INVARIANT DIFFERENTIAL OPERATORS ASSOCIATED WITH THE PICK'S MATRICES

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Abstract. In this notes, motivated by the fact the determinant of 2×2 Pick's matrices being greater or equal to zero implies the Schwarz-Pick lemma which carries interesting geometric interpretations, we investigate the geometric information encoded in Pick's matrices of all dimensions. We manage to define a geometric differential operator with respect to $n \times n$ Pick's matrices and find a connection between the n dimensional case and $n-1$ dimensional case by the classical yet mysterious trick of pulling out the z.

1. Introduction

In this section we investigate the 2×2 Pick's matrices and the geometric information encoded in them.

We first state the version of the Schwarz-Pick theorem from [McC03] that we will be using throughout.

Theorem 1.1 (Schwarz-Pick). *Given $z_1, \ldots, z_n \in \mathbb{D}$, $w_1, \ldots, w_n \in \mathbb{D}$, there exits a holomorphic map $f : \mathbb{D} \to \mathbb{D}$ such that $f(z_i) = w_i, i = \{1, \ldots, n\}$ if and only if*

$$\left(\tfrac{1-w_i\bar{w}_j}{1-z_i\bar{z}_j}\right)_{n\times n} \tag{1.1}$$

is positive definite. Moreover, f is unique if and only if the pick's matrix (1.1) *is singular; in this event, f is the blaschke product of degree $n-1$.*

We will refer to matrices of form (1.1) as $n \times n$ Pick's matrices.

The existence problem described in the theorem is the so-called Pick's interpolation problem, and this theorem is interesting in the sense that an algebraic condition is used to characterize the existence of certain holomorphic maps.

Now given $f : \mathbb{D} \to \mathbb{D}$ holomorphic, $f(0) = 0$, and $z \neq 0$, by the Schwarz-Pick theorem, we know that

$$\begin{pmatrix} \frac{1-f(0)\overline{f(0)}}{1} & \frac{1-f(0)\overline{f(z)}}{1} \\ \frac{1-\overline{f(0)}f(z)}{1} & \frac{1-f(z)\overline{f(z)}}{1-z\bar{z}} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & \frac{1-f(z)\overline{f(z)}}{1-z\bar{z}} \end{pmatrix} \tag{1.2}$$

is positive definite. Which is equivalent to the determinant of the given matrix being non-negative.

So we have

$$\begin{aligned} \frac{1-f(z)\overline{f(z)}}{1-z\bar{z}} - 1 \geq 0 &\iff |z|^2 - |f(z)|^2 \geq 0 \\ &\iff 1 - \frac{|f(z)|^2}{|z|^2} \geq 0 \\ &\implies 1 - |f'(0)|^2 \geq 0 \end{aligned}$$

when pushing z to zero.

The last line is just the differential condition in the usual Schwarz Lemma and it carries the geometric interpretation that any holomorphic map between disks has to decrease the hyperbolic distance between two given points where the hyperbolic metric on the unit disk is defined to be $h_{\mathbb{D}} := \frac{1}{1-|f|^2}d|z|$.

Note if we globalize the differential condition using fractional linear transformations we will get the following Schwarz-Pick Lemma.

Lemma 1.2 (Schwarz-Pick)**.** *Given $f : \mathbb{D} \to \mathbb{D}$, holomorphic, then*

$$|f'|^2 \frac{1-|z|^2}{1-|f|^2} \leq 1$$

with equality iff f is a fractional linear transformation.

$|f|^2\frac{1-|z|^2}{1-|f|^2}$ is a geometric quantity in the sense that it is invariant under fractional linear transformations.

To elaborate on this point of view further, consider the differential df of f, it can be viewed as a section of $T^*(\mathbb{D}) \otimes f^*T\mathbb{D}$, that is, $df = dz \otimes \frac{\partial}{\partial w}$ written in coordinates. Note that $|df|^2 = |f|^2\frac{1-|z|^2}{1-|f|^2}$ with respect to the hyperbolic metric on $\mathbb{D}$. So the Schwarz-Pick lemma just says that

$$|df|^2 \leq 1$$

Furthermore, $|df|^2$ is invariant in the sense that given T_1, T_2, both fractional linear transformations, $|d(T_2 \circ f \circ T_1)|^2 = |df|^2 \circ T_1$.

Putting together all the information above, we see that by manipulating the determinant of 2×2 Pick's matrix with respect to a given holomorphic function f, we can somehow get an invariant geometric differential operator and an inequality concerning it.

Thus, our strategy is to first generalize the method described above to Pick's matrices of all dimensions and define a differential operator which satisfies certain invariant properties. Then we look for possible geometric intepretations of it.

2. Existence

In this section, we will define the differential operators associated with Pick's matrices of all dimensions. As we mentioned before, the stratey is to imitate the 2-dimensional case in a systematic way.

Let $P(z_1, \ldots, z_n; f)$ denote the Pick's matrix with respect to f, i.e.

$$P(z_1, \ldots, z_n; f) = \left(\frac{1-f(z_i)\overline{f(z_j)}}{1-z_i\bar{z}_j}\right)_{n\times n}$$

Define a function $h_{(f,n)} : \mathbb{D}^n - \{(z_1, \ldots, z_n) \in \mathbb{D}^n : z_i \neq z_j, \forall i \neq j\} \to \mathbb{R}$ by

$$h_{(f,n)}(z_1, \ldots, z_n) = \frac{\left(\prod\limits_{1\leq i\leq n, 1\leq j\leq n}(1-z_i\bar{z}_j)\right) det P(z_1, z_2, \ldots, z_n; f)}{\prod\limits_{1\leq i<j\leq n}|z_i - z_j|^2}$$

This definition of $h_{(f,n)}$ is a straightforward generalization of the two dimensional case.

Theorem 2.1.

$$\lim_{z_n\to z}\dots\lim_{z_1\to z} h_{(f,n)}(z_1,\dots,z_n) \text{ exits, } \forall(z,\dots,z)\in\mathbb{D}^n.$$

Before proceeding with the proof of the theorem, we pause to introduce some results concerning the invariant property of the limit defined above.

3. Invariance

In this section, we explore certain invariant property of our set-up in the last section. Specifically, we would like to see how it interacts with fractional linear transformations.

Proposition 3.1. *Consider a fractional linear transformation $T(z)=e^{i\theta}\frac{z-w}{1-\bar{w}z}$, then $\forall n\in\mathbb{N}$,*

$$\lim_{z_n\to z}\dots\lim_{z_1\to z} h_{(T\circ f,n)}(z_1,\dots,z_n)=\frac{(1-|w|^2)^n}{|1-\bar{w}f(z)|^{2n}}\lim_{z_n\to z}\dots\lim_{z_1\to z} h_{(f,n)}(z_1,\dots,z_n)$$

, providing either side is defined.

Proof. Note that

$$h_{(T\circ f,n)}(z_1,\dots,z_n)=\frac{\Big(\prod\limits_{1\le i\le n,1\le j\le n}(1-z_i\bar{z}_j)\Big)\det P(z_1,z_2,\dots,z_n;T\circ f)}{\prod\limits_{1\le i<j\le n}|z_i-z_j|^2}$$

and

$$\begin{aligned}\det P(z_1,z_2,\dots,z_n;T\circ f)&=\det\left(\frac{1-T\circ f(z_i)\overline{T\circ f(z_j)}}{1-z_i\bar{z}_j}\right)_{n\times n}\\&=\det\left(\frac{\frac{(1-|w|^2)(1-f(z_i)\overline{f(z_j)})}{(1-\bar{w}f(z_i))(1-w\overline{f(z_j)})}}{1-z_i\bar{z}_j}\right)_{n\times n}\\&=(1-|w|^2)^n\prod_{1\le m\le n}\frac{1}{|1-\bar{w}f(z_m)|^2}det\left(\frac{1-f(z_i)\overline{f(z_j)}}{1-z_i\bar{z}_j}\right)_{n\times n}\end{aligned}$$

Then Proposition 3.1 is immediate if we plug this relation back into the definition of $h_{(f,n)}$.

□

Now we consider the situation where we compose f with a fractional linear transformation on the right-hand side.

Proposition 3.2. *Consider a fractional linear transformation $T(z)=e^{i\theta}\frac{z-w}{1-\bar{w}z}$, then $\forall n\in\mathbb{N}$*

$$\lim_{z_n\to z}\dots\lim_{z_1\to z} h_{(f\circ T,n)}(z_1,\dots,z_n)=\lim_{z_n\to T(z)}\dots\lim_{z_1\to T(z)} h_{(f,n)}(z_1,\dots,z_n)$$

,providing either side of the limit exists.

Before proving the proposition we first need the following lemma as an intermediate step.

Lemma 3.3. *Consider a fractional linear transformation $T(z) = e^{i\theta}\frac{z-w}{1-\bar{w}z}$, then $\forall n \in \mathbb{N}$,*

$$\lim_{z_n\to z}\dots\lim_{z_1\to z} h_{(f,n)}(T(z_1),\dots,T(z_n)) = \lim_{z_n'\to T(z)}\dots\lim_{z_1'\to T(z)} h_{(f,n)}(z_1',\dots,z_n')$$

providing either side is defined.

Proof. First suppose that the left-hand side of the equality is defined. Now note that by continuity

$$\lim_{z_1'\to T(z)} h_{(f,n)}(z_1',\dots,z_n') = h_{(f,n)}(T(z),z_2',\dots,z_n'), \forall z_2',\dots,z_n' \in \mathbb{D}-\{T(z)\}$$

Given $z_i' \in \mathbb{D}-\{T(z)\}, i = \{3,\dots,n\}$, $z_i' = T\circ(T^{-1}(z_i'))$ and $T^{-1}(z_i') \neq z$, we know that

$$\lim_{z_2\to z} h_{(f,n)}(T(z),T(z_2),z_3',\dots,z_n') = \lim_{z_2\to z} h_{(f,n)}(T(z),T(z_2),T\circ(T^{-1}(z_3')),\dots,T\circ(T^{-1}(z_n')))$$

exists by assumption and we denote it by a_2, where $T^{-1}(z) = \frac{z+e^{i\theta}w}{e^{i\theta}+\bar{w}z}$. It means that given any $\epsilon > 0$, there exists $\delta > 0$, if $|z_2 - z| < \delta$, then

$$|h_{(f,n)}(T(z),T(z_2),z_3',\dots,z_n') - a_2| < \epsilon \tag{3.1}$$

Choose $\delta' = \frac{\delta}{1+\delta}(1-|\frac{z-w}{1-\bar{w}z}|)$, then $\forall z_2'$, such that $|z_2' - e^{i\theta}\frac{z-w}{1-\bar{w}z}| < \delta'$, we have $|z_2' + e^{i\theta}w - e^{i\theta}z - z\bar{w}z_2'| < \delta'$. and $1-|w||z_2'| \geq 1-(|\frac{z-w}{1-\bar{w}z}|+\delta') > 0$. So

$$\left|\frac{z_2'+e^{i\theta}w}{e^{i\theta}+\bar{w}z_2'} - z\right| \leq \frac{|z_2'+e^{i\theta}w - e^{i\theta}z - z\bar{w}z_2'|}{1-|w||z_2'|} \leq \frac{1}{1-(|\frac{z-w}{1-\bar{w}z}|+\delta')}\delta' < \delta$$

by our choice of δ'. Thus,

$$|h_{(f,n)}(T(z),T\circ(T^{-1}(z_2')),z_3',\dots,z_n') - a_2| < \epsilon$$

which means that

$$|h_{(f,n)}(T(z),z_2',z_3',\dots,z_n') - a_2| < \epsilon$$

In other words,

$$\lim_{z_2'\to T(z)} h_{(f,n)}(T(z),z_2',z_3',\dots,z_n') = \lim_{z_2\to z} h_{(f,n)}(T(z),T(z_2),z_3',\dots,z_n')$$

for any $z_2',\dots,z_n' \in \mathbb{D}-\{\frac{z-w}{1-\bar{w}z}\}$.

Repeat this process finite times we will get the result we want.

If we suppose that the left hand-side is defined in the first place then we just need to consider f as $(f\circ T)\circ T^{-1}$, and we can apply the result we just proved. □

Now we are going to prove Proposition 3.2,

proof of Proposition 3.2. Note that

$$\begin{aligned}\det P(T(z_1),\dots,T(z_n)) &= \det\left(\frac{1-f(T(z_1))\overline{f(T(z_1))}}{1-\frac{z_i-w}{1-\bar{w}z_i}\overline{\frac{z_j-w}{1-\bar{w}z_j}}}\right)_{n\times n}\\ &= \frac{1}{(1-|w|^2)^n}\prod_{1\leq m\leq n}(|1-\bar{w}z_m|^2) det P(z_1,\dots,z_n; f\circ T)\end{aligned}$$

So $h_{(f,n)}(T(z_1), \ldots, T(z_n))$

$$= \frac{\frac{1}{(1-|w|^2)^n} \prod_{1\le m\le n}(|1-\bar{w}z_m|^2) \prod_{1\le i\le n, 1\le j\le n}(1 - \frac{z_i - w}{1-\bar{w}z_j}\frac{\bar{z}_j-\bar{w}}{1-w\bar{z}_j}) detP(z_1,\ldots,z_n; f\circ T)}{\prod_{1\le i<j\le n} |\frac{z_i - w}{1-\bar{w}z_i} - \frac{z_j - w}{1-\bar{w}z_j}|^2}$$

$$= \frac{\frac{(1-|w|^2)^{n^2}}{(1-|w|^2)^n} \prod_{1\le m\le n}(|1-\bar{w}z_m|^2) \prod_{1\le i\le n, 1\le j\le n}(\frac{1-\bar{z}_i z_j}{(1-\bar{w}z_i)(1-w\bar{z}_j)}) detP(z_1,\ldots,z_n; f\circ T)}{(1-|w|^2)^{n(n-1)} \prod_{1\le i<j\le n} |\frac{z_i - z_j}{(1-\bar{w}z_i)(1-\bar{w}z_j)}|^2}$$

Taking limit on both sides and apply Lemma 3.3, then we are done. □

Lemma 3.4. *Given $f : \mathbb{D} \to \mathbb{D}$ holomorphic, $f(0) = 0$, then $f = zg(z)$ for some $g : \mathbb{D} \to \mathbb{D}$ holomorphic,*

$$\lim_{z_n\to 0} \ldots \lim_{z_1\to 0} h_{(f,n)}(z_1,\ldots,z_n) = \lim_{z_n\to 0} \ldots \lim_{z_2\to 0} h_{(g,n-1)}(z_2,\ldots,z_n)$$

If either side is defined.

Proof. Note that

$$\det P(0, z_2, \ldots, z_n; f) = \det \begin{pmatrix} 1 & 1 & \ldots & 1 \\ 1 & \frac{1-f(z_2)\overline{f(z_2)}}{1-z_2\bar{z}_2} & \ldots & \frac{1-f(z_2)\overline{f(z_n)}}{1-z_2\bar{z}_n} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \frac{1-f(z_i)\overline{f(z_2)}}{1-z_i\bar{z}_2} & \ldots & \frac{1-f(z_i)\overline{f(z_n)}}{1-z_i\bar{z}_n} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \frac{1-f(z_n)\overline{f(z_2)}}{1-z_n\bar{z}_2} & \ldots & \frac{1-f(z_n)\overline{f(z_n)}}{1-z_n\bar{z}_n} \end{pmatrix}$$

$$= \det \begin{pmatrix} 1 & 1 & \dots & 1 \\ 0 & \frac{1-f(z_2)\overline{f(z_2)}}{1-z_2\bar{z}_2} - 1 & \dots & \frac{1-f(z_2)\overline{f(z_n)}}{1-z_2\bar{z}_n} - 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{1-f(z_i)\overline{f(z_2)}}{1-z_i\bar{z}_2} - 1 & \dots & \frac{1-f(z_i)\overline{f(z_n)}}{1-z_i\bar{z}_n} - 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \frac{1-f(z_n)\overline{f(z_2)}}{1-z_n\bar{z}_2} - 1 & \dots & \frac{1-f(z_n)\overline{f(z_n)}}{1-z_n\bar{z}_n} - 1 \end{pmatrix}$$

$$= \det \begin{pmatrix} 1 & 1 & \dots & 1 \\ 0 & z_2\bar{z}_2\frac{1-g(z_2)\overline{g(z_2)}}{1-z_2\bar{z}_2} & \dots & z_2\bar{z}_n\frac{1-g(z_2)\overline{g(z_n)}}{1-z_2\bar{z}_n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & z_i\bar{z}_2\frac{1-g(z_i)\overline{g(z_2)}}{1-z_i\bar{z}_2} & z_i\bar{z}_n\frac{1-g(z_i)\overline{g(z_n)}}{1-z_i\bar{z}_n} & \\ \vdots & \vdots & \ddots & \vdots \\ 0 & z_n\bar{z}_2\frac{1-g(z_n)\overline{g(z_2)}}{1-z_n\bar{z}_2} & \dots & z_n\bar{z}_n\frac{1-g(z_n)\overline{g(z_n)}}{1-z_n\bar{z}_n} \end{pmatrix}$$

$$= |z_2|^2 \dots |z_n|^2 \begin{pmatrix} \frac{1-g(z_2)\overline{g(z_2)}}{1-z_2\bar{z}_2} & \dots & \frac{1-g(z_2)\overline{g(z_n)}}{1-z_2\bar{z}_n} \\ \vdots & \ddots & \vdots \\ \frac{1-g(z_i)\overline{g(z_2)}}{1-z_i\bar{z}_2} & \dots & \frac{1-g(z_i)\overline{g(z_n)}}{1-z_i\bar{z}_n} \\ \vdots & \ddots & \vdots \\ \frac{1-g(z_n)\overline{g(z_2)}}{1-z_n\bar{z}_2} & \dots & \frac{1-g(z_n)\overline{g(z_n)}}{1-z_n\bar{z}_n} \end{pmatrix}$$

$$= |z_2|^2 \dots |z_n|^2 detP(z_2, \dots, z_n; g)$$

So we conclude that

$$\frac{(\prod_{1\le i\le n, 1\le j\le n}(1-z_i\bar{z}_j))detP(0,\dots,z_n;f)}{\prod_{1\le i<j\le n}|z_i-z_j|^2} = \frac{(\prod_{2\le i\le n, 2\le j\le n}(1-z_i\bar{z}_j))detP(z_2,\dots,z_n;g)}{\prod_{2\le i<j\le n}|z_i-z_j|^2}$$

And this concludes the proof of the Lemma 3.4.

□

We are able to prove Theorem 2.1 now.

proof of Theorem 2.1. We prove this theorem by induction. For $n = 1$, $\forall g : \mathbb{D} \to \mathbb{D}$ holomorphic, it is true that

$$\lim_{z_1 \to z} 1 - g(z_1)\overline{g(z_1)} = 1 - |g(z)|^2, \forall z \in \mathbb{D}$$

Now suppose that the theorem is true for $n = k - 1$, we want to show that it is true for $n = k$. Given $f : \mathbb{D} \to \mathbb{D}$ holomorphic, and $z_0 \in \mathbb{D}$, consider $T_1 = \frac{z+z_0}{1-\bar{z}_0 z}$ and $T_2 = \frac{z-f(z_0)}{1-\overline{f(z_0)}z}$, note that $T_2 \circ f \circ T_1(0) = 0$. According to Lemma 3.4 and the induction hypothesis, $\lim_{z_k \to 0} \dots \lim_{z_1 \to 0} h_{(T_2\circ f\circ T_1, k)}$ exits. Then by Proposition 3.2 and Proposition 3.1

$$\lim_{z_k \to 0} \dots \lim_{z_1 \to 0} h_{(T_2\circ f\circ T_1, k)} = \frac{1}{(1-|f(z_0)|^2)^k} \lim_{z_k \to z_0} \dots \lim_{z_1 \to z_0} h_{(f,k)}$$

We are done. □

References

[McC03] John E. McCarthy. "Pick's theorem-what's the big deal?" In: *The American mathematical monthly* 110.1 (2003), pp. 36–45.