frac{-2\pi i(n+j)}{N}}\right)^k = \sum_{k=0}^{N-1} 1 = N.$$

Thus, using Kronecker delta notation, we can now write

$$(F_N(F_N\mathbf{x}))_j = \sum_{n=0}^{N-1} x_n \; N\delta_{n+j}.$$

The Kronecker delta δ_{n+j} here is defined as

$$\delta_{n+j} = \begin{cases} 1, & n = -j \\ 0, & \text{otherwise} \end{cases}.$$

Therefore, we have

$$(F_N(F_N\mathbf{x}))_j = Nx_{-j} = N\widetilde{x}_j.$$

$\square$

Property 2(iv.). *2D Time Reversal of the Shift:*

$$(\widetilde{S_{\boldsymbol{\ell}}\mathcal{X}})_{\mathbf{n}} = (S_{-\boldsymbol{\ell}}\widetilde{\mathcal{X}})_{\mathbf{n}}.$$

Proof. Starting with the definition of time reversal before applying the 2-dimensional shift, we have

$$(\widetilde{S_{\boldsymbol{\ell}}\mathcal{X}})_{\mathbf{n}} = (S_{\boldsymbol{\ell}}\mathcal{X})_{-\mathbf{n}} = x_{-\mathbf{n}-\boldsymbol{\ell}} = x_{-(\mathbf{n}+\boldsymbol{\ell})} = \widetilde{x}_{\mathbf{n}+\boldsymbol{\ell}} = (S_{-\boldsymbol{\ell}}\widetilde{\mathcal{X}})_{\mathbf{n}}.$$

$\square$

Property 2(viii.). *The DFT of the DFT in 2D:*

$$(F_N(F_N\mathcal{X}))_{\mathbf{j}} = (\det N)\; \widetilde{x}_{\mathbf{j}}.$$

Proof.

$$\begin{aligned}
(F_N(F_N\mathcal{X}))_{\mathbf{j}} &= \left(F_N\left(\sum_{\mathbf{n}\in R_N} x_{\mathbf{n}} e^{-2\pi i\mathbf{n}^T N^{-1}\mathbf{k}}\right)\right)_{\mathbf{j}} \\
&= \sum_{\mathbf{k}\in R_N}\left(\sum_{\mathbf{n}\in R_N} x_{\mathbf{n}} e^{-2\pi i\mathbf{n}^T N^{-1}\mathbf{k}}\right) e^{-2\pi i\mathbf{k}^T N^{-1}\mathbf{j}} \\
&= \sum_{\mathbf{n}\in R_N} x_{\mathbf{n}} \sum_{\mathbf{k}\in R_N} e^{-2\pi i\mathbf{n}^T N^{-1}\mathbf{k}} e^{-2\pi i\mathbf{k}^T N^{-1}\mathbf{j}} \\
&= \sum_{\mathbf{n}\in R_N} x_{\mathbf{n}} \sum_{\mathbf{k}\in R_N} e^{-2\pi i\mathbf{k}^T N^{-1}\mathbf{n}} e^{-2\pi i\mathbf{k}^T N^{-1}\mathbf{j}} \\
&= \sum_{\mathbf{n}\in R_N} x_{\mathbf{n}} \sum_{\mathbf{k}\in R_N} e^{-2\pi i\mathbf{k}^T N^{-1}(\mathbf{n}+\mathbf{j})} \\
&= \sum_{\mathbf{n}\in R_N} x_{\mathbf{n}}\; (\det N)\,\delta_{\mathbf{n}+\mathbf{j}}.
\end{aligned}$$

The last equality follows from the second summation over R_N, equivalent to a double sum of 2 complex exponentials, yielding the product $\delta_{n_1+j_1}\delta_{n_2+j_2} = \delta_{\mathbf{n}+\mathbf{j}}$. The Kronecker delta $\delta_{\mathbf{n}+\mathbf{j}}$ is defined here as

$$\delta_{\mathbf{n}+\mathbf{j}} = \begin{cases} 1, & \text{if } \mathbf{n} = -\mathbf{j} \\ 0, & \text{otherwise.} \end{cases}$$

Therefore, we have

$$\begin{aligned} (F_N(F_N \mathcal{X}))_{\mathbf{j}} &= (\det N)\ x_{-\mathbf{j}} \\ &= (\det N)\ \widetilde{x}_{\mathbf{j}}. \end{aligned}$$

□

B Algorithm 1 Pseudocode

Algorithm 1: Signal Recovery from Phaseless Measurements for Bandlimited Masks

Inputs

1. Noisy measurements matrix $Y \in \mathbb{R}^{N \times L}$ with entries

$$Y_{j,\ell} = \left| \sum_{n=0}^{N-1} x_n m_{n-\ell} e^{-\frac{2\pi i n j}{N}} \right|^2 + \eta_{j,\ell}, \quad \text{for } j \in \mathbb{Z}_N,\ \ell \in \left\{0, \frac{N}{L}, \frac{2N}{L}, \dots, \frac{(L-1)N}{L}\right\}.$$

2. Bandlimited Mask $\mathbf{m} \in \mathbb{C}^N$ with $\operatorname{supp}(\widehat{\mathbf{m}}) = \{0, 1, \dots, \delta - 1\}$, where $L = 2\delta - 1$.

Steps

1. Estimate $\left(F_N\left(\widehat{\mathbf{x}} \circ S_\omega \overline{\widehat{\mathbf{x}}}\right)\right)_k$ for $k \in \mathbb{Z}_N$ from Corollary 1 result (2.4).
2. Invert the Fourier transforms above to recover estimates of the $L = 2\delta - 1$ vectors $\widehat{\mathbf{x}} \circ S_{\omega - pL} \overline{\widehat{\mathbf{x}}}$.
3. Form the banded matrix $T_\delta \mathbf{X}$ from estimates in Step 2 where

$$(T_\delta \mathbf{X})_{i,j} := \begin{cases} \left(\widehat{\mathbf{x}} \circ S_j \overline{\widehat{\mathbf{x}}}\right)_i, & \text{if } |i - j| \bmod N < \delta \\ 0, & \text{otherwise} \end{cases}.$$

4. Hermitianize the matrix above: $T_\delta \mathbf{X} \longleftarrow \frac{1}{2}\left(T_\delta \mathbf{X} + (T_\delta \mathbf{X})^*\right)$.
5. Estimate $|\widehat{\mathbf{x}}|$ from the diagonal of $T_\delta \mathbf{X}$.
6. Normalize $T_\delta \mathbf{X}$ component-wise to form relative phase matrix $T_\delta \mathring{\mathbf{X}}$.
7. Compute the leading normalized eigenvector of $T_\delta \mathring{\mathbf{X}}$, $\mathbf{u}$.

Output

An estimate of $\mathbf{x}$, $\mathbf{x}_{\text{rec}} := F_N^{-1} \widehat{\mathbf{x}}_{\text{rec}}$, where $\widehat{\mathbf{x}}_{\text{rec}}$ is given component-wise by

$$\left(\widehat{\mathbf{x}}_{\text{rec}}\right)_j := \sqrt{(T_\delta \mathbf{X})_{j,j}}\ u_j.$$

C Algorithm 2 Pseudocode

Algorithm 2:
2D Signal Recovery from Phaseless Measurements for Bandlimited Masks

Inputs

1. $N_1 \cdot N_2 \cdot L_1 \cdot L_2$ noisy measurements 4-tensor $\mathcal{Y} \in \mathbb{R}^{N_1 \times N_2 \times L_1 \times L_2}$ with entries

$$\mathcal{Y}_{j_1,j_2,\ell_1,\ell_2} = \left| \sum_{\mathbf{n} \in R_N} x_{\mathbf{n}} m_{\mathbf{n}-\boldsymbol{\ell}} e^{-2\pi i \mathbf{n}^T N^{-1} \mathbf{j}} \right|^2 + \eta_{j_1,j_2,\ell_1,\ell_2},$$

for $j_1 \in \mathbb{Z}_{N_1}$, $j_2 \in \mathbb{Z}_{N_2}$ and $\ell_1 \in \left\{0, \frac{N_1}{L_1}, \frac{2N_1}{L_1}, \dots, \frac{(L_1-1)N_1}{L_1}\right\}$, $\ell_2 \in \left\{0, \frac{N_2}{L_2}, \frac{2N_2}{L_2}, \dots, \frac{(L_2-1)N_2}{L_2}\right\}$.

2. Bandlimited Mask $\mathcal{M} \in \mathbb{C}^{N_1 \times N_2}$ with $\text{supp}(\widehat{\mathcal{M}}) = \{\mathbf{n} \in \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} : 0 \leq n_1 \leq \delta_1 - 1, 0 \leq n_2 \leq \delta_2 - 1\}$ and horizontal and vertical shifts $L_1 = 2\delta_1 - 1$ and $L_2 = 2\delta_2 - 1$, respectively.

Steps

1. Estimate $\left(F_N \left(\widehat{\mathcal{X}} \circ S_{\boldsymbol{\omega}} \overline{\widehat{\mathcal{X}}}\right)\right)_{\mathbf{k}}$ for $\mathbf{k} \in \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2}$ from Theorem 2 result (3.3).
2. Invert the Fourier transforms above to recover estimates of the $L_1 \cdot L_2$ matrices $\widehat{\mathcal{X}} \circ S_{\boldsymbol{\omega}-L\mathbf{p}} \overline{\widehat{\mathcal{X}}}$.
3. Vectorize the recovered matrices to $N_1 \cdot N_2$ length vectors $\text{vec}(\widehat{\mathcal{X}} \circ S_{\boldsymbol{\omega}-L\mathbf{p}} \overline{\widehat{\mathcal{X}}})$.
4. Form the banded matrix $T_{\delta_1 \times \delta_2}\mathbf{X}$ from the vectorized estimates in Step 3
5. Hermitianize the matrix above: $T_{\delta_1 \times \delta_2}\mathbf{X} \longleftarrow \frac{1}{2}\left(T_{\delta_1 \times \delta_2}\mathbf{X} + (T_{\delta_1 \times \delta_2}\mathbf{X})^*\right)$.
6. Estimate $\left|\text{vec}(\widehat{\mathcal{X}})\right|$ from the diagonal of $T_{\delta_1 \times \delta_2}\mathbf{X}$.
7. Normalize $T_{\delta_1 \times \delta_2}\mathbf{X}$ component-wise to form relative phase matrix $T_{\delta_1 \times \delta_2}\mathring{\mathbf{X}}$.
8. Compute the leading normalized eigenvector of $T_{\delta_1 \times \delta_2}\mathring{\mathbf{X}}$, $\mathbf{u}$.

Output
An estimate of $\mathcal{X}$ (converted to matrix form from a vector), $\mathcal{X}_{\text{rec}} := F_N^{-1} \widehat{\mathcal{X}}_{\text{rec}}$, where $\widehat{\mathcal{X}}_{\text{rec}}$ derives from the vectorized quantities

$$\left(\text{vec}\left(\widehat{\mathcal{X}}_{\text{rec}}\right)\right)_j := \sqrt{(T_{\delta_1 \times \delta_2}\mathbf{X})_{j,j}}\; u_j.$$

D Algorithm 2 MATLAB Code

```
%% 2D (Ptychographic) Phase Retrieval for Bandlimited Masks
%
% MATLAB Script to implement phase retrieval for bandlimited masks using
% discrete Fourier analysis and angular synchronization. An adaptation of the
% 1D phase retrieval code presented in
%
% 'Inverting spectrogram measurements via aliased Wigner distribution
% deconvolution and angular synchronization' by Michael Perlmutter, Sami
% Merhi, Aditya Viswanathan, and Mark Iwen. arXiv pre-print (Jul 2019),
% https://arxiv.org/abs/1907.10773.
%
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% University of Michigan-Dearborn and the CHARMS Research Group)
%
% Cyril Cordor, Brendan Williams, Aditya Viswanathan, and Yulia Hristova, 2019
%

clear; close all; clc

% For repeatability, set random seed to ...
rng(1234);


%% Signal Parameters

% Signal dimension
N1 = 2^8;                       % Choose signal horizontal size
N2 = 2^7;                       % Choose signal vertical size

% Mask parameters: choose random or exponential deterministic
maskType = 'random';            % random mask
%maskType = 'exp';              % (Non-symmetric) exponential mask

% Support of m^hat, bandlimited parameter
delta1 = ceil(1.25*log2(N1));
delta2 = ceil(1.25*log2(N2));

% Sub-sampling in space - no. of shifts
L1 = 2*delta1 - 1;              % No. of horizontal shifts
L2 = 2*delta2 - 1;              % No. of vertical shifts

% Noise parameters
addnoise = false;               % Add noise?
snr = 30;                          % SNR of noise to be added


% Fix N1, N2 to satisfy divisibility requirements
N1 = L1 * ceil(N1/L1);
N2 = L2 * ceil(N2/L2);


% Print out problem parameters
fprintf( '\n\n ------------------------------------------------------------ \n' );
fprintf(     '     2D Phase Retrieval from Masked Fourier Measurements  \n' );
fprintf(     ' ------------------------------------------------------------ \n' );

fprintf( ' Problem size, N1 = %d , N2 = %d , \n', N1, N2 );
fprintf( ' Bandwidth of mask (horiz./vert. card. of support of m^hat),');
fprintf( ' delta1 = %d, delta2 = %d \n', delta1, delta2 );

fprintf( ' No. of shifts (in space), L1 = %d , L2 = %d \n', L1, L2 );
fprintf( ' Total no. of measurements, N1xN2xL1xL2 = %d\n\n', N1*N2*L1*L2 );
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switch lower(maskType)
    case 'random'
        fprintf( ' Using random masks \n\n' );
    case 'exp'
        fprintf( ' Using (non-symmetric) exponential masks from BlockPR method \n\n' );
end

if( addnoise )
    fprintf( ' Noisy simulation? - Yes \n' );
    fprintf( ' Added noise (SNR, dB) = %3.2f \n\n', snr );
else
    fprintf( ' Noisy simulation? - No \n\n' );
end


%% Choice of mask m

% define m^hat
m_hat = zeros(N1,N2);

% Generate mask depending on type
switch lower(maskType)
    case 'random'
        % Random Gaussian mask
        m_hat(1:delta1, 1:delta2) = randn(delta1,delta2) .* ...
                    exp(1i*2*pi*rand(delta1,delta2));

    case 'exp'
        % Exponential mask (deterministic, real parameter)
        a = max(4, (delta1-1)/2);
        m1_hat(1:delta1,1) = exp(-(0:delta1-1).'/a) / ((2*delta1-1)^.25);
        b = max(4, (delta2-1)/2);
        m2_hat(1:delta2,1) = exp(-(0:delta2-1).'/a) / ((2*delta2-1)^.25);

        m_hat(1:delta1,1:delta2) = m1_hat*m2_hat';
end

% here is the mask in physical space (for reference)
m = ifft2(m_hat);


% Pre-computation (of terms involving masks)
mask_precomp_1 = zeros( N1,N2,delta1*delta2 );
mask_precomp_2 = zeros( N1,N2,delta1*(L2 - delta2) );
mask_precomp_3 = zeros( N1,N2,(L1 - delta1)*delta2 );
mask_precomp_4 = zeros( N1,N2,(L1 - delta1)*(L2 - delta2) );

for w1 = 0:delta1-1
    for w2 = 0:delta2 - 1
        mask_precomp_1(:,:,delta1 * w2 + w1+1) = fft2( ...
                    m_hat.*circshift(conj(m_hat),-[w1 w2]), N1, N2);
    end
end

for w1 = 0: delta1 -1
    for w2 = delta2: L2 - 1
        mask_precomp_2(:,:,delta1*(w2-delta2) + w1 + 1) = fft2( ...
                    m_hat.*circshift(conj(m_hat),-[w1 w2-L2]), N1, N2);
    end
end

for w1 = delta1:L1 - 1
    for w2 = 0:delta2 - 1
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            mask_precomp_3(:,:,(delta1-1)*w2 + (w1-delta1) + 1) = fft2( ...
                        m_hat.*circshift(conj(m_hat),-[w1-L1 w2]), N1, N2);
        end
end

for w1 = delta1: L1 - 1
        for w2 = delta2: L2 - 1
            mask_precomp_4(:,:,(delta1-1)*(w2-delta2) + (w1-delta1) + 1) = fft2( ...
                        m_hat.*circshift(conj(m_hat),-[w1-L1 w2-L2]), N1, N2);
        end
end

%% Choice of signal x
x = randn(N1,N2) + 1i*randn(N1,N2);      % random complex signal x


%% Measurements
Y = zeros(N1,N2,L1,L2);                  % 4-tensor of (spectrogram) measurements

for p1=0:N1/L1:N1-1
        for p2=0:N2/L2:N2-1
            Y(:,:,p1*(L1/N1) + 1,p2*(L2/N2) + 1) = abs( ...
                        fft2( x.*circshift(m,[p1 p2]),N1,N2) ).^2;
        end
end

% Adding noise
if( addnoise )
        signal_power = norm( Y(:) )^2 / ( N1*N2*L1*L2 );
        noise_power = signal_power / ( 10^(snr/10) );

        % Add (real) Gaussian noise of desired variance
        noise = sqrt( noise_power )*randn( size(Y) );
        Y = Y + noise;
else
        noise_power = 0;
end


%% Solve for diagonal of x^x^*
tic;      % Start execution time measurement

% First, compute the left-hand side of double-aliasing formulation
LHS = permute( fft2( permute( fft2(Y), [3, 4, 1, 2] ) ), [3, 4, 1, 2] );

% Initialize indexing arrays and counters
Indexmatcol = zeros(N1*N2,L1*L2);
Indexmatrow = zeros(N1*N2,L1*L2);
Tvals = zeros(N1*N2,L1*L2);
counter = 1;
indvals = reshape(1:N1*N2, N1, N2);

% Next, solve for diagonals
for w1 = 0:delta1-1
        for w2 = 0:delta2 - 1
            tmp = (N1*N2)^2 / (L1*L2)*ifft2( LHS(:,:,w1 + 1,w2 + 1) ./ ...
                        mask_precomp_1(:,:,delta1*w2 + w1 + 1) );
            Indexmatrow(:, counter) = indvals(:);
            shftd_idx = circshift(indvals, [w1 w2]);
            Indexmatcol(:, counter) = shftd_idx(:);
            Tvals(:, counter) = tmp(:);
            counter = counter + 1;
        end
end
```

```
for w1 = 0:delta1 - 1
    for w2 = delta2:L2 - 1
        tmp = (N1*N2)^2/(L1*L2)*ifft2( LHS(:,:,w1+1,w2+1) ./ ...
                    mask_precomp_2(:,:,delta1*(w2-delta2) + w1 + 1) );
        Indexmatrow(:, counter) = indvals(:);
        shftd_idx = circshift(indvals, [w1 w2-L2]);
        Indexmatcol(:, counter) = shftd_idx(:);
        Tvals(:, counter) = tmp(:);
        counter = counter + 1;
    end
end

for w1 = delta1:L1 - 1
    for w2 = 0:delta2 - 1
        tmp = (N1*N2)^2/(L1*L2)*ifft2( LHS(:,:,w1+1,w2+1) ./ ...
                    mask_precomp_3(:,:,(delta1-1)*w2 + (w1-delta1) + 1) );
        Indexmatrow(:, counter) = indvals(:);
        shftd_idx = circshift(indvals, [w1-L1 w2]);
        Indexmatcol(:, counter) = shftd_idx(:);
        Tvals(:, counter) = tmp(:);
        counter = counter + 1;
    end
end

for w1 = delta1:L1 - 1
    for w2 = delta2:L2 - 1
        tmp = (N1*N2)^2 / (L1*L2)*ifft2( LHS(:,:,w1+1,w2+1) ./ ...
                    mask_precomp_4(:,:,(delta1-1)*(w2-delta2) + (w1-delta1) + 1) );
        Indexmatrow(:, counter) = indvals(:);
        shftd_idx = circshift(indvals, [w1-L1 w2-L2]);
        Indexmatcol(:, counter) = shftd_idx(:);
        Tvals(:, counter) = tmp(:);
        counter = counter + 1;
    end
end


%% Angular synchronization

% Form T_(delta1 by delta2) matrix
T_del = sparse(Indexmatrow(:), Indexmatcol(:), Tvals(:), N1*N2, N1*N2);

% Hermitian symmetrize
T_del = T_del/2 + T_del'/2;

% View sparsity structure of matrix
spy(T_del)

% Magnitudes
mags = sqrt(abs(diag(T_del)));

% Compute eig. values
nz_idx = find(T_del);   % non zero locations

% Entry-wise normalization to get relative phases
T_del(nz_idx) = sign(T_del(nz_idx));

% Find leading e-vector
[xv, ~, ~] = eigs(T_del, 1, 'LM');
xv = sign(xv);

% Reconstruction
xrec = ifft2(reshape(full(mags.*xv), [N1 N2]));
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```
xrec = xrec(:);
vecx = x(:);            % true signal

% Correct for global phase factor
phaseOffset = angle( (xrec'*vecx) / (vecx'*vecx) );
xrec = xrec * exp(1i*phaseOffset);

% Record execution time for signal recovery
etime = toc;

fprintf( ' Execution time is %3.3e secs.\n', etime );

% Reshape to a matrix
recon = reshape(xrec, [N1, N2]);

% Reconstruction error in decibels (dB)
errordB = 10*log10( norm(x - recon, 'fro')^2 / norm(x, 'fro')^2 );

fprintf( '\n 2-Norm Error in reconstruction is %3.3e or %3.2f dB', ...
norm(recon - x, 'fro') / norm(x, 'fro'), errordB );

fprintf( '\n Inf. Norm Error in reconstruction is %3.3e\n\n', ...
norm(x(:) - xrec,inf) / norm(x(:),inf) );
```