THE FUNDAMENTAL THEOREM OF AFFINE GEOMETRY ON TORI

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Abstract. The classical Fundamental Theorem of Affine Geometry states that for $n \geq 2$, any bijection of n-dimensional Euclidean space that maps lines to lines (as sets) is given by an affine map. Analogous statements have since also been proven the 'model spaces' for projective and hyperbolic geometry. We consider an analogue of the above problem for compact quotients, and solve it for tori: Any bijection of an n-dimensional torus ($n \geq 2$) that maps lines to lines, is given by an affine map.

Contents

1. Introduction

1.1. **Main result.** The classical Fundamental Theorem of Affine Geometry (FTAG), going back to Von Staudt's work in the 1840s [Sta47], characterizes invertible affine maps of $\mathbb{R}^n$ for $n \geq 2$: Namely, if $f : \mathbb{R}^n \to \mathbb{R}^n$ is a bijection so that for any line ℓ, the image $f(\ell)$ is also a line, then f is affine.

Since then, numerous generalizations and variations have been proven, some algebraic and some geometric. For example, on the algebraic side, there is a version for for projective spaces (see e.g. [Art88]), general fields k, and for noninvertible maps (by Chubarev-Pinelis [CP99]). On the geometric side, there is an analogous result in hyperbolic geomety (see e.g. [Jef00]), and a Lorentzian version with lightcones instead of lines (by Alexandrov [Ale67]).

All of these versions have in common that they are characterizations of self-maps of the model space of the corresponding geometry (e.g. $\mathbb{E}^n, \mathbb{H}^n$, or $\mathbb{P}^n$). On the other hand, if X is some topological space, and $\mathcal{C}$ is a sufficiently complicated family of curves on X, then one can expect rigidity

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for self-maps of X preserving $\mathcal{C}$. In this light it seems reasonable to ask for the following generalized version of the FTAG:

Question 1.1 (Generalized FTAG)**.** Let M be an affine manifold of dimension > 1. Suppose that $f : M \to M$ is a bijection such that for any affine line ℓ in M, the image $f(\ell)$ is also an affine line (as a set). Is f affine?

Remark 1.2. Here by *affine manifold* and *affine line*, we use the language of geometric structures: If M is a smooth manifold of dimension n, then an *affine structure* on M is a covering of M by $\mathbb{R}^n$-valued charts whose transition functions are locally affine. An *affine line* in an affine manifold M is a curve in M that coincides with an affine line segment in affine charts. A map is *affine* if in local affine coordinate the map is affine. See [Gol16] for more information on affine manifolds and geometric structures.

Remark 1.3. Of course one can formulate analogues of Question 1.1 for the variations of FTAG cited above, e.g. with affine manifolds (affine lines) replaced with projective manifolds (projective lines) or hyperbolic manifolds (geodesics).

The only case in which Question 1.1 known is the classical FTAG (i.e. $M = \mathbb{A}^n$ is affine n-space). Our main result is that Question 1.1 has a positive answer for the standard affine torus:

Theorem 1.4. *Let $n \geq 2$ and let $T = \mathbb{R}^n/\mathbb{Z}^n$ denote the standard n-torus. Let $f : T \to T$ be any bijection that maps lines to lines (as sets). Then f is affine.*

Remark 1.5. Note that f is not assumed to be continuous! Therefore it is not possible to lift f to a map of $\mathbb{R}^n$ and apply the classical FTAG. In addition, the proof of FTAG does not generalize to the torus setting: The proof starts by showing that f maps midpoints to midpoints. Iteration of this property gives some version of continuity for f, which is crucial to the rest of the proof.

The method for showing midpoints go to midpoints is as follows: Let $P, Q \in \mathbb{R}^n$ and let M be the midpoint. Take any two points R, S that are not collinear with P, Q but such that P, Q, R, S lie in a single plane, and R, S lie on different sides of the line PQ. Then M is the (unique) point of intersection of the diagonals AB and PQ of the parallellogram $APBQ$. Let $A' := f(A), B' := f(B)$, etc. Then M' is the point of intersection of the diagonals $A'B'$ and $P'Q'$ of the parallellogram $A'P'B'Q'$. This shows M' is the midpoint of $A'B'$.

There are several reasons this argument fails on the torus: First, any two lines may intersect multiple, even infinitely many, times. Hence there is no hope of characterizing the midpoint as the unique intersection of two diagonals. And second, any two points are joined by infinitely many distinct lines. Therefore there is no way to talk about the diagonals of a parallellogram.

It is amusing then that these two geometric differences (multiple intersections and multiple lines between points) will play a crucial role in our proof.

Remark 1.6. The standard affine structure on the torus is the unique Euclidean structure. However, there are other affine structures on the torus, both complete and incomplete. We refer to [Gol16] for more information. We do not know whether a similar characterization of affine maps holds for these other, non-Euclidean structures.

Finally, let us mention some related questions in (pseudo-)Riemannian geometry. If M is a smooth manifold, then two metrics g_1 and g_2 on M are called *geodesically equivalent* if they have the same geodesics (as sets). Of course if M is a product, then scaling any factor will not affect the geodesics. Are any two geodesically equivalent metrics isometric up to scaling on factors?

This is of course false for spheres, but Matveev essentially gave a positive answer for Riemannian manifolds with infinite fundamental group [Mat03]: If M admits two Riemannian metrics that are geodesically equivalent but not homothetic, and $\pi_1(M)$ is infinite, then M supports a metric such that the universal cover of M splits as a Riemannian product.

Of course Matveev's result makes no reference to maps that preserve geodesics. The related problem for maps has been considered with a regularity assumption, and has been called the "Projective Lichnerowicz Conjecture" (PLC): First we say that a smooth map $f : M \to M$ of a closed (pseudo-)Riemannian manifold M is *affine* if f preserves the Levi-Civita connection ∇. Further f is called *projective* if ∇ and $f^*\nabla$ have the same (unparametrized) geodesics. PLC then states that unless M is covered by a round sphere, the group of affine transformations has finite index in the group of projective transformations.

For Riemannian manifolds, Zeghib has proven PLC [Zeg16]. See also [Mat07] for an earlier proof by Matveev of a variant of this conjecture. In view of these results, and Question 1.1, let us ask:

Question 1.7. Let M be a closed nonpositively curved manifold of dimension > 1 and let ∇ be the Levi-Civita connection. Suppose $f : M \to M$ maps geodesics to geodesics (as sets). Is f affine (i.e. smooth with $f^*\nabla = \nabla$)?

As far as we are aware, the answer to Question 1.7 is not known for any choice of M. Theorem 1.4 is of course a positive answer to Question 1.7 for the case that $M = T^n$ is a flat torus.

1.2. **Outline of the proof.** Let $f : T^n \to T^n$ be a bijection preserving lines. The engine of the the proof is that f preserves the number of intersections of two objects. A key observation is that affine geometry of T^n allows for affine objects (lines, planes, etc.) to intersect in interesting ways (e.g. unlike in $\mathbb{R}^n$, lines can intersect multiple, or even infinitely many, times).

In Section 2, we give a characterization of rational subtori in terms of intersections with lines. This is used to prove that f maps rational subtori to rational subtori.

Then we start the proof of Theorem 1.4. The proof is by induction on dimension, and the base case (i.e. dimension 2) is completed in Section 3. We start by recalling a characterization of the homology class of a rational line in terms of intersections with other lines. This allows us to associate to f an induced map A on $H_1(T^2)$, even though f is not necessarily continuous. We regard A as the linear model for f, and the rest of the section is devoted to proving $f = A$, up to a translation.

Finally in Section 4, we use the base case and the fact that f preserves rational subtori (proven in Section 2), to complete the proof in all dimensions.

1.3. **Notation.** We will use the following notation for the rest of the paper. Let $n \geq 1$. Then $T = \mathbb{R}^n/\mathbb{Z}^n$ will be the standard affine n-torus. If $x \in \mathbb{R}^n$, then $[x]$ denotes the image of x in T. Similarly, if $X \subseteq \mathbb{R}^n$ is any subset, then $[X]$ denotes the image of X in T.

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2. Rational subtori are preserved

Definition 2.1. Let $n \geq 1$ and $1 \leq k \leq n$. Then a *k-plane* W in $T = \mathbb{R}^n/\mathbb{Z}^n$ is the image in T of an affine k-dimensional subspace $V \subseteq \mathbb{R}^n$. Let V_0 be the translate of V containing the origin. If V_0 is spanned by $V_0 \cap \mathbb{Q}^n$, we say W is *rational*. In this case we also say W is a *rational subtorus* of dimension k. If $k = 1$, we also say W is a *rational line*.

The goal of this section is to show:

Proposition 2.2. *Let $n \geq 2$ and $1 \leq k \leq n$. Write $T = \mathbb{R}^n/\mathbb{Z}^n$ and let $f : T \to T$ be a bijection that maps lines to lines. Then the image under f of any rational k-dimensional subtorus is again a rational k-dimensional subtorus.*

For the rest of this section, we retain the notation of Proposition 2.2. We start with the following criterion for subtori to be rational:

Lemma 2.3. *Let V be a plane in T. Then the following are equivalent:*

(i) V is rational.
(ii) V is compact.

(iii) Every rational line not contained in V intersects V at most finitely many times.

Proof. The equivalence of (i) and (ii) is well-known. We prove that (ii) and (iii) are equivalent. First suppose that V is compact and let ℓ be any rational line not contained in V. If $\ell \cap V = \emptyset$, we are done. If $v_0 \in \ell \cap V$, we can translate ℓ and V by $-x_0$, so that we can assume without loss of generality that ℓ and V intersect at 0. Then $\ell \cap V$ is a compact subgroup of T and not equal to ℓ. Because ℓ is 1-dimensional, any proper compact subgroup of ℓ is finite. This proves ℓ and V intersect only finitely many times.

Now we prove (iii) $\implies$ (ii). Suppose that V is not compact. Then $\overline{V}$ is a compact torus foliated by parallel copies of V. Let

$$\psi : U \to \mathbb{R}^k \times \mathbb{R}^{n-k}$$

be a chart near $0 \in T$ such that the slices $\mathbb{R}^k \times \{y\}$, $y \in \mathbb{R}^{n-k}$, correspond to the local leaves of the foliation. Since V is dense in $\overline{V}$, there are infinitely many values of y such that $\mathbb{R}^k \times y \subsetneq \psi(V)$.

Now choose a rational line ℓ in $\overline{V}$ that is not contained in V but with $0 \in \ell$. If the neighborhood U above is chosen sufficiently small, then $\psi(\ell \cap U)$ intersects all leaves $\mathbb{R}^k \times y$. Since there are infinitely many values of y such that $\mathbb{R}^k \times y \subseteq \psi(V)$, it follows that $\ell \cap V$ is infinite. □

Recall that we are trying to show that the image under f of any rational subtorus is again a rational subtorus. We first show the image is a plane.

Claim 2.4. Let S be any rational subtorus of dimension k. Then $f(S)$ is a k-plane.

Proof. We induct on $k = \dim(S)$. The base case $k = 1$ is just the assertion that f maps lines to lines.

Suppose now the claim holds for l-planes where $l < k$, and let S be a rational k-dimensional subtorus. Choose some $x_0 \in S$ and let $S_0 := S - x_0$ be the translate of S passing through 0. Let $V_0 \subseteq \mathbb{R}^n$ be the subspace that projects to S_0. Since S is rational, we can choose a basis $v_1, \ldots, v_k$ of V with $v_i \in \mathbb{Q}^n$ for every i.

To use the inductive hypothesis, consider the $(k-1)$-plane

$$V_1 = \text{span}\{v_1, ..., v_{k-1}\}$$

and similarly $V_2 = \text{span}\{v_2, ..., v_k\}$. For $i = 1, 2$, set $S_i := [V_i + x_0]$. Since v_i are rational vectors for each i, we clearly have that S_1 and S_2 are rational $(k-1)$-dimensional subtori. Also let $S_{12} := S_1 \cap S_2$ be the intersection, which is a $(k-2)$-dimensional rational subtorus. The inductive hypothesis implies that $f(S_1)$ and $f(S_2)$ and $f(S_{12})$ are all rational subtori of T.

Since $f(S_1)$ and $f(S_2)$ are $(k-1)$-planes that intersect in a $(k-2)$-plane, they span a k-dimensional plane. More precisely, let W_1 (resp. W_2) be the subspace of $\mathbb{R}^n$ that projects to $f(S_1) - f(x_0)$ (resp. $f(S_2) - f(x_0)$). Then W_1 and W_2 are $(k-1)$-dimensional subspaces of $\mathbb{R}^n$ that intersect in a $(k-2)$-dimensional subspace, and hence span a k-dimensional subspace W_0. Then the k-plane $W := [W_0] + f(x_0)$ contains both $f(S_1)$ and $f(S_2)$.

We claim that $f(S) = W$. We start by showing the inclusion "$\subseteq$". Let $x \in S$. If $x \in S_1 \cup S_2$, then clearly $f(x) \in W$, so we will assume that $x \in S$ but $x \notin S_1 \cup S_2$.

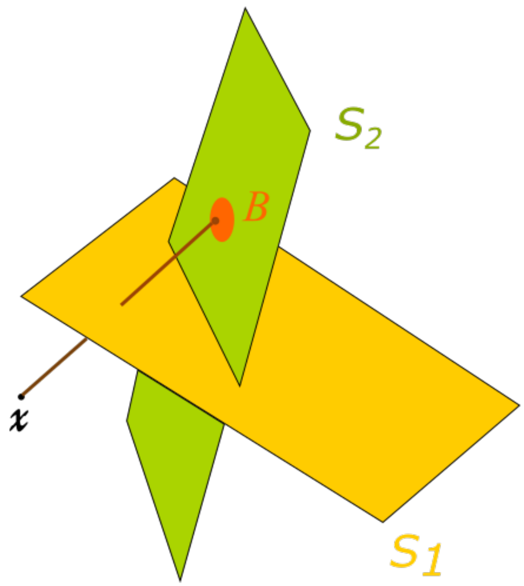

FIGURE 1. A ball $B \subseteq S_2$ such that lines from x to B meet S_1.

Since S_1 and S_2 are closed codimension 1 submanifolds of S, there is an open ball $B \subseteq S_2$ such that for any $y \in B$, there is a line joining x and y that intersects S_1 (see Figure 1). Set $Y := f(B)$. Since f sends lines to lines and preserves intersections, $f(x)$ is a point with the property that for any $y \in Y$, there is a line ℓ_y joining $f(x)$ and y that intersects $f(S_1)$. We claim that this implies that $f(x) \in W$.

Write $U_1 := f(S_1) - f(x_0)$ for the translate of $f(S_1)$ that passes through the origin. We can regard U_1 as a subgroup of T, and consider the projection

$$\pi : T \to T/U_1.$$

The image $\pi(W)$ of W is a line because $f(S_1)$ has codimension 1 in W. Further $\pi(f(x))$ is a point with the property that for any $y \in Y$, the line $\pi(\ell_y)$ joins $\pi(f(x_0))$ and $\pi(f(x))$ and intersects $\pi(W)$ at the point $\pi(y)$.

Note that there are only countably many lines from $\pi(f(x_0))$ to $\pi(f(x))$, and unless one of them is contained in $\pi(W)$, each one has at most countably many intersections with the line $\pi(W)$. However, since $\pi(Y)$ is uncountable, we conclude that not every point $\pi(y)$ can lie on a line that passes through both $\pi(f(x_0))$ and $\pi(f(x))$, unless $\pi(f(x)) \in \pi(W)$. Therefore we must have that $\pi(f(x)) \in \pi(W)$, so that $f(x) \in W$. This proves that $f(S) \subseteq W$.

To establish the reverse inclusion, we just apply the same argument to f^{-1}. The above argument then yields that $f^{-1}(W) \subseteq S$. Applying f gives $W \subseteq f(S)$, as desired. □

Actually the above proof also shows the following more technical statement, which basically states that linearly independent lines are mapped to linearly independent lines. We will not need this until Section 4, but it is most convenient to state it here.

Lemma 2.5. *Let $S = [V]$ be a rational subtorus containing 0, where $V \subseteq \mathbb{R}^n$ is a subspace. For $1 \leq i \leq k := \dim(S)$, let $v_i \in \mathbb{Q}^n$ such that $v_1, \dots, v_k$ is a basis for V. Set $\ell_i := [\mathbb{R}v_i]$ and choose $w_i \in \mathbb{Q}^n$ such that $f(\ell_i) = [\mathbb{R}w_i]$. Then*

$$f(S) = [\operatorname{span}(w_1, \dots, w_k)].$$

Proof. Just recall that (with the notation of the proof of Claim 2.4), we have

$$W = \operatorname{span}(w_1, \dots, w_k),$$

and we have shown $f(S) = [W]$, as desired. □

It remains to show that if S is a rational subtorus, then $f(S)$ is also rational. We first show this for S of codimension 1.

Claim 2.6. Let $S \subset T$ be a rational codimension 1 subtorus. Then $f(S)$ is also rational.

Proof. Let $S \subseteq T$ be a codimension 1 rational subtorus and let ℓ be any rational line not contained in S. By Lemma 2.3 applied to S, we see that $\ell \cap S$ is finite. Then $f(\ell)$ and $f(S)$ also intersect only finitely many times.

Suppose now that $f(S)$ is not rational. Then $\overline{f(S)}$ is a compact torus properly containing the codimension 1 plane $f(S)$, and therefore $\overline{f(S)} = T$, i.e. $f(S)$ is dense in T. But if $f(S)$ is dense, then it intersects any line that is not parallel to S infinitely many times. We know that $f(\ell)$ is not parallel to $f(S)$, because $f(\ell)$ and $f(S)$ intersect at least once. On the other hand, $f(\ell)$ and $f(S)$ intersect finitely many times. This is a contradiction. □

Finally we can finish the proof of Proposition 2.2.

Claim 2.7. Let S be a rational subtorus. Then $f(S)$ is a rational subtorus.

Proof. First note that if S_1 and S_2 are rational subtori, then any component of $S_1 \cap S_2$ is also rational (e.g. by using that rationality is equivalent to compactness).

Now let S be any rational subtorus of codimension l. Then we can choose l rational codimension 1 subtori $S_1, \dots, S_l$ such that S is a component of $\cap_i S_i$. Since f is a bijection, we have

$$f(S) \subseteq f\left(\bigcap_{1\leq i\leq l} S_i\right) = \bigcap_{1\leq i\leq l} f(S_i)$$

and by Claim 2.6, we know that $f(S_i)$ are rational. Therefore $f(S)$ is a codimension l-plane contained in $\cap_i f(S_i)$. The components of $\cap_i f(S_i)$ have codimension l, so we must have that $f(S)$ is a component, and hence is rational. □

3. The two-dimensional case

The goal of this section is to prove Theorem 1.4 in the two-dimensional case. First recall the following elementary computation of the number of intersections of a pair of rational lines.

Proposition 3.1. *Let ℓ_1 and ℓ_2 be two affine rational lines in the torus. For $i = 1, 2$, let $v_i \in \mathbb{Z}^2$ be a primitive tangent vector to the translate of ℓ_i passing through $0 \in T^2$. Then the number of intersections of ℓ_1 and ℓ_2 is given by*

$$|\ell_1 \cap \ell_2| = \left| \det \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix} \right|.$$

We now turn towards proving Theorem 1.4 in the two-dimensional case. For the rest of this section, suppose $f : T^2 \to T^2$ is a bijection that maps lines to lines. Also let us fix the following notation: For $i = 1, 2$, set $\ell_i := [\mathbb{R} e_i]$. We first make some initial reductions: By replacing f with

$$x \mapsto f(x) - f(0),$$

we can assume that $f(0) = 0$. For the next reduction, we need the following claim.

Claim 3.2. There is a linear automorphism $A : T^2 \to T^2$ such that $A\ell_i = f(\ell_i)$.

Proof. Since $f(0) = 0$, the lines $f(\ell_i)$ pass through 0. In addition, because ℓ_i are rational, so are $f(\ell_i)$ (see Proposition 2.2). Therefore there are coprime integers p_i and q_i such that

$$f(\ell_i) = \left[\mathbb{R} \begin{pmatrix} p_i \\ q_i \end{pmatrix} \right].$$

Note that ℓ_1 and ℓ_2 intersect exactly once, and hence so do $f(\ell_1)$ and $f(\ell_2)$. By Proposition 3.1, the number of intersections is also given by

$$|f(\ell_1) \cap f(\ell_2)| = \left| \det \begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix} \right|,$$

so the linear transformation $A : T^2 \to T^2$ with matrix

$$A = \begin{pmatrix} p_1 & p_2 \\ q_1 & q_2 \end{pmatrix}$$

is an automorphism, and clearly satisfies $A\ell_i = f(\ell_i)$. □

Let A be as in Claim 3.2. Then by replacing f by $A^{-1} \circ f$, we can assume that $f(\ell_i) = \ell_i$ for $i = 1, 2$. Note that because each v_i is only unique up to sign, the matrix A is not canonically associated to f. Therefore we cannot expect that $A^{-1} \circ f = \mathrm{id}$, and we have to make one further reduction to deal with the ambiguity in the definition of A. Let us introduce the following notation: For a rational line $\ell = \left[\mathbb{R}\begin{pmatrix} p \\ q \end{pmatrix}\right]$, let $\reflectbox{\ensuremath{\ell}}$ be the line $\left[\mathbb{R}\begin{pmatrix} p \\ -q \end{pmatrix}\right]$. We will make a final reduction below, for which we need the following result.

Claim 3.3. Let ℓ be a rational line in T^2 passing through the origin. Then $f(\ell) = \ell$ or $f(\ell) = \reflectbox{\ensuremath{\ell}}$.

Proof. Let p, q be coprime integers such that $\ell = \left[\mathbb{R}\begin{pmatrix} p \\ q \end{pmatrix}\right]$. Also choose coprime integers r, s such that $f(\ell) = \left[\mathbb{R}\begin{pmatrix} r \\ s \end{pmatrix}\right]$. We can compute p, q, r, s as suitable intersection numbers. Indeed, we have

$$|p| = \left|\det\begin{pmatrix} p & 0 \\ q & 1 \end{pmatrix}\right| = |\ell \cap \ell_2|.$$

Since $f(\ell_2) = \ell_2$ and f preserves the number of intersections of a pair of lines, we see that

$$|p| = |\ell \cap \ell_2| = |f(\ell) \cap \ell_2| = |r|,$$

and similarly $|q| = |s|$. Therefore we find that

$$\mathbb{R}\begin{pmatrix} p \\ q \end{pmatrix} = \mathbb{R}\begin{pmatrix} r \\ s \end{pmatrix} \qquad \text{or} \qquad \mathbb{R}\begin{pmatrix} p \\ -q \end{pmatrix} = \mathbb{R}\begin{pmatrix} r \\ s \end{pmatrix},$$

which exactly corresponds to $\ell = f(\ell)$ or $\reflectbox{\ensuremath{\ell}} = f(\ell)$. □

We will now show that whichever of the two alternatives of Claim 3.3 occurs does not depend on the line ℓ chosen.

Claim 3.4. Let ℓ be a rational line passing through the origin that is neither horizontal nor vertical (i.e. $\ell \neq \ell_i$ for $i = 1, 2$). Suppose that $f(\ell) = \ell$. Then for any rational line m passing through the origin, we have $f(m) = m$.

Proof. Let p, q coprime integers such that $\ell = \left[\mathbb{R}\begin{pmatrix} p \\ q \end{pmatrix}\right]$. Since ℓ is neither vertical nor horizontal, we know that p and q are nonzero.

Now let m be any other rational line, and choose coprime integers r, s such that $m = \left[\mathbb{R}\begin{pmatrix} r \\ s \end{pmatrix}\right]$. The number of intersections of ℓ and m is given by

$$|\ell \cap m| = \left|\det\begin{pmatrix} p & r \\ q & s \end{pmatrix}\right| = |ps - qr|.$$

We will argue by contradiction, so suppose that $f(m) \neq m$. Using the other alternative of Claim 3.3 for m, we have that

$$|f(\ell) \cap f(m)| = \left|\det \begin{pmatrix} p & r \\ q & -s \end{pmatrix}\right| = |ps + qr|.$$

Since f preserves the number of intersections, we therefore have

$$|ps - qr| = |ps + qr|$$

which can only happen if $ps = 0$ or $qr = 0$. Since we know that p and q are nonzero, we must have $r = 0$ or $s = 0$. This means that m is horizontal or vertical, but in those cases the alternatives provided by Claim 3.3 coincide. □

We can now make the final reduction: If $f(\ell) = \ell$ for any rational line ℓ, then we leave f as is. In the other case, i.e. $f(\ell) = \reflectbox{$\ell$}$ for any rational line ℓ, we replace f by

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \circ f,$$

after which we have $f(\ell) = \ell$ for any rational line ℓ passing through the origin. After making these initial reductions, our goal is now to prove that $f = \mathrm{id}$ on T^2. We will first show that this holds on the rational points $T^2(\mathbb{Q}) = \mathbb{Q}^2/\mathbb{Z}^2$. We start with the following easy observation:

Claim 3.5. If ℓ and m are parallel lines in T, then $f(\ell)$ and $f(m)$ are also parallel.

Proof. Since $\dim(T) = 2$, two lines in T are parallel if and only if they do not intersect. This property is obviously preserved by f. □

Combining our reductions on f so far with Claim 3.5 yields the following quite useful property for f: Given any rational line ℓ, the line $f(\ell)$ is parallel to ℓ.

However, we still do not know the behavior of f in the direction of ℓ or transverse to ℓ. In particular we do not know

- whether or not f leaves invariant every line ℓ; we only know this for ℓ passing through the origin, and
- whether or not $f = \mathrm{id}$ on rational lines ℓ passing through the origin.

We can now prove:

Theorem 3.6. *For any rational point $x \in \mathbb{Q}^2/\mathbb{Z}^2$ we have $f(x) = x$.*

Proof. For $n > 2$, set

$$G_n = \left[\frac{1}{n}\mathbb{Z}^2\right].$$

Clearly we have $T^2(\mathbb{Q}) = \cup_n G_n$, so it suffices to show that $f = \mathrm{id}$ on G_n for every $n > 2$. We purposefully exclude the case $n = 2$ because G_2 only consists of 4 points, and this is not enough for the argument below. However, since we have $G_2 \subseteq G_6$, this does not cause any problems. Fix $n \geq 2$.

Set $Q := \left[\frac{1}{n}\begin{pmatrix}1\\1\end{pmatrix}\right]$.

Claim 3.7. If $f(Q) = Q$, then $f = \mathrm{id}$ on G_n.

Proof. Suppose that $x = \begin{bmatrix}x_1\\y_1\end{bmatrix}$ is any point of G_n with $f(x) \neq x$. We can choose x_1 and y_1 that lie in $[0, 1)$. We will show that $f(Q) \neq Q$.

Let $x_2, y_2 \in [0, 1)$ be such that $f(x) = \begin{bmatrix}x_2\\y_2\end{bmatrix}$. Then there are two cases: Either $x_1 \neq x_2$ or $y_1 \neq y_2$. Clearly the entire setup is symmetric in the two coordinates, so we will just consider the case $x_1 \neq x_2$. See Figure 2 for an illustration of the various lines introduced below.

For $a \in [0, 1)$, let v_a (resp. h_a) denote the vertical (resp. horizontal) line $\big[x = a\big]$ (resp. $\big[y = a\big]$). Since f maps parallel lines to parallel lines by Claim 3.5 and $f(\ell_i) = \ell_i$ by assumption, we know that f maps vertical (resp. horizontal) lines to vertical (resp. horizontal) lines. In particular we have that $f(v_{x_1}) = v_{x_2}$ and $f(h_{y_1}) = h_{y_2}$. We will show that $f(Q) \neq Q$ by showing that $f(h_{1/n}) \neq h_{1/n}$.

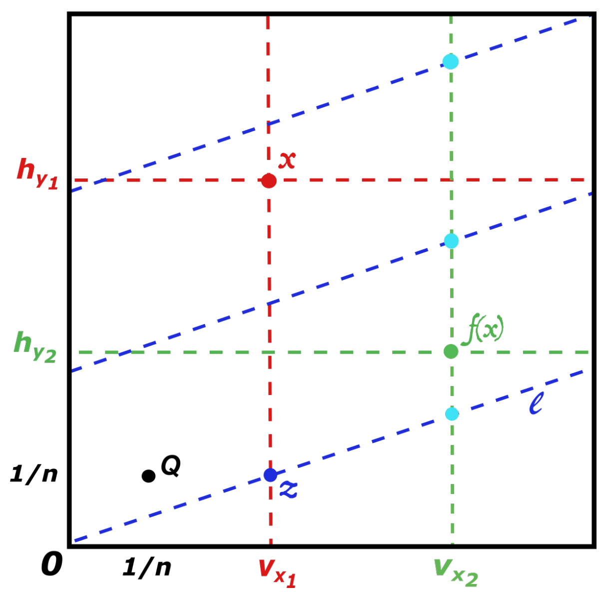

FIGURE 2. Illustration of the proof of Claim 3.7. The points in aqua are the possibilities for $f(z)$. The proof amounts to showing all of these lie at different heights than z does.

Consider the point $z := \left(x_1, \frac{1}{n}\right) \in T^2$. Since $x \in G_n$, there is an integer $0 \leq a < n$ such that $x_1 = \frac{a}{n}$. Hence z lies on the line ℓ of slope $\frac{1}{a}$ going through the origin. By Claim 3.4 we have $f(\ell) = \ell$, so $f(z) \in \ell$.

In addition, since $z \in v_{x_1}$, we have $f(z) \in v_{x_2}$. An easy computation yields

$$\ell \cap v_{x_2} = \begin{bmatrix} x_2 \\ \frac{x_2}{a} \end{bmatrix} + \frac{1}{a} \begin{bmatrix} 0 \\ \mathbb{Z} \end{bmatrix}.$$

Therefore we automatically have $f(h_{\frac{1}{n}}) \neq h_{\frac{1}{n}}$ unless

$$\frac{1}{n} x_2 + \frac{k}{a} = \frac{1}{n}$$

for some integer k. Solving for k yields

$$k = \frac{a}{n}(1 - x_2).$$

Since $0 \leq a < n$ and $0 \leq x_2 < 1$, we see that the only option is that $k = 0$. This means that $x_2 = 1$ or $a = 0$. Since $x_2 < 1$, we must therefore have $a = 0$. Hence

$$x_1 = \frac{a}{n} = 0,$$

so $x \in \ell_2$. Since $f(\ell_2) = \ell_2$, we see that $x_2 = 0$ as well. This contradicts the initial assumption that $x_1 \neq x_2$. □

It remains to show that $f(Q) = Q$. Consider the points P, R, S as shown in Figure 3. Let us give a brief sketch of the idea so that the subsequent algebraic manipulations are more transparent.

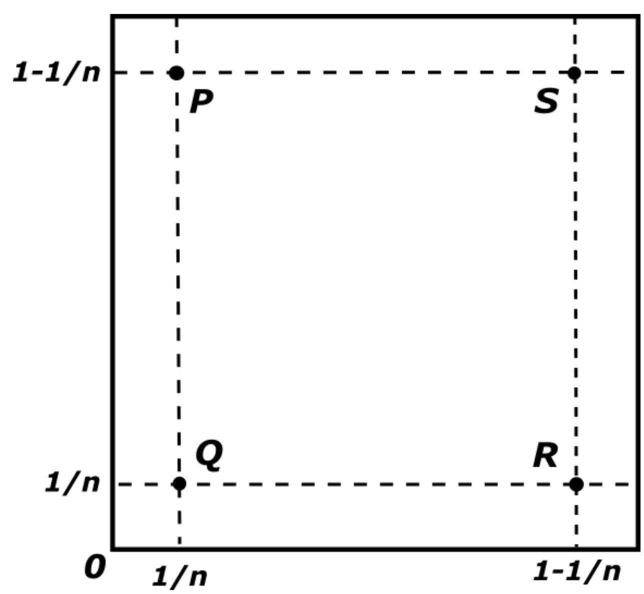

FIGURE 3. Quadrilateral in T^2 with vertices P, Q, R, S.

Because P and Q lie on a vertical line, and f maps vertical lines to vertical lines, $f(P)$ and $f(Q)$ have to lie on the same vertical line. Because P lies on the line of slope $n - 1$ through the origin, $f(P)$ also has to lie on this line. Therefore however much P and $f(P)$ differ in the horizontal direction determines how much they differ in the vertical direction. Finally since P and S lie on a horizontal line, the vertical difference between P and $f(P)$ equals the vertical difference between S and $f(S)$. The upshot of this is that any difference between Q and $f(Q)$ translates into information about the vertical difference between S and $f(S)$.

The same reasoning with P replaced by R yields information about the horizontal difference between S and $f(S)$. Finally because S lies on a line of slope 1 through the origin, any horizontal difference between S and $f(S)$ matches the vertical difference. Therefore the information gained about the difference between S and $f(S)$ using the two methods (once using P and once using R) has to match. We will see that this forces $Q = f(Q)$.

Let us now carry out the calculations. Choose $\delta \in \left(\frac{-1}{n}, \frac{n-1}{n}\right)$ such that

$$f(v_{PQ}) = v_{PQ} + \begin{bmatrix} \delta \\ 0 \end{bmatrix},$$

so that $f(P)$ and $f(Q)$ have first coordinate $\frac{1}{n} + \delta$. Note that the boundary cases $\delta = -\frac{1}{n}$ and $\delta = \frac{n-1}{n}$ are impossible since these correspond to $f(v_{PQ}) = \ell_2$. Since Q lies on the line of slope 1 through the origin, we have

$$f(Q) = \begin{bmatrix} \frac{1}{n} + \delta \\ \frac{1}{n} + \delta \end{bmatrix} \tag{3.1}$$

Since P lies on the line of slope $n-1$ through the origin, there is some integer k_P such that

$$f(P) = \begin{bmatrix} \frac{1}{n} + \delta \\ \frac{n-1}{n} + \delta(n-1) - k_P \end{bmatrix}. \tag{3.2}$$

Here we choose k_P such that

$$0 \leq \frac{n-1}{n} + \delta(n-1) - k_P < 1.$$

Using that R lies on the line of slope $\frac{1}{n-1}$ through the origin, we similarly find an integer k_R such that

$$f(R) = \begin{bmatrix} \frac{n-1}{n} + \frac{\delta}{n-1} - k_R \\ \frac{1}{n} + \delta \end{bmatrix}.$$

Finally, using that S lies on the same horizontal line as P, and on the same vertical line as R, we find

$$f(S) = \begin{bmatrix} \frac{n-1}{n} + \frac{\delta}{n-1} - k_R \\ \frac{n-1}{n} + \delta(n-1) - k_P \end{bmatrix}. \tag{3.3}$$

Since S lies on the line of slope 1 through the origin, so does $f(S)$. Setting the two coordinates of $f(S)$ given by Equation 3.3 equal to each other, we find (after some simple algebraic manipulations):

$$k_P - k_R = \delta\left(n - 1 - \frac{1}{n-1}\right). \tag{3.4}$$

We can obtain one more equation relating k_P and δ by noting that P lies on the line of slope -1 through the origin, and hence so must $f(P)$. This means that $f(P)$ is of the form $(x, 1-x)$. Using Equation 3.2, this gives after some algebraic manipulations:

$$k_P = n\delta \tag{3.5}$$

We can use Equation 3.5 to eliminate k_P from Equation 3.4 to obtain:

$$k_R = \delta\left(1 + \frac{1}{n-1}\right). \tag{3.6}$$

Using that

$$-\frac{1}{n} < \delta < \frac{n-1}{n},$$

and $n > 2$ one easily sees that

$$-1 < k_R < 1.$$

Since k_R is also an integer, this forces $k_R = 0$. Combining this with Equation 3.6, we see that $\delta = 0$. Therefore $f(Q) = Q$, as desired. □

At this point we have $f = \mathrm{id}$ on the dense set $T^2(\mathbb{Q})$. We need to promote this to the entirety of the torus. To do so, we will establish a weak version of continuity for f, namely a homogeneity property (see Claim 3.10 below). Before doing so, we need to introduce a bit more notation.

We identify the torus (as a set) with $[0,1)^2$ via the obvious map

$$[0,1)^2 \to T^2, \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{bmatrix} x \\ y \end{bmatrix}.$$

Suppose $x = (x_0, y_0) \in T^2$ and write

$$f(x) = (x_1, y_1).$$

Since f maps vertical lines to vertical lines, the value of x_1 does not depend on y_0. Similarly, the value of y_1 does not depend on x_0. Hence for $i = 1, 2$, there are functions $f_i : [0,1) \to [0,1)$ such that

$$f(x, y) = (f_1(x), f_2(y)).$$

We actually have $f_1 = f_2$: Indeed, for any $x \in [0,1)$, consider the point $p := (x,x) \in T^2$. Since p lies on the line of slope 1 through the origin, so does $f(p) = (f_1(x), f_2(x))$. Hence $f_1 = f_2$. We will write $\sigma : [0,1) \to [0,1)$ for this map, so that

$$f(x,y) = (\sigma(x), \sigma(y))$$

for any $x, y \in [0,1)$. We can use this symmetry to show f preserves another geometric configuration, in addition to collinearity of points:

Definition 3.8. A *block* $B \subseteq [0,1)^2$ is a square all of whose sides are either horizontal or vertical.

Claim 3.9. Let $x_1, \dots, x_4$ be the vertices of a block B. Then $f(x_1), \dots, f(x_4)$ are also the vertices of a block.

Proof. Since f maps vertical lines to vertical lines, and horizontal lines to horizontal lines, it is clear that $f(x_1), \dots, f(x_4)$ are the vertices of a rectangle R all of whose sides are horizontal or vertical. We need to show R is a square. To do so, it suffices to show that one of the diagonals is a line of slope 1, i.e. two of the vertices lie on a line of slope 1.

Since B is a block, two of its vertices lie on a line of slope 1. f maps lines of slope 1 to lines of slope 1, so two of the vertices of R also lie on a line of slope 1. Hence R is a square. □

We are now able to establish a weak version of homogeneity for f:

Claim 3.10. Let $n \geq 1$ and $z \in [0, \frac{1}{n})$. Then

$$\sigma(nz) = n\,\sigma(z).$$

Proof. We induct on n, treating $n = 1$ and $n = 2$ as base cases. For $n = 1$, the statement is trivial. Now let $n = 2$ and $z \in [0, \frac{1}{2})$. Consider the block B with side lengths $2z$ and left bottom vertex placed at the origin (see Figure 4). We divide B into 4 smaller blocks $B_1, \dots, B_4$ as shown in the figure.

Then $f(B_1)$ is also a block by Claim 3.9. Its vertices are

$$(0, \sigma(z)), (0, \sigma(2z)), (\sigma(z), \sigma(z)), \text{ and } (\sigma(z), \sigma(2z)).$$

Hence we see one side has length $\sigma(z)$ and the other has length $|\sigma(2z) - \sigma(z)|$. Since $f(B_1)$ is a block, we must have

$$\sigma(z) = |\sigma(2z) - \sigma(z)|.$$

Therefore we either have $\sigma(2z) = 0$ or $\sigma(2z) = 2\sigma(z)$. If $\sigma(2z) = 2\sigma(z)$ then we are done, so let us assume that $\sigma(2z) = 0$. Since we already know that $\sigma(0) = 0$, and σ is a bijection, we must have $z = 0$. Therefore we also have $\sigma(2z) = 2\sigma(z)$ in this case. This completes the proof of the base cases.

Suppose now that $n > 2$, and assume the statement is true for all $m < n$. The proof is similar to the proof of the case $n = 2$ above. Let $z \in [0, \frac{1}{n})$ and let $B \subseteq [0,1)^2$ be the block with side lengths nz and left bottom vertex at

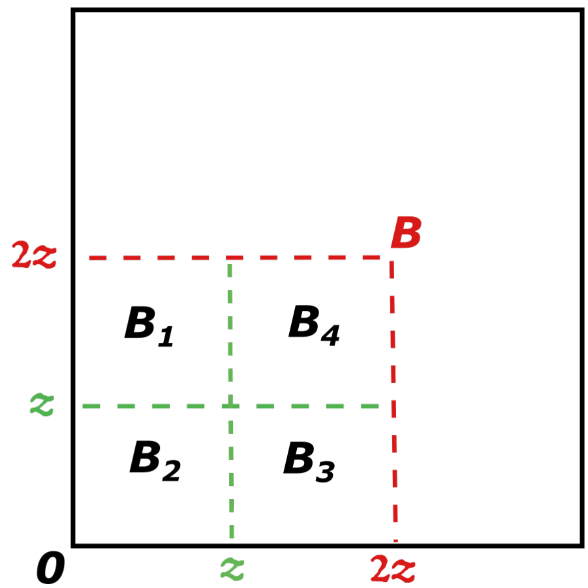

FIGURE 4. The block B with side lengths $2z$ and the origin as lower left vertex is divided into four smaller blocks.

0. Divide B into n^2 smaller blocks of equal size, each with side lengths z. Let B_1 be the top left one, i.e. the block with vertices

$$(0,(n-1)z),(0,nz),(z,(n-1)z),\text{ and }(z,nz).$$

Then $f(B_1)$ is the block with vertices

$$(0,\sigma((n-1)z)),(0,\sigma(nz)),(\sigma(z),\sigma((n-1)z)),\text{ and }(\sigma(z),\sigma(nz)).$$

Using that $\sigma((n-1)z) = (n-1)\sigma(z)$ by the inductive hypothesis, and comparing the side lengths of the block $f(B_1)$, we find that

$$\sigma(z) = |\sigma(nz) - (n-1)\sigma(z)|.$$

Hence either $\sigma(nz) = n\sigma(z)$ or $\sigma(nz) = (n-2)\sigma(z)$. In the former case, we are done. In the latter case, we show that $z = 0$ just as in the case $n = 2$. Hence either way we have $\sigma(nz) = \sigma(z)$. □

We now use the homogeneity property to establish a weak version of continuity of f near 0:

Corollary 3.11. *Let $n \geq 1$. Then $\sigma\left(\left[0,\frac{1}{n}\right)\right) \subseteq \left[0,\frac{1}{n}\right)$.*

Proof. Let $x \in \left[0,\frac{1}{n}\right)$. By Claim 3.10, we have

$$n\sigma(x) = \sigma(nx) \leq 1,$$

so $\sigma(x) \leq \frac{1}{n}$, as desired. □

To obtain a version of continuity at every point, instead of just at 0, we need the following equivariance property of f with respect to $T^2(\mathbb{Q})$:

Claim 3.12. Let $x, y \in [0,1)$ such that $x+y \in [0,1)$. Then

$$\sigma(x+y) = \sigma(x) + \sigma(y).$$

In particular, if y is in addition rational, then $\sigma(x+y) = \sigma(x) + y$.

Proof. The statement is trivial if $x = 0$ or $y = 0$, so we will assume that $x > 0$ and $y > 0$. Consider the block B with lower left-vertex $(0, y)$ and side lengths x. Then $f(B)$ is the block with vertices

$$(0, \sigma(y)), (\sigma(x), \sigma(y)), (0, \sigma(x+y)), \text{ and } (\sigma(x), \sigma(x+y)).$$

Therefore the side lengths of $f(B)$ are

$$\sigma(x) = |\sigma(x+y) - \sigma(y)|.$$

Hence either $\sigma(x+y) = \sigma(x)+\sigma(y)$ or $\sigma(x+y) = \sigma(y)-\sigma(x)$. We claim the latter case cannot occur. So suppose that we have $\sigma(x+y) = \sigma(y) - \sigma(x)$. By switching the roles of x and y in the above argument, we find that $\sigma(x+y) = \sigma(x) - \sigma(y)$. But σ only takes nonnegative values, so we must have $\sigma(x) = \sigma(y)$. Since σ is a bijection, we see that $x = y$. Then

$$\sigma(2x) = \sigma(x+y) = \sigma(x) - \sigma(y) = 0.$$

Hence $2x = 0$, so $x = 0$. But we assumed $x > 0$ to begin with. This is a contradiction.

Finally we note that the second statement in the claim (the case $y \in \mathbb{Q}$) immediately follows since $f = \mathrm{id}$ on $\mathbb{Q}^2/\mathbb{Z}^2$. □

Using all of the above, we can finish the proof of the two-dimensional case of the main theorem:

Proof of two-dimensional case of Theorem 1.4. We need to show that $f = \mathrm{id}$ on T^2. We have already shown that there exists a function $\sigma : [0,1) \to [0,1)$ such that

(1) $f(x,y) = (\sigma(x), \sigma(y))$ for all $x, y \in [0,1)$,
(2) If $n \geq 1$ and $0 \leq x < \frac{1}{n}$, then $\sigma(x) < \frac{1}{n}$, and
(3) If $x, y \in [0,1)$ such that $x+y \in [0,1)$ and y is rational, then $\sigma(x+y) = \sigma(x) + y$.

Let $x \in [0,1)$. Choose rational numbers x_n, $n \geq 1$ with

$$0 \leq x - x_n < \frac{1}{n}.$$

By Property (3), we have

$$\begin{aligned} \sigma(x) &= \sigma(x - x_n + x_n) \\ &= \sigma(x - x_n) + x_n. \end{aligned} \tag{3.7}$$

Property (2) gives bounds on $\sigma(x - x_n)$:

$$0 \leq \sigma(x - x_n) < \frac{1}{n}.$$

Combined with Equation 3.7, we then find

$$x_n \leq \sigma(x) < x_n + \frac{1}{n}.$$

Taking the limit as $n \to \infty$ gives $\sigma(x) = x$. □

4. The n-dimensional case

We finish the proof of Theorem 1.4 that any bijection of $T = \mathbb{R}^n/\mathbb{Z}^n$, $n \geq 2$, that maps lines to lines, is an affine map. We argue by induction on n. The base case $n = 2$ has been proven in the previous section. Let $f : T \to T$ be a bijection that maps lines to lines. We recall that in Section 2, we showed that for any rational subtorus $S \subseteq T$, the image $f(S)$ is also a rational subtorus.

Proof of Theorem 1.4. Without loss of generality, we assume $f(0) = 0$. For $1 \leq i \leq n$, we let

$$\ell_i := \left[\mathbb{R}e_i\right]$$

denote the (image of the) coordinate line. Each ℓ_i is a rational line, so $f(\ell_i)$ is also a rational line. Choose primitive integral vectors $v_i \in \mathbb{Z}^n$ such that

$$f(\ell_i) = \left[\mathbb{R}v_i\right].$$

Note that v_i is unique up to sign. Let A be the linear map of $\mathbb{R}^n$ with $Ae_i = v_i$ for every i. Our goal is to show that $f = A$ as maps of the torus.

Claim 4.1. *A is invertible with integer inverse.*

Proof. To show that A is invertible, we need to show that $v_1, \ldots, v_n$ span $\mathbb{R}^n$. To show that A^{-1} has integer entries, we need to show that in addition, $v_1, \ldots, v_n$ generate $\mathbb{Z}^n$ (as a group). For $1 \leq j \leq n$, set

$$U_j := \operatorname{span}\{e_i \mid i \leq j\},$$

and

$$V_j := \operatorname{span}\{v_i \mid i \leq j\}.$$

Note that $f[U_j] = [V_j]$ by Lemma 2.5. Taking $j = n$, this already shows that $v_1, \ldots, v_n$ span $\mathbb{R}^n$, so A is invertible. We argue by induction on j that $\{v_1, \ldots, v_j\}$ generate $\pi_1[V_j] \subseteq \pi_1 T$. For $j = n$, this exactly means that $v_1, \ldots, v_n$ generate $\pi_1 T$, which would finish the proof.

The base case $j = 1$ is exactly the assertion that v_1 is a primitive vector. Now suppose the statement is true for some j. Consider the composition

$$f(\ell_j) \hookrightarrow V_j \to V_j/V_{j-1}. \tag{4.1}$$

This composition is a homomorphism of $f(\ell_j) \cong S^1$ to $V_j/V_{j-1} \cong S^1$. The kernel is given by

$$f(\ell_j) \cap V_{j-1} = f(\ell_j \cap U_{j-1}) = f(0) = 0.$$

Therefore the map $f(\ell_j) \to V_j/V_{j-1}$ is an isomorphism, and

$$\pi_1(V_j) \cong \pi_1(V_{j-1}) \oplus \pi_1(f(\ell_j)).$$

Since we know by the inductive hypothesis that $\pi_1(V_{j-1})$ is generated by $v_1, \ldots, v_{j-1}$, and that $\pi_1(f(\ell_j))$ is generated by v_j (again because v_j is primitive), we find that $\pi_1(V_j)$ is generated by $v_1, \ldots, v_j$, as desired. □

For the remainder of the proof we set $g := A^{-1} \circ f$. Our goal is to show that $g = \mathrm{id}$. Note that g is a bijection of T with $g(0) = 0$ and $g(\ell_i) = \ell_i$ for every i. Let

$$H_i := \big[\mathrm{span}(e_j \mid j \neq i)\big]$$

denote the (image of the) coordinate hyperplane. Since $g(\ell_i) = \ell_i$ for every i, we know (using Lemma 2.5 again) that $g(H_i) = H_i$. By the inductive hypothesis, g is given by a linear map A_i on H_i. But any linear automorphism of H_i that leaves the coordinate lines ℓ_j, $j \neq i$, invariant, must be of the form

$$\begin{pmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{pmatrix}.$$

Now recall that v_i was unique up to sign. Replace v_i by $-v_i$ whenever $g|_{\ell_i} = -\mathrm{id}$. After this modification, we have that $g = \mathrm{id}$ on every coordinate hyperplane H_i. Our goal is to show that $g = \mathrm{id}$ on T.

Claim 4.2. Fix $1 \leq i \leq n$ and let ℓ be any line parallel to ℓ_i. Then $g(\ell) = \ell$.

Proof. First recall the following elementary general fact about the number of intersections of a line with a coordinate hyperplane: If m is any rational line and

$$v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

is a primitive integer vector with $m = [\mathbb{R}v]$, then

$$|v \cap H_i| = |v_i|.$$

Now let ℓ be parallel to ℓ_i. Then

$$|\ell \cap H_j| = \delta_{ij},$$

and hence also

$$|g(\ell) \cap H_j| = |g(\ell) \cap g(H_j)| = |\ell \cap H_j| = \delta_{ij}.$$

Therefore we see that if v is a primitive integer vector with $g(\ell) = [\mathbb{R}v]$, then $v = e_i$. This exactly means that ℓ and $g(\ell)$ are parallel. Hence to show that $\ell = g(\ell)$, it suffices to show that $\ell \cap g(\ell)$ is nonempty.

Since ℓ is parallel to ℓ_i, there is a unique point of intersection $x_\ell := \ell \cap H_i$. Since $x_\ell \in H_i$, we have $g(x_\ell) = x_\ell$, so $x_\ell \in \ell \cap g(\ell)$, as desired. □

We are now able to finish the proof of the main theorem. Let $x \in T$ and let $\ell_i(x)$ be the unique line parallel to ℓ_i that passes through x. For any two distinct indices $i \neq j$, the point x is the unique point of intersection of $\ell_i(x)$ and $\ell_j(x)$. On the other hand,

$$g(x) \in g(\ell_i(x)) \cap g(\ell_j(x)) = \ell_i(x) \cap \ell_j(x),$$

so we must have $g(x) = x$. □

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