A Polarization Operation for Pseudomonomial Ideals

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Abstract

Pseudomonomials and ideals generated by pseudomonomials (pseudomonomial ideals) are a central object of study in the theory of neural rings and neural codes. In the setting of a polynomial ring, we define the polarization operation ρ sending pseudomonomials to squarefree monomials and a further polarization operation $\mathcal{P}$ sending pseudomonomial ideals to squarefree monomial ideals. We show for a pseudomonomial ideal I, in a polynomial ring R, that

- A pseudomonomial f is in I if and only if $\rho(f)$ is in $\mathcal{P}(I)$.
- $\mathcal{P}(I)$ is generated by the polarizations of the minimal pseudomonomials in I.
- The prime ideals in the unique minimal primary decomposition of $\mathcal{P}(I)$ are the polarizations $\mathcal{P}(\mathbf{p})$ of the prime ideals $\mathbf{p}$ in the unique minimal primary decomposition of I.

Furthermore, I is Cohen-Macaulay if and only if $\mathcal{P}(I)$ is.

1 Introduction

1.1 Combinatorial Commutative Algebra

Combinatorial structures often arise in algebraic objects. A quintessential example is that abstract simplicial complexes can be encoded in squarefree monomial ideals over polynomial rings. The theory of this connection was developed by Stanley, Hochster, and Reisner in the 1970s, and is known as Stanley-Reisner theory. For more on Stanley-Reisner theory, see [2].

Definition 1.1. *Let $R = \mathbb{F}_2[x_1, \dots x_n]$ be a polynomial ring over $\mathbb{F}_2$. A* ***squarefree monomial*** *in R is a monomial $x_1^{e_1} x_2^{e_2} \cdots x_n^{e_n}$ with each $e_i \in \{0, 1\}$. Equivalently, it is an element of the form $\prod_{i \in \sigma} x_i$ for some $\sigma \subseteq \{x_1, \dots, x_n\}$. A* ***squarefree monomial ideal*** *is an ideal generated by squarefree monomials.*

Definition 1.2. *An* ***abstract simplicial complex*** *on a set S is a subset $\Delta \subseteq \mathcal{P}(S)$ of the power set of S, with the property that if $\sigma \in \Delta$, and $\tau \subset \sigma$, then $\tau \in \Delta$.*

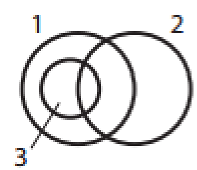
If R is the polynomial ring above, and $I \subset R$ is a squarefree monomial ideal, then the collection of subsets of $\{x_1, \ldots, x_n\}$,

$$\Delta = \{\sigma \subseteq \{x_1, \ldots, x_n\} | \prod_{i \in \sigma} x_i \notin I\},$$

whose product does not lie in I, forms an abstract simplicial complex on the set $\{x_1, \ldots, x_n\}$.

1.2 Neural Rings

In studying the firing patterns of neurons relating to spatial processing in rats, researchers noticed that individual neurons corresponded to convex regions in space. Each neuron would fire when the rat was in the region in space corresponding to it. The way the neurons fire together have a combinatorial structure with a topological interpretation. Consider the following configuration, taken from [4], which we will return to.



Any subset of the regions intersect, but because region 3 is contained in region 1, neuron 3 cannot fire without neuron 1. The collection of sets of neurons that can fire together forms a simplicial complex. This is the **intersection complex** $\{\{123\}, \{12\}, \{13\}, \{23\}, \{1\}, \{2\}, \{3\}, \emptyset\}$. Observe that the intersection complex does not encode the information that region 3 is contained within region 1. The full combinatorial information is captured by the collection of sets of neurons that can fire together *exclusively*. We call this the **neural code**. The neural code of the above configuration is $\mathcal{C}_e = \{\{1,2,3\}, \{1,3\}, \{1,2\}, \{2\}, \{1\}, \emptyset\}$.

Despite the fact that some information is lost in the intersection complex, it still encodes quite a bit of topological information, as the following theorem shows.

Theorem 1.3 (Nerve Theorem)**.** *Viewed as a geometric simplicial complex, the intersection complex of a neural code is homotopy-equivalent to the union of the convex regions producing it.*

Pseudomonomial ideals, a class of ideals that generalizes the class of squarefree monomial ideals, provide a setting in which we can apply something analagous to the Stanley-Reisner connection. Neural codes can be encoded in pseudomonomial ideals over polynomial rings, in a way that loses no information about the code. Where $\mathcal{C}$ is a neural code on the set $\{1, \ldots, n\}$, the **neural ideal** is the ideal

$$I = \left\langle \left\{ \prod_{i \in v} x_i \prod_{j \notin v} (1 - x_j) \mid v \notin \mathcal{C} \right\} \right\rangle.$$

The **neural ring** is the quotient R/I. We lose no information because a set of neurons is in the code if and only if its corresponding pseudomonomial is not in I. That is,

$$\sigma \in \mathcal{C} \Leftrightarrow \prod_{i\in\sigma} x_i \prod_{j\notin\sigma}(1-x_j) \notin I.$$

To return to our previous example, the code $\mathcal{C}_e = \{\{1\},\{2\},\{1,2\},\{1,3\},\{1,2,3\}\}$ has the following corresponding ideal.

$$I_e = \langle (1-x_1)x_2x_3, (1-x_1)(1-x_2)x_3 \rangle \subset R$$

For more about pseudomonomial ideals, see [4]

1.3 Polarization of Monomial Ideals

When dealing with ideals generated by monomials that are not squarefree, the polarization operation is often used to "turn them into" squarefree monomial ideals. Given a monomial $M = x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$, we define its polarization as the squarefree monomial

$$\rho(M) = x_{1,1}x_{1,a_1}x_{2,1}x_{2,a_2}\cdots x_{n,1}x_{n,a_n},$$

in the polynomial ring with all the required elements adjoined. There is also a polarization operation for ideals. Given a monomial ideal $I = \langle M_1, \dots M_\ell\rangle$ in a polynomial ring, we define its polarization as $\mathcal{P}(I) = \langle \rho(M_1), \dots \rho(M_\ell)\rangle$, in the polynomial ring S over $\mathbb{F}_2$ with all the required elements adjoined.

The polarization operation has many useful properties. It allows us to only deal with squarefree monomial ideals, for which the techniques of Stanley-Reisner theory apply. Many properties of the polarized ideal transfer to the original ideal, such as the following[1].

- I and $\mathcal{P}(I)$ have equal height.
- $\mathcal{P}(I+J) = \mathcal{P}(I) + \mathcal{P}(J)$.
- $\mathcal{P}(I\cap J) = \mathcal{P}(I) \cap \mathcal{P}(J)$.
- If $\mathbf{p}$ is a (minimal) prime ideal containing I, then $\mathcal{P}(\mathbf{p})$ is a (minimal) prime ideal containing $\mathcal{P}(I)$.
- The quotient ring R/I is Cohen-Macaulay if and only if $S/\mathcal{P}(I)$ is. (Fröberg [3])

2 Polarization of Pseudomonomials

Let $R = \mathbb{F}_2[x_1, \dots x_n]$ and $S = \mathbb{F}_2[x_1,\dots x_n, y_1, \dots y_n]$. We want each y_i to act as an alias for $1-x_i$, and we encode this by defining the **depolarization ideal**,

$$D = \langle \{x_i + y_i - 1 \mid i \in [n]\}\rangle,$$

so that $S/D = R$. We denote the corresponding quotient map by $\pi : S \to S/D = R$, so that π identifies y_i and $1-x_i$ for each i.

A **pseudomonomial** in R is a product

$$\prod_{i\in\sigma} x_i \prod_{j\in\tau}(1-x_j) \in R$$

where $\sigma, \tau \subseteq [n]$ and $\sigma \cap \tau = \emptyset$. A **pseudomonomial ideal** is an ideal generated by a finite set of pseudomonomials. We say a pseudomonomial f is **minimal** in I if no proper divisor of f is contained in I. Furthermore, if $\{f_1, \ldots, f_\ell\}$ is the set of *every* minimal pseudomonomial in I, and we write I in the following form,

$$I = \langle f_1, \ldots f_\ell \rangle$$

we say that I is in **canonical form**.

To return to the running example, the ideal I_e has canonical form

$$CFI_e = \langle (1-x_1)x_3 \rangle.$$

Intuitively, what the canonical form is showing us here is that the only place the rat can't be is in region 3 but not in region 1, because region 1 contains region 3.

For a pseudomonomial $f = \prod_{i\in\sigma} x_i \prod_{j\in\tau}(1-x_j) \in R$, we define its **polarization** to be the squarefree monomial

$$\rho(f) = \prod_{i\in\sigma} x_i \prod_{j\in\tau} y_j \in S.$$

Lemma 2.1. *The polarization operation on pseudomonomials, ρ has the property that if $f, g \in R$ are pseudomonomials then $f|g \Leftrightarrow \rho(f)|\rho(g)$ in S.*

Proof. Let

$$f = \prod_{i\in\sigma_f} x_i \prod_{j\in\tau_f}(1-x_j), g = \prod_{i\in\sigma_g} x_i \prod_{j\in\tau_g}(1-x_j).$$

The condition that $f|g$, is equivalent to the condition that $\sigma_f \subset \sigma_g$ and $\tau_f \subset \tau_g$. The condition that $\rho(f)|\rho(g)$, that is, that

$$\prod_{i\in\sigma_f} x_i \prod_{j\in\tau_f} y_j \text{ divides } \prod_{i\in\sigma_g} x_i \prod_{j\in\tau_g} y_j$$

is the same. Thus, $f|g$ if and only if $\rho(f)|\rho(g)$.

□

Let $I \subset R$ be a pseudomonomial ideal. As above, I can be defined by the set of minimal pseudomonomials in I. Abstractly, I can also be defined by the set of pseudomonomials contained in I. A theorem of [4], I can also be defined by a primary decomposition constructed in the following way.

For any $\alpha \in \{0, 1, *\}^n$, define

$$\mathbf{p}_\alpha = \langle \{x_i \,|\, \alpha_i = 0\}, \{1 - x_i \,|\, \alpha_i = 1\} \rangle.$$

For the following two theorems, let $\mathcal{A} = \{\alpha \,|\, \mathbf{p}_\alpha$ is minimal with respect to the property that $I \subseteq \mathbf{p}_\alpha\}$.

Theorem 2.2 (Youngs [4]).

$$I = \bigcap_{\alpha \in \mathcal{A}} \mathbf{p}_\alpha.$$

is the unique irredundant primary decomposition of I.

In the running example, the primary decomposition of I_e is

$$I_e = (x_3) \cap (1 - x_1).$$

We define the polarization of I by showing that all three definitions of I are compatible with polarization in the following way.

Theorem 2.3. *For an pseudomonomial ideal $I = \langle f_1, \dots f_\ell \rangle \subset R$ in canonical form, and a squarefree monomial ideal $J \subset S$, the following are equivalent:*

1. *J is the smallest ideal in S such that, for any pseudomonomial f in R, $f \in I$ if and only if $\rho(f) \in J$.*
2. *$J = \bigcap_{\alpha \in \mathcal{A}} \mathbf{q}_\alpha$, where $\mathbf{q}_\alpha = \langle \{x_i \,|\, \alpha_i = 0\}, \{y_i \,|\, \alpha_i = 1\} \rangle = \mathcal{P}(\mathbf{p}_\alpha)$.*
3. *$J = \langle \rho(f_1), \dots \rho(f_\ell) \rangle \subset S$.*

Proof.

- (2) $\Leftrightarrow$ (3) Let $J = \langle \rho(f_1), \dots \rho(f_\ell) \rangle \subset S$.

  We show that $J = \bigcap_{\alpha \in \mathcal{A}} \mathbf{q}_\alpha$.

  - $J \subseteq \bigcap_{\alpha \in \mathcal{A}} \mathbf{q}_\alpha$
    Let $J = \langle \rho(f_1), \dots \rho(f_\ell) \rangle \subset S$.
    Define a function $\phi : \{0, 1, *\}$ by $\phi(0) = x_i, \phi(1) = y_i$, and $\phi(*) = 0$. Since for each $\alpha \in \mathcal{A}$, we have $I \subseteq \mathbf{p}_\alpha$, then for each f_i there must exist a j such that $x_j - \alpha_j \,|\, f_i$. But then $\phi(\alpha_j) \,|\, \mathcal{P}(f_i)$ and $f_i \in \mathbf{q}_j$. Since i was arbitrary, each generator of J is contained in $\bigcap_{\alpha \in \mathcal{A}} \mathbf{q}_\alpha$.
  - $J \subseteq \bigcap_{\alpha \in \mathcal{A}} \mathbf{q}_\alpha$
    Because $\bigcap_{\alpha \in \mathcal{A}} \mathbf{q}_\alpha$ is the intersection of monomial ideals, it is a monomial ideal. If two monomial ideals contain the same monomials, then they are equal. So it suffices to check that every monomial in $\bigcap_{\alpha \in \mathcal{A}} \mathbf{q}_\alpha$ is in J. Let $g \in \bigcap_{\alpha \in \mathcal{A}} \mathbf{q}_\alpha$ a monomial. Then for every $\alpha \in \mathcal{A}$ there is some k such that $\phi(\alpha_k) \,|\, g$. But then $1 - \alpha_k$ divides $\pi(g)$, where π is the quotient map by D. Since k is arbitrary, this implies that $\pi(g) \in \bigcap_{\alpha \in \mathcal{A}} \mathbf{p}_\alpha$, which implies that $\pi(g) \in I$. Since g was a monomial, we automatically get a factorization of $\pi(g)$ into linear factors of the form x_i or $1 - x_i$. Furthermore, some $f_i \,|\, \pi(g)$, which implies that $\mathcal{P}(f_i) \,|\, g$. Thus $g \in J$.

- (1) $\implies$ (3)

  Suppose that $J_0 \subset S$ is the smallest ideal in S such that for every pseudomonomial $f \in R$, $f \in I \Leftrightarrow \rho(f) \in J_0$. Then $J_0 = \langle \{\rho(f) | f \in I, f \text{ is a pseudomonomial}\} \rangle$. Let $J = \langle \rho(f_1), \dots \rho(f_\ell) \rangle \subset S$, where the f_i are the set of *all* minimal pseudomonomials in I. Then it suffices to show that $J_0 = J$.

  Obviously $J \subseteq J_0$ because the generators of J are a subset of the generators of J_0.

Recall from lemma 2.1 that ρ preserves divisibility. Let $f \in I$ a pseudomonomial. Since f is a pseudomonomial in I, some factor of f is a minimal pseudomonomial in I. Denote it by f_i so that $f_i|f$. Then $\rho(f_i)|\rho(f)$, so $\rho(f_i)|\rho(f)$, and $\rho(f) \in J$. Since f was an arbitrary pseudomonomial in I, this shows that every generator of J_0 is contained in J and $J_0 \subseteq J$. Therefore, $J = J_0$.

- (3) $\implies$ (1).
  Let $J = \langle \rho(f_1), \ldots \rho(f_\ell) \rangle \subset S$, where the f_i are the set of *all* minimal pseudomonomials in I. We need to show that J is the smallest ideal in S such that for every pseudomonomial $f \in R$, $f \in I \Leftrightarrow \rho(f) \in J$.

  First we show that indeed, for every pseudomonomial $f \in R$, $f \in I \Leftrightarrow \rho(f) \in J$. From the above, if a pseudomonomial $f \in I$, then $\rho(f) \in J$. Suppose f is a pseudomonomial not in I. Suppose, for the sake of contradiction, that $\rho(f) \in J$. Then, since J is a monomial ideal $\rho(f)$ is a monomial, some generator $\rho(f_i)$ of J divides $\rho(f)$. But by Lemma 1, if $\rho(f_i)|\rho(f)$,then $f_i|f$, so $f \in I$, a contradiction. This concludes the proof.

  □

Now that we have shown that these three definitions produce the same ideal, we can use them equivalently to refer to a well-defined polarization $\mathcal{P}(I)$ of any pseudomonomial ideal $I \subset R$.

In the running example, we now have

$$\mathcal{P}(I) = (x_1 y_3) = (x_1) \cap (y_3).$$

2.1 The Polar Simplicial Complex

Now that squarefree monomial ideals can be made out of neural codes in a way that loses no information, we can use the Stanley-Reisner connection to associate to it a simplicial complex on a set corresponding to $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$. Returning to our example, if $\{x_1, x_2, x_3, y_1, y_2, y_3\}$ corresponds to the vertex set $\{1_x, 2_x, 3_x, 1_y, 2_y, 3_y\}$, the polarized ideal $\mathcal{P}(I)$ corresponds to the simplicial complex

$$\Delta\mathcal{P}(I) = \{\{1_x 2_x 3_x 1_y 2_y\}, \{2_x 3_x 1_y 2_y 3_y\}\} \cup \{\text{subsets thereof}\}.$$

Combinatorially, this simplicial complex can be interpreted in a couple of different ways. $\Delta\mathcal{P}(I)$ is:

- isomorphic to the collection of squarefree monomials $h = x_1^{e_1} \cdots x_n^{e_n} y_1^{d_1} \cdots y_n^{d_n}$ such that if $g|h$ and $g = \rho(f)$ for some pseudomonomial $f \in R$, we have $f \notin I$. Note that the condition that $g = \rho(f)$ for some pseudomonomial $f \in R$ is equivalent to the condition that $x_i y_i \not| g$ for all i.
- isomorphic to completion of the intersection complex of the original regions $\{1, \ldots n\}$ and their complements $\{1', \ldots, n'\}$ by the rule that for a region m and an element z of the intersection complex Δ, that if $\{m\} \cup z \in \Delta$ and $\{m'\} \cup z \in \Delta$, then $\{m, m'\} \cup z \in \Delta$.

3 Cohen-Macaulayness

We begin with a few definitions. The reader is assumed to be familiar with local rings and localization.

Definition 3.1. *Let R a ring, and $A = (a_1, \dots a_k)$ a sequence of elements in R. Then A is called a* ***regular sequence*** *if a_i is a non-zero divisor in the quotient ring $R/\langle a_{i+1}, \dots a_k \rangle$.*

Definition 3.2. *The* ***Krull dimension*** *of a ring R is the supremum of all integers n such that there exists a chain of prime ideals $\mathbf{p}_0 \subset \mathbf{p}_1 \subset \cdots \mathbf{p}_n \subset R$.*

Definition 3.3. *A local ring L is said to be* ***Cohen-Macaulay*** *if the maximum length of a regular sequence in the maximal ideal of L is equal to its Krull dimension. A general ring is said to be Cohen-Macaulay its localizations at all prime ideals are Cohen-Macaulay.*

With R and S as above, let $I \subset R$ a pseudomonomial ideal. We have the following theorem.

Theorem 3.4. *R/I is Cohen-Macaulay if and only if $S/\mathcal{P}(I)$ is.*

Proof. The quotient of a ring T by an ideal generated by a regular sequence is Cohen-Macaulay if and only if T is. The proof of this is omitted. Let D be the depolarization ideal defined above, and set $J = \mathcal{P}(I)$. Then $S/D = R$, and $\pi(J) = I$. Then $R/J = (S/D)/J = S/(I+D) = (S/J)/D$. By the following lemma, D is generated by a regular sequence in S/J, so R/I is Cohen-Macaulay if and only if S/J is. $\square$

As in the previous proof, denote $\mathcal{P}(I)$ by J.

Lemma 3.5. *The sequence $(x_1 + y_1 - 1, \dots x_n + y_n - 1)$ is a regular sequence in S/J.*

Proof. We are trying to show that $(x_1+y_1-1, \dots x_n+y_n-1)$ is a regular sequence in S/J. This is equivalent to showing that x_t+y_t-1 is a non-zero divisor in S/J_t, where

$$J_t = D_t + J, \text{ and } D_t = \langle x_{t+1} + y_{t+1} - 1, \dots x_n + y_n - 1 \rangle.$$

By Theorem 2.3, we know that

$$J_t = D_t + \bigcap_{\alpha \in \mathcal{A}} \mathbf{q}_\alpha.$$

Regrouping the intersection, we get

$$J_t = \bigcap_{\alpha \in \mathcal{A}} (\mathbf{q}_\alpha + D_t).$$

So to show that $x_t + y_t - 1$ is a non-zero divisor in S/J_t, i.e. that

$$(x_t + y_t - 1) f \in J_t \implies f \in J_t$$

it suffices to show that

$$(x_t + y_t - 1) f \in \mathbf{q}_\alpha + D_t \implies f \in \mathbf{q}_\alpha + D_t$$

for each $\alpha \in \mathcal{A}$. That is, that $x_t + y_t - 1$ is a non-zero divisor in $\mathbf{q}_\alpha + D_t$. First, we expand $\mathbf{q}_\alpha + D_t$,

$$\begin{aligned}\mathbf{q}_\alpha + D_t &= \langle x_{t+1} + y_{t+1} - 1, \dots x_n + y_n - 1\rangle + \mathbf{q}_\alpha \\ &= \langle x_{t+1} + y_{t+1} - 1, \dots x_n + y_n - 1\rangle + \langle \phi(\alpha_1), \dots \phi(\alpha_n)\rangle \\ &= \langle x_{t+1} + y_{t+1} - 1, \dots x_n + y_n - 1, \phi(\alpha_1), \dots \phi(\alpha_n)\rangle.\end{aligned}$$

To show that $(x_t + y_t - 1)$ is a non-zero divisor in $\mathbf{q}_\alpha + D_t$, it (more than) suffices to show that

$$(x_t + y_t - 1, x_{t+1} + y_{t+1} - 1, \dots x_n + y_n - 1, \phi(\alpha_1), \dots \phi(\alpha_n)) = \lambda$$

is a regular sequence in S. We change variables, setting $\tilde{x}_i = x_i + y_i - 1$, and $\tilde{y}_i = y_i$ if $\phi(\alpha_i) = 0$, and $\tilde{y}_i = \phi(\alpha_i)$ otherwise. Then the sequence

$$(\tilde{x}_t, \tilde{x}_{t+1}, \dots \tilde{x}_n, \tilde{y}_1, \dots \tilde{y}_n)$$

is a subsequence of λ. The above sequence is a sequence of independent variables, and is clearly regular. Therefore λ is regular. This concludes the proof. □

Now that we know that the Cohen-Macaulayness of the neural ring depends only on the Cohen-Macaulayness of the polarized ring, we can use the methods of Stanley-Resiner theory, particularly Reisner's criterion, to computationally determine when a neural ring is Cohen-Macaulay.

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