Symmetric and Asymmetric Centroid Bodies of the Regular Simplex

Student: Ryan Britton
Advisor: Beatrice-Helen Vritsiou

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Abstract

The centroid bodies of an n-dimensional convex body K are a family of bodies that are useful in studying the uniform distribution on K, since they encode information about its moments. In this report we find the approximate form of both the symmetric and the asymmetric centroid bodies of an n-dimensional regular simplex, which is an n-dimensional convex body with n+1 evenly spaced vertices. The methods used are probabilistic and are largely inspired by methods that have been used for bodies with more symmetries than the simplex.

1 Basic Notions

We begin by reviewing a few key terms. A convex body is a compact convex set with nonempty interior; convex bodies are the main object of study in convex geometry. A common operation is to take the convex hull, here denoted *conv*, which produces the smallest convex set containing the input set. If $x_1, ..., x_m$ are points, then the point y is a convex combination of them if

$$y = \sum_{j=1}^{m} \lambda_j x_j,$$

where for all j, $\lambda_j \geq 0$ and

$$\sum_{j=1}^{m} \lambda_j = 1.$$

Taking the convex hull of a set of points is the same as taking all convex combinations of those points. If $K \subset \mathbb{R}^m$ is a convex body, then the support function of K, here denoted h_K, is defined by

$$h_K(a) = \max_{x \in K} \langle a, x \rangle$$

for any $a \in \mathbb{R}^m$ (here and elsewhere, angled brackets denote the standard inner product on $\mathbb{R}^m$). It is a well-known fact that if two such convex bodies have the same support function, then they are identical.

In general, an n-dimensional simplex is an n-dimensional convex body with n+1 vertices, and the simplex is regular if the vertices are evenly spaced, i.e. the distance between any two of them is the same. For our purposes, it is helpful to consider, in particular, the following sets.

Definition 1.1. Define

$$A_n := conv\{e_1, ..., e_{n+1}\} \subset \mathbb{R}^{n+1}$$

(here $e_1, ..., e_{n+1}$ refer to the standard basis vectors). Furthermore, define

$$S_n := A_n - \left(\frac{1}{n+1}, ..., \frac{1}{n+1}\right).$$

For volume normalization, define

$$c_n := \left(\frac{n!}{\sqrt{n+1}} \right)^{\frac{1}{n}}$$

and

$$\tilde{S_n} := c_n S_n.$$

For any $x \in S_n$,

$$\langle x, (1, ..., 1) \rangle = \sum_{j=1}^{n+1} \left(x_j - \frac{1}{n+1} \right),$$

where

$$\sum_{j=1}^{n+1} x_j = 1$$

(this is because x is a convex combination of the standard basis vectors). Therefore, $\langle x, (1, ..., 1) \rangle = 0$ and $S_n \subset H_n := (1, ..., 1)^{\perp}$. Henceforth, we think of H_n and $H_n + \left(\frac{1}{n+1}, ..., \frac{1}{n+1} \right)$ as n-dimensional Euclidean space with the Lebesgue measure. Thus, we can speak of certain subsets of them as convex bodies and measure their n-dimensional volume, as well as integrate over H_n. In particular, we can say that A_n, S_n, and $\tilde{S_n}$ are regular simplices.

The following facts are not original to this report, but they and their proofs are included for completeness.

Proposition 1.2. $vol(\tilde{S_n}) = 1$

Proof. It is enough to show that $vol(S_n) = \frac{\sqrt{n+1}}{n!}$. For this, first recall that the m-dimensional cross-polytope of radius 1, which we denote B_1^m is the set containing every point in $\mathbb{R}^m$ whose l_1-norm, i.e. the sum of the absolute value of its coordinates, is less than or equal to 1. Now consider the pyramids $P_n := B_1^n \cap \mathbb{R}_+^n$. Note that $P_{n+1} = conv\{P_n, (0, ..., 0, 1)\}$ (where we embed P_n in $\mathbb{R}^{n+1}$ as a subset of $(0, ..., 0, 1)^{\perp}$). Since $vol(P_0) = 1$ and $vol(P_{n+1}) = \frac{1}{n+1} vol(P_n)$, $vol(P_n) = \frac{1}{n!}$. We also have that $P_{n+1} = conv\{A_n, (0, ..., 0)\}$, so $vol(P_{n+1}) = \frac{1}{n+1} vol(A_n) \sqrt{(\frac{1}{n+1})^2 + ... + (\frac{1}{n+1})^2} = \frac{1}{(n+1)^{\frac{3}{2}}} vol(A_n)$. Thus, $vol(A_n) = \frac{(n+1)!}{(n+1)^{\frac{3}{2}}} = \frac{n!}{\sqrt{n+1}}$. Since S_n is a translation of A_n, $vol(S_n) = vol(A_n)$, which completes the proof. □

Proposition 1.3. *S_n is centered at the origin, i.e. $bar(S_n) = (0, ..., 0)$*

Proof. It is enough to show that $b := bar(A_n) = (\frac{1}{n+1}, ..., \frac{1}{n+1})$. For this, note that if T is a linear transformation that permutes the standard basis vectors, then $T(A_n) = A_n$, since A_n is the same as $\{x \in \mathbb{R}_+^n : \sum_{j=1}^{n+1} x_j = 1\}$. Thus, b is a fixed point of T. The only point in A_n that is fixed by any such T is $(\frac{1}{n+1}, ..., \frac{1}{n+1})$. □

Corollary 1.3.1. *$\tilde{S_n}$ is centered at the origin.*

Now we introduce the symmetric and asymmetric centroid bodies. Their form for the regular simplex will be the subject of the rest of this report.

Definition 1.4. Given a probability measure μ on $\mathbb{R}^m$ and $p \geq 1$, define the p^{th} centroid body of μ, denoted $\mathcal{Z}_p(\mu)$, to be the convex body whose support function is given, for $a \in \mathbb{R}^m$, by

$$h_{\mathcal{Z}_p(\mu)}(a) = \left(\int |\langle a, y \rangle|^p d\mu(y) \right)^{\frac{1}{p}}$$

If $K \subset \mathbb{R}^m$ is a convex body with volume equal to 1, then let $\mathcal{Z}_p(K)$ denote $\mathcal{Z}_p(\mu_K)$, where μ_K is the Lebesgue measure restricted to K.

Definition 1.5. Given a probability measure μ on $\mathbb{R}^m$ and $p \geq 1$, define the p^{th} asymmetric centroid body of μ, denoted $\mathcal{Z}_p^+(\mu)$, to be the convex body whose support function is given, for $a \in \mathbb{R}^m$, by

$$h_{\mathcal{Z}_p^+(\mu)}(a) = \left(\int_{\langle a,y\rangle \geq 0} \langle a,y\rangle^p d\mu(y) \right)^{\frac{1}{p}}$$

If $K \subset \mathbb{R}^m$ is a convex body with volume equal to 1, then let $\mathcal{Z}_p^+(K)$ denote $\mathcal{Z}_p^+(\mu_K)$, where μ_K is the Lebesgue measure restricted to K.

The following useful equations are an easy consequence of the fact that $A_n \subset H_n + \left(\frac{1}{n+1}, ..., \frac{1}{n+1}\right)$. For $a \in H_n$,

$$h_{\mathcal{Z}_p(\tilde{S_n})}(a) = c_n \left(\frac{1}{vol(A_n)} \int_{A_n} |\langle a,y\rangle|^p dy \right)^{\frac{1}{p}} \tag{1}$$

and

$$h_{\mathcal{Z}_p^+(\tilde{S_n})}(a) = c_n \left(\frac{1}{vol(A_n)} \int_{\{y \in A_n : \langle a,y\rangle \geq 0\}} |\langle a,y\rangle|^p dy \right)^{\frac{1}{p}} \tag{2}$$

2 Probabilistic Representation

In this section, we introduce some helpful random variables and prove a key result relating the centroid bodies of the simplex to another family of centroid bodies.

Definition 2.1. Let $X_1, ..., X_{n+1}$ be independent random variables, whose distributions are given by

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ e^{-x} & \text{if } x \geq 0 \end{cases}$$

and let X be the random vector $(X_1, ..., X_{n+1})$. Furthermore, let ν be the probability measure on $\mathbb{R}$ whose density is given by $f(x+1)$.

Note that ν is centered, i.e. its mean is equal to 0. This follows from the fact that the mean of X_j is

$$\int_0^\infty x e^{-x} dx = \Gamma(2) = 1! = 1.$$

The following proposition illustrates the utility of the random variables just introduced.

Proposition 2.2. *The random vector $Y := \frac{X}{\|X\|_1}$ generates the uniform probability measure on A_n.*

Proof. Let $g_1, ..., g_{n+1}$ be independent, identically distributed random variables with density $\frac{e^{-|x|}}{2}$, and let $G := (g_1, ..., g_{n+1})$. Then, [2] and [3], the random vector $\frac{G}{\|G\|_1}$ generates the cone measure on the boundary of B_1^{n+1}, the cross-polytope. (For a symmetric convex body K in $\mathbb{R}^m$, the cone measure, which we call μ_K for now, is a measure on ∂K, defined for $A \subset \partial K$ by $\mu_K(A) = \frac{vol\{ta : a \in A, t \in [0,1]\}}{vol(K)}$.) The boundary of the cross-polytope lies in finitely many hyperplanes, and the perpendicular distance between each hyperplane and the origin is the same. Thus, the cone measure is the same as the uniform measure, if we consider subsets of the boundary to be subsets of n-dimensional Euclidean space. The result then follows from the fact that $A_n = \left(\partial B_1^{n+1}\right) \cap R_+^{n+1}$. ☐

Corollary 2.2.1. *For $a \in H_n$,*

$$h_{\mathcal{Z}_p(\tilde{S_n})}(a) = c_n (E|\langle a,Y\rangle|^p)^{\frac{1}{p}} \tag{3}$$

$$h_{\mathcal{Z}_p^+(\tilde{S_n})}(a) = c_n \left(\nu^{n+1}\left\{x \in \mathbb{R}^{n+1} : \langle a,x\rangle \geq 0\right\}\right)^{\frac{1}{p}} (E\langle a,Y\rangle^p | \langle a,Y\rangle \geq 0)^{\frac{1}{p}} \tag{4}$$

Proof. (3) follows immediately from (1), and (4) follows from (2), along with the fact that $P(\langle a, Y\rangle \geq 0) = P(\langle a, X\rangle \geq 0) = P(\langle a, X-(1,...,1)\rangle \geq 0) = \nu^{n+1}\{x \in \mathbb{R}^{n+1} : \langle a, x\rangle \geq 0\}$. □

We are about to give the main result for this section, but first we prove an easy proposition that will be used to prove the main result.

Proposition 2.3. *Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $Z_1 : \Omega \to S_1$ and $Z_2 : \Omega \to S_2$ be independent random variables (here S_1 and S_2 can be any measure spaces). Let $A \subset \Omega$ such that $P(A) \neq 0$ be the preimage under Z_1 of some measurable subset of S_1 and define a new probability measure $\tilde{P}$ by*

$$\tilde{P}(B) = \frac{P(A \cap B)}{P(A)},$$

i.e. by conditioning on A. Then Z_1 and Z_2 are independent with respect to $\tilde{P}$.

Proof. For measurable $C \subset S_1$ and $D \subset S_2$, we must show that $\tilde{P}\big(Z_1^{-1}(C)\big)\tilde{P}\big(Z_2^{-1}(D)\big) = \tilde{P}\big(Z_1^{-1}(C) \cap Z_2^{-1}(D)\big)$. This is easy to check. By independence, we have

$$\tilde{P}\big(Z_2^{-1}(D)\big) = \frac{P\big(Z_2^{-1}(D) \cap A\big)}{P(A)} = P\big(Z_2^{-1}(D)\big).$$

Thus,

$$\tilde{P}\big(Z_1^{-1}(C) \cap Z_2^{-1}(D)\big) = \frac{P\big(Z_1^{-1}(C) \cap Z_2^{-1}(D) \cap A\big)}{P(A)} = \frac{P\big(Z_1^{-1}(C) \cap A\big)P\big(Z_2^{-1}(D)\big)}{P(A)}$$
$$= \tilde{P}\big(Z_1^{-1}(C)\big)\tilde{P}\big(Z_2^{-1}(D)\big).$$

□

The following result will be used crucially in the remaining sections. In its statement, the $\sim$ symbol denotes an equivalence relation where for two indexed families of sets, A_i and B_i, there is some absolute constant C such that for all i, $\frac{1}{C}A_i \subset B_i \subset CA_i$. The same idea and notation are applied to families of numbers, with $\leq$ instead of $\subset$.

Proposition 2.4.

$$\mathcal{Z}_p\left(\tilde{S_n}\right) \sim \frac{c_n}{max\{n,p\}} proj_{H_n}\big(\mathcal{Z}_p(\nu^{n+1})\big) \tag{5}$$

$$\mathcal{Z}_p^+\left(\tilde{S_n}\right) \sim \frac{c_n}{max\{n,p\}} proj_{H_n}\big(\mathcal{Z}_p^+(\nu^{n+1})\big) \tag{6}$$

Proof. Much of this proof is based on the proof given in [1] of a similar result (Lemma 6). First of all, note that for any convex body $K \subset \mathbb{R}^{n+1}$ and any $a \in H_n$,

$$h_{proj_{H_n}(K)}(a) = h_K(a).$$

This is because

$$\max_{x\in K}\langle a, x\rangle = \max_{x\in K}\langle a, proj_{H_n}(x)\rangle = \max_{x\in proj_{H_n}(K)}\langle a, x\rangle.$$

By the one-to-one correspondence between support functions and convex bodies and the homogeneity of support functions, it is therefore enough to show that for any $a \in H_n$,

$$h_{\mathcal{Z}_p(\tilde{S_n})}(a) \sim \frac{c_n}{max\{n,p\}} h_{\mathcal{Z}_p(\nu^{n+1})}(a)$$

and

$$h_{\mathcal{Z}_p^+(\tilde{S_n})(a)} \sim \frac{c_n}{max\{n,p\}} h_{\mathcal{Z}_p^+(\nu^{n+1})}(a).$$

With notation as in the proof of Proposition 2.2, by [2] and [3], the random variable $\|G\|_1$ is independent from the random vector $\frac{G}{\|G\|_1}$. By Proposition 2.3, this implies that $\|X\|_1$ and $\frac{X}{\|X\|_1}$ are independent.

Furthermore, for a fixed $a \in \mathbb{R}^{n+1}$, if X_a is the random vector that comes from conditioning on $\left\langle \frac{X}{\|X\|_1}, a \right\rangle \geq 0$ (this condition is equivalent to $\langle X, a\rangle \geq 0$), then $\frac{X_a}{\|X_a\|_1}$ and $\|X_a\|_1$ are independent (this also follows from Proposition 2.3). Also, by independence of $\|X\|_1$ and $\left\langle \frac{X}{\|X\|_1}, a \right\rangle$, $E(\|X_a\|_1) = E(\|X\|_1)$.

For any $a \in \mathbb{R}^n$, we therefore have the following:

$$E|\langle a, Y\rangle|^p = E\left|\left\langle a, \frac{X}{\|X\|_1}\right\rangle\right|^p = \frac{1}{E\|X\|_1^p} E|\langle a, X\rangle|^p$$

and, similarly,

$$E\left(\langle a, Y\rangle^p | \langle a, Y\rangle \geq 0\right) = \frac{1}{E\|X\|_1^p} E\left(\langle a, X\rangle^p | \langle a, X\rangle \geq 0\right)$$

Now to calculate $\frac{1}{E\|X\|_1^p}$, fix $a = (1, 0, ..., 0)$. Then, by Proposition 2.2 and the geometry of A_n,

$$\begin{aligned} E|\langle a, Y\rangle|^p = \frac{1}{vol(A_n)} \int_{A_n} |\langle (1,0,...,0), y\rangle|^p dy &= \frac{1}{vol(A_n)} \int_0^1 t^p (1-t)^{n-1} vol(A_{n-1}) dt \\ &= \frac{vol(A_{n-1})}{vol(A_n)} B(p+1, n) = \frac{vol(A_{n-1})}{vol(A_n)} \frac{\Gamma(p+1)\Gamma(n)}{\Gamma(p+n+1)}. \end{aligned}$$

Also,

$$E|\langle (1,0,...,0), X\rangle|^p = \int_0^\infty t^p e^{-t} dt = \Gamma(p+1)$$

. Thus,

$$\frac{vol(A_{n-1})}{vol(A)} \frac{\Gamma(p+1)\Gamma(n)}{\Gamma(p+n+1)} = \frac{1}{E\|X\|_1^p} \Gamma(p+1)$$

and therefore,

$$\begin{aligned} \frac{1}{E\|X\|_1^p} = \frac{vol(A_{n-1})}{vol(A)} \frac{\Gamma(n)}{\Gamma(p+n+1)} = \frac{c_n^n}{c_{n-1}^{n-1}} \frac{\Gamma(n)}{\Gamma(p+n+1)} &= \sqrt{\frac{n}{n+1}}\, n \frac{\Gamma(n)}{\Gamma(p+n+1)} \\ &= \sqrt{\frac{n}{n+1}} \frac{\Gamma(n+1)}{\Gamma(p+n+1)}. \end{aligned}$$

By Stirling's Approximation,

$$\left(\sqrt{\frac{n}{n+1}} \frac{\Gamma(n+1)}{\Gamma(p+n+1)}\right)^{\frac{1}{p}} \sim \frac{1}{max\{n, p\}}.$$

Now, for any $a \in H_n$,

$$\begin{aligned} h_{\mathcal{Z}_p(\tilde{S_n})}(a) = c_n \left(E|\langle a, Y\rangle|^p\right)^{\frac{1}{p}} \sim \frac{c_n}{max\{n,p\}} \left(E|\langle a, X\rangle|^p\right)^{\frac{1}{p}} &= \frac{c_n}{max\{n,p\}} \left(E|\langle a, X - (1,...,1)\rangle|^p\right)^{\frac{1}{p}} \\ &= \frac{c_n}{max\{n,p\}} h_{\mathcal{Z}_p(\nu^{n+1})}(a). \end{aligned}$$

For the asymmetric centroid bodies, the computation is similar:

$$\begin{aligned} h_{\mathcal{Z}_p^+(\tilde{S_n})}(a) &= c_n \left(\nu^{n+1}\{x \in \mathbb{R}^{n+1} : \langle a, x\rangle \geq 0\} E\left(\langle a, Y\rangle^p | \langle a, x\rangle \geq 0\right)\right)^{\frac{1}{p}} \\ &\sim \frac{c_n}{max\{n,p\}} \left(\nu^{n+1}\{x \in \mathbb{R}^{n+1} : \langle a, x\rangle \geq 0\} E\left(\langle a, X\rangle^p | \langle a, X\rangle \geq 0\right)\right)^{\frac{1}{p}} \\ &= \frac{c_n}{max\{n,p\}} \left(\nu^{n+1}\{x \in \mathbb{R}^{n+1} : \langle a, x\rangle \geq 0\} E\left(\langle a, X\rangle^p | \langle a, X - (1,...,1)\rangle \geq 0\right)\right)^{\frac{1}{p}} \\ &= \frac{c_n}{max\{n,p\}} h_{\mathcal{Z}_p^+(\nu^{n+1})}(a). \end{aligned}$$

□

3 Symmetric Centroid Bodies

In this section, we find the form of the symmetric centroid bodies by first finding the form of $\mathcal{Z}_p\big(\nu^{n+1}\big)$ and using Proposition 2.3. To this end, we introduce the following notion.

Definition 3.1. Let μ be a probability measure on $\mathbb{R}^m$ that is absolutely continuous with respect to the Lebesgue measure. Let f denote the density of μ, and define $\tilde{\mu}$, the symmetrization of μ to be the probability measure whose density is given by

$$\tilde{f}(x) = \int_{\mathbb{R}^m} f(x-y)f(-y)dy$$

In other words, $\tilde{\mu}$ is the convolution of μ with its symmetric image.

The utility of this notion comes from the following proposition, which is helpful because the symmetrization of ν^{n+1} has been studied more extensively than ν^{n+1}.

Proposition 3.2. *If μ is a centered, absolutely continuous, log-concave probability measure on $\mathbb{R}^m$, then $\mathcal{Z}_p(\mu) \sim \mathcal{Z}_p(\tilde{\mu})$.*

Proof. Let f be the density of μ. Then, for any $a \in \mathbb{R}^m$,

$$\big(h_{\mathcal{Z}_p(\tilde{\mu})}\big)^p = \int |\langle a, x\rangle|^p d\tilde{\mu}(x) = \int_{\mathbb{R}^m} |\langle a, x\rangle|^p \int_{\mathbb{R}^m} f(x-y)f(-y)dydx$$
$$= \int_{\mathbb{R}^{2m}} |\langle a, z_1 + z_2\rangle|^p f(z_1)f(-z_2)d(z_1, z_2). \quad (7)$$

The final equality comes from the change of variables $z_1 = x - y$, $z_2 = y$, which preserves volume.

Note that

$$|\langle a, z_1 + z_2\rangle|^p \le (|\langle a, z_1\rangle| + |\langle a, z_2\rangle|)^p \le (2max\{|\langle a, z_1\rangle|, |\langle a, z_2\rangle|\})^p \le 2^p|\langle a, z_1\rangle|^p + 2^p|\langle a, z_2\rangle|^p.$$

From (7), we then have that

$$\big(h_{\mathcal{Z}_p(\tilde{\mu})}\big)^p \le 2^p \int_{\mathbb{R}^m} |\langle a, z_1\rangle|^p f(z_1) \int_{\mathbb{R}^m} f(-z_2)dz_2dz_1 + 2^p \int_{\mathbb{R}^m} |\langle a, z_2\rangle|^p f(-z_2) \int_{\mathbb{R}^m} f(z_1)dz_1dz_2$$
$$= 2^p \left(\int_{\mathbb{R}^m} |\langle z_1, a\rangle|^p f(z_1) + \int_{\mathbb{R}^m} |\langle -z_2, a\rangle|^p f(-z_2)\right) = 2^{p+1}\big(h_{\mathcal{Z}_p(\tilde{\mu}}\big)^p$$

Thus, $h_{\mathcal{Z}_p(\tilde{\mu})} \le 4h_{\mathcal{Z}_p(\mu)}$.

For the other inequality, define $A := \{(z_1, z_2) \in \mathbb{R}^{2m} : \langle a, z_1\rangle$ and $\langle a, z_2\rangle$ have the same sign or one of them is $0\}$ and note that for any positive $b, c \in \mathbb{R}$, $(b + c)^p \ge b^p + c^p$. In the following chain of inequalities, we use the fact that the measure of any halfspace containing the centroid (here the origin) is greater than or equal to $\frac{1}{e}$ (this is a direct consequence of Grünbaum's Lemma, originally proved in [5]). We use ϵ_b to denote a number (equal to 0, 1, or -1) that depends only on b. Now, from (7), we have that

$$\Big(h_{\mathcal{Z}_p(\tilde{\mu})}\Big)^p \ge \int_{\mathbb{R}^{2m}\cap A} |\langle a, z_1\rangle + \langle a, z_2\rangle|^p f(z_1)f(-z_2)dz_1dz_2$$
$$\ge \int_{\mathbb{R}^{2m}\cap A} |\langle a, z_1\rangle|^p f(z_1)f(-z_2)dz_1dz_2 + \int_{\mathbb{R}^{2m}\cap A} |\langle a, z_2\rangle|^p f(z_1)f(-z_2)dz_1dz_2$$
$$\ge \int_{\mathbb{R}^m} |\langle a, z_1\rangle|^p f(z_1) \int_{\langle \tilde{z_2}, \epsilon_{z_1} a\rangle \ge 0} f(\tilde{z_2})d\tilde{z_2}dz_1 + \int_{\mathbb{R}^m} |\langle a, \tilde{z_2}\rangle|^p f(\tilde{z_2}) \int_{\langle z_1, \epsilon_{z_2} a\rangle \ge 0} f(z_1)dz_1d\tilde{z_2}$$
$$\ge \frac{1}{e}\int_{\mathbb{R}^m} |\langle a, z_1\rangle|^p f(z_1)dz_1 + \frac{1}{e}\int_{\mathbb{R}^m} |\langle a, \tilde{z_2}\rangle|^p f(\tilde{z_2})d\tilde{z_2} = \frac{2}{e}(h_{\mathcal{Z}_p(\mu)(a)})^p$$

Thus, $h_{\mathcal{Z}_p(\tilde{\mu})} \ge \frac{2}{e}h_{\mathcal{Z}_p(\mu)}$. This and the previous inequality show that $h_{\mathcal{Z}_p(\tilde{\mu})} \sim h_{\mathcal{Z}_p(\mu)}$, which completes the proof. □

We are about to describe the symmetrization of ν^{n+1}, but first, let ξ denote the two-sided exponential probability measure, i.e. the probability measure whose density is given by $\frac{1}{2}e^{-|x|}$.

Proposition 3.3. $\widetilde{\nu^{n+1}} = \xi^{n+1}$

Proof. We just compute the density of $\widetilde{\nu^{n+1}}$. Let f denote the density of ν^{n+1} and let $\tilde{f}$ denote the density of $\widetilde{\nu^{n+1}}$. Then for $x \in \mathbb{R}^{n+1}$,

$$\tilde{f}(x) = \int f(x-y)f(-y)dy = \int_{x-y+(1,...,1),-y+(1,...,1)\in\mathbb{R}_+^{n+1}} e^{-\|x-y+(1,...,1)\|_1}e^{-\|-y+(1,...,1)\|_1}dy$$
$$= \int_{x+z,z\in\mathbb{R}_+^{n+1}} e^{-\|x+z\|_1}e^{-\|z\|_1}dz.$$

(The final equality came from a straightforward change of variables.) To finish, we define

$$b_j := \begin{cases} 0 & \text{if } x_j \geq 0 \\ -x_j & \text{if } x_j < 0 \end{cases}$$

Now, continuing the previous calculation, we have

$$\tilde{f}(x) = \int_{b_{n+1}}^{\infty} ... \int_{b_1}^{\infty} \prod_{j=1}^{n+1} e^{-x_j-2z_j}dz_1...dz_{n+1} = \prod_{j=1}^{n+1}\frac{1}{2}e^{-|x_j|}$$

Thus, $\tilde{f}$ is precisely the density of ξ^{n+1}. □

Now we are ready to state and prove the main theorem regarding the symmetric centroid bodies:

Theorem 3.4. *For* $p \geq 2$,

$$\mathcal{Z}_p\left(\tilde{S_n}\right) \sim \frac{c_n\sqrt{p}}{max\{n,p\}}B_2^{H_n} + \frac{p}{max\{n,p\}}conv\left\{\tilde{S_n}, -\tilde{S_n}\right\}$$

Here, $B_2^{H_n}$ *denotes the Euclidean ball of radius 1 inside of* H_n.

Proof. Since ν is centered, log-concave, and absolutely continuous, propositions 2.4, 3.2, and 3.3 give that

$$\mathcal{Z}_p\left(\tilde{S_n}\right) \sim \frac{c_n}{max\{n,p\}}proj_{H_n}\mathcal{Z}_p\left(\xi^{n+1}\right).$$

In [4], the form of $\mathcal{Z}_p\left(\xi^{n+1}\right)$ is found:

$$\mathcal{Z}_p\left(\xi^{n+1}\right) \sim \sqrt{p}B_2^{n+1} + pB_1^{n+1}$$

Here, B_2^{n+1} and B_1^{n+1} denote the Euclidean ball and cross-polytope of radius 1. (The paper does not state this fact explicitly, but it is an immediate consequence of Propositions 3.2, 3.5, and 3.8 and Corollary 5.2.) Now the projection of a Euclidean ball onto any hyperplane through its center is just the intersection of the ball with that hyperplane. Thus, $proj_{H_n}\left(B_2^{n+1}\right) = B_2^{H_n}$. For the projection of the cross-polytope, note that

$$B_1^{n+1} = conv\{e_j, -e_j : j = 1, ..., n+1\} = conv\{conv\{e_j : j = 1, ..., n+1\}, conv\{-e_j : j = 1, ..., n+1\}\}$$
$$= conv\{A_n, -A_n\}.$$

The linearity of multiplying by -1 and projecting onto H_n means that these operations commute with taking the convex hull. Thus,

$$proj_{H_n}\left(B_1^{n+1}\right) = conv\{proj_{H_n}(A_n), -proj_{H_n}(A_n)\}$$

Since A_n is a translation of S_n in a direction orthogonal to H_n, $proj_{H_n}(A_n)$ is just S_n. Thus,

$$proj_{H_n}\left(B_1^{n+1}\right) = conv\{S_n, -S_n\}$$

and

$$c_n proj_{H_n}\left(B_1^{n+1}\right) = conv\left\{\tilde{S_n}, -\tilde{S_n}\right\}$$

The result follows. □

4 Asymmetric Centroid Bodies

In this section, we find the form of the asymmetric centroid bodies. For this, we adapt methods used in [4] to find the centroid bodies of the exponential distribution and then use Proposition 2.4. First, as in [4], we introduce the following function.

Definition 4.1. Let μ be a probability measure on $\mathbb{R}^m$. Then we define the following functions:

$$M_\mu(x) := \int_{\mathbb{R}^m} e^{\langle x,y\rangle} d\mu(y)$$

$$\Lambda_\mu(x) := log M_\mu(x)$$

$$\Lambda^*_\mu(x) := \mathcal{L}\Lambda_\mu(x) = \underset{y\in\mathbb{R}^m}{sup}\ \{\langle x,y\rangle - \Lambda_\mu(y)\}$$

Furthermore, define $B_p(\mu) := \{x \in \mathbb{R}^m : \Lambda_\mu(x) \le p\}$.

There is a close relationship between the level sets and the centroid bodies, which is reflected in the following two propositions. These are adaptations of Propositions 3.2 and 3.5 in [4], which concern the symmetric centroid bodies (Proposition 3.2 also assumes symmetry of the measure). The proofs presented here are very similar to the proofs given in [4].

Proposition 4.2. *For any centered probability measure μ on $\mathbb{R}^m$ and $p \ge 1$,*

$$\mathcal{Z}^+_p(\mu) \subset eB_p(\mu)$$

Proof. Let $x \in \mathcal{Z}^+_p(\mu)$. We must show that $\Lambda^*_\mu\left(\frac{x}{e}\right) \le p$, i.e. for any $y \in \mathbb{R}^m$,

$$\frac{\langle x,y\rangle}{e} - \Lambda_\mu(y) \le p.$$

Fix $y \in \mathbb{R}^m$ and let β be such that $\int_{\langle y,z\rangle\ge 0}\langle y,z\rangle^p d\mu(z) = \beta^p$. Then $\langle y,x\rangle \le \beta$ (this is because, by definition of asymmetric centroid body and also of the support function, $\underset{z\in\mathcal{Z}^+_p(\mu)}{max}\ \langle y,z\rangle = \beta$). Now we consider two cases, $\beta \le ep$ and $\beta > ep$.

If $\beta \le ep$, then since $\Lambda_\mu(x) \ge \int\langle x,z\rangle d\mu(x) = 0$ (by Jensen's inequality), we have

$$\frac{\langle x,y\rangle}{e} - \Lambda_\mu(x) \le \frac{\beta}{e} \le p.$$

If $\beta > ep$, then

$$\int e^{\langle y,z\rangle} d\mu(z) \ge \int_{\langle y,z\rangle\ge 0} \left|e^{\frac{\langle y,z\rangle}{p}}\right|^p d\mu(z) \ge \int_{\langle y,z\rangle\ge 0}\left(\frac{\langle y,z\rangle}{p}\right)^p d\mu(z).$$

Thus,

$$\int e^{\frac{ep\langle y,z\rangle}{\beta}} d\mu(z) \ge \int_{\langle\frac{epy}{\beta},z\rangle\ge 0}\left|\frac{e\langle y,z\rangle}{\beta}\right|^p d\mu(z) = e^p$$

Therefore, $\Lambda_\mu\left(\frac{epy}{\beta}\right) \ge p$, so

$$\Lambda_\mu(y) \ge \frac{\beta}{ep}\Lambda_\mu\left(\frac{epy}{\beta}\right) \ge \frac{\beta}{e}$$

(the second inequality is another application of Jensen's inequality). Finally,

$$\frac{\langle x,y\rangle}{e} - \Lambda_\mu(y) \le \frac{\beta}{e}\left\langle \frac{y}{\beta}, x\right\rangle - \frac{\beta}{e} \le 0.$$

□

Proposition 4.3. *If μ is a centered, log-concave probability measure on $\mathbb{R}^m$, then for $p \geq 2$, $B_p(\mu) \subset 4e\alpha \mathcal{Z}_p^+(\mu)$, where α is an absolute constant.*

Proof. By equation (2.3) from [6], which is a corollary of Grünbaum's Lemma from [5], there is an absolute constant α such that for any $p \geq q \geq 2$ and for any $x \in \mathbb{R}^m$,

$$\left(\int_{\langle x,y\rangle \geq 0} \langle x, y\rangle^p d\mu(y)\right)^{\frac{1}{p}} \leq \alpha \frac{p}{q} \left(\int_{\langle x,y\rangle \geq 0} \langle x, y\rangle^q d\mu(y)\right)^{\frac{1}{q}}.$$

With this fact in hand, we show that if $h^+_{\mathcal{Z}_p(\mu)}(y) \leq 1$, then $\Lambda_\mu\left(\frac{py}{2e\alpha}\right) \leq p$. For such y, set $\tilde{y} := \frac{py}{2e\alpha}$. Then, for any $k \in \mathbb{N}$,

$$\left(\int_{\langle \tilde{y},z\rangle \geq 0} \langle \tilde{y}, z\rangle^k d\mu(z)\right)^{\frac{1}{k}} = \frac{p}{2e\alpha} \left(\int_{\langle y,z\rangle \geq 0} \langle y, z\rangle^k d\mu(z)\right)^{\frac{1}{k}} \leq \begin{cases} \frac{p}{2e\alpha} & \text{if } k \leq p \\ \frac{k}{2e} & \text{if } k > p \end{cases}$$

Therefore,

$$\int e^{\langle \tilde{y},z\rangle} d\mu(z) \leq \mu\{z \in \mathbb{R}^m : \langle \tilde{y}, z\rangle < 0\} + \int_{\langle \tilde{y},z\rangle \geq 0} e^{\langle \tilde{y},z\rangle} d\mu(z) \leq 1 + \sum_{k=0}^{\infty} \frac{1}{k!} \int_{\langle \tilde{y},z\rangle \geq 0} \langle \tilde{y}, z\rangle^k d\mu(z)$$
$$\leq 1 + \sum_{k \leq p} \frac{1}{k!}\left(\frac{p}{2e\alpha}\right)^k + \sum_{k>p} \frac{1}{k!}\left(\frac{k}{2e}\right)^k \leq 1 + e^{\frac{p}{2e\alpha}} + \frac{1}{2} \leq (e-1) + e^{\frac{p}{2e\alpha}} \leq e^{1+\frac{p}{2e\alpha}} \leq e^p$$

(for the second inequality on the second line, we used that $\left(\frac{k}{e}\right)^k \leq k!$, which comes from a variant of Stirling's Approximation). Thus, $\Lambda_\mu\left(\frac{py}{2e\alpha}\right) \leq p$.

Now we prove the proposition. If $x \notin 4e\alpha\mathcal{Z}_p^+(\mu)$, then there exists a $y \in \mathbb{R}^m$ such that $\langle y, x\rangle > h_{4e\alpha\mathcal{Z}_p^+(\mu)}(y)$. By homogeneity of the support function, we may assume that $h_{\mathcal{Z}_p^+(\mu)}(y) = 1$, so $\langle x, y\rangle > 4e\alpha$. Then, using the fact above, we have that,

$$\Lambda_\mu^*(x) \geq \left\langle x, \frac{py}{2e\alpha}\right\rangle - \Lambda_\mu\left(\frac{py}{2e\alpha}\right) > \frac{p}{2e\alpha} 4e\alpha - p = p$$

□

Because of the close relationship between $\Lambda^*_{\nu^{n+1}}$ and the asymmetric centroid bodies, we now investigate $\Lambda^*_{\nu^{n+1}}$.

Proposition 4.4. *There exists an absolute constant C such that for all $x \in \mathbb{R}$, $\Lambda_\nu^*\left(\frac{x}{C}\right) \leq f(x) \leq \Lambda_\nu^*(Cx)$, where f is defined by*

$$f(x) = \begin{cases} \infty & \text{if } x < -1 \\ x^2 & \text{if } -1 \leq x \leq 1 \\ x & \text{if } x > 1 \end{cases}$$

Proof. We begin by simply computing Λ_ν^*. For $y \in \mathbb{R}$,

$$M_\nu(y) = \int_{-1}^{\infty} e^{yz} e^{-z-1} dz = \frac{1}{e} \int_{-1}^{\infty} e^{(y-1)z} dz = \begin{cases} \frac{e^{-y}}{1-y} & \text{if } y < 1 \\ \infty & \text{if } y \geq 1 \end{cases}$$

Thus,

$$\Lambda_\nu(y) = log M_\nu(y) = \begin{cases} -y - log(1-y) & \text{if } y < 1 \\ \infty & \text{if } y \geq 1 \end{cases}$$

and

$$\Lambda_\nu^*(x) = \sup_{y \in \mathbb{R}}\{xy - \Lambda_\nu(y)\} = \sup_{y<1}\{(x+1)y + log(1-y)\}.$$

If $x+1 \leq 0$, then the supremum is equal to infinity. For $x > -1$, we can use calculus to find that $xy + y + log(1-y)$ is maximized when $y = 1 - \frac{1}{x+1}$. Thus,

$$\Lambda_\nu^*(x) = \begin{cases} \infty & \text{if } x \leq -1 \\ x - log(x+1) & \text{if } x > -1 \end{cases}$$

We will show that we can take $C = 3$ by using a second-order Taylor approximation for $\log(1+x)$: for every $x \in (-1, 1]$ we can find some y_x between 0 and x so that

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3(1+y_x)^3}.$$

Thus, for all such x,

$$\Lambda_\nu^*(x) = \frac{x^2}{2} - \frac{x^3}{3(1+y_x)^3}.$$

First we prove that $\Lambda_\nu^*\big(\frac{x}{3}\big) \leq f(x)$ by considering the cases $x < 1$, $x \in [-1, 0)$, $x \in [0, 1]$, $x \in (1, 3]$, and $x > 3$.

If $x < -1$, then $f(x) = \infty$, so the claim is trivially true.

If $x \in [-1, 0)$, then $\frac{x}{3} \in (-1, 0)$ and $y_{x/3} \in (x, 0) \subset (-\frac{1}{3}, 0)$. Thus

$$\Lambda_\nu^*\big(\frac{x}{3}\big) = \frac{x^2}{2\cdot 9} - \frac{x^3}{3\cdot 27(1+y_{x/3})^3} < \frac{x^2}{2\cdot 9} + x^2\frac{|x|}{3\cdot 27(1-\frac{1}{3})^3} \leq \frac{x^2}{2\cdot 9} + \frac{x^2}{3\cdot 8} < x^2 = f(x).$$

If $x \in [0, 1]$, then

$$\Lambda_\nu^*\big(\frac{x}{3}\big) = \frac{x^2}{2\cdot 9} - \frac{x^3}{3\cdot 27(1+y_{x/3})^3} \leq \frac{x^2}{2\cdot 9} \leq x^2 = f(x).$$

If $x \in (1, 3]$, then again

$$\Lambda_\nu^*\big(\frac{x}{3}\big) = \frac{x^2}{2\cdot 9} - \frac{x^3}{3\cdot 27(1+y_{x/3})^3} < \frac{x^2}{2\cdot 9} \leq \frac{1}{2} < x = f(x).$$

Finally, if $x > 3$, then

$$\Lambda_\nu^*\big(\frac{x}{3}\big) = \frac{x}{3} - \log(1+\frac{x}{3}) < \frac{x}{3} < x = f(x).$$

We now prove that $\Lambda_\nu^*(3x) \geq f(x)$ by considering the cases $x \leq -\frac{1}{3}$, $x \in (-\frac{1}{3}, 0]$, $x \in (0, \frac{1}{3}]$, $x \in (\frac{1}{3}, 1]$, and $x > 1$.

If $x \leq -\frac{1}{3}$, then $\Lambda_\nu^*(3x) = \infty$, so the claim is trivially true.

If $x \in (-\frac{1}{3}, 0]$, then $3x \in (-1, 0]$. Thus

$$\Lambda_\nu^*(3x) = \frac{9x^2}{2} - \frac{27x^3}{3(1+y_{3x})^3} \geq \frac{9x^2}{2} \geq x^2 = f(x).$$

If $x \in (0, \frac{1}{3}]$, then again

$$\Lambda_\nu^*(3x) = \frac{9x^2}{2} - \frac{27x^3}{3(1+y_{3x})^3} = x^2 + \frac{7x^2}{2} - \frac{27x^3}{3(1+y_{3x})^3} \geq x^2 + \frac{7x^2}{2} - 9x^3 \geq x^2 + x^2\Big(\frac{7}{2} - \frac{9}{3}\Big) > x^2 = f(x).$$

If $x \in (\frac{1}{3}, 1]$, then we can check, using calculus, that the function

$$\Lambda_\nu^*(3x) - f(x) = 3x - \log(1+3x) - x^2$$

is increasing on $(\frac{1}{3}, 1]$. Since $\Lambda_\nu^*(1) - \big(\frac{1}{3}\big)^2 > 0$, we conclude that, for all $x \in (\frac{1}{3}, 1]$, $\Lambda_\nu^*(3x) - f(x) > 0$.

If $x > 1$, then

$$\Lambda_\nu^*(3x) - f(x) = 3x - \log(1+3x) - x$$

and again using calculus we can check that this function is increasing on $[1, \infty)$ and hence $\Lambda_\nu^*(3x) - f(x) > \Lambda_\nu^*(3) - f(1) > 0$.

We conclude that, for all $x \in \mathbb{R}$, $\Lambda_\nu^*\big(\frac{x}{3}\big) \leq f(x) \leq \Lambda_\nu^*(3x)$. □

We are now ready to give the form of the asymmetric centroid bodies.

Theorem 4.5. *For $p \geq 2$,*

$$\mathcal{Z}_p^+\left(\tilde{S}_n\right) \sim \frac{p}{max\{n,p\}}\tilde{S}_n + \frac{c_n}{\max\{n,p\}} proj_{H_n}\left(B_\infty^{n+1} \cap \sqrt{p}B_2^{n+1}\right),$$

where B_∞^{n+1} is the cube $[-1,1]^{n+1}$.

Proof. Since ν^{n+1} is log-concave, we can apply Propositions 4.2 and 4.3 to conclude that $\mathcal{Z}_p^+\left(\nu^{n+1}\right) \sim B_p\left(\nu^{n+1}\right)$. By Proposition 2.4, we therefore have that

$$\mathcal{Z}_p^+\left(\tilde{S}_n\right) \sim \frac{c_n}{max\{n,p\}} proj_{H_n}\left(B_p\left(\nu^{n+1}\right)\right).$$

Because ν^{n+1} is a product measure, for any $x \in \mathbb{R}^{n+1}$,

$$\Lambda_{\nu^{n+1}}(x) = \sum_{j=1}^{n+1} \Lambda_\nu(x_j)$$

and

$$\Lambda^*_{\nu^{n+1}}(x) = \sum_{j=1}^{n+1} \Lambda^*_\nu(x_j)$$

(this follows immediately from the way these functions are defined and the way product measures behave with respect to iterated integrals). Thus, by Proposition 4.4,

$$B_p\left(\nu^{n+1}\right) \sim B_p(f) := \left\{ x \in \mathbb{R}^{n+1} : \sum_{j=1}^{n+1} f(x_j) \leq p \right\}.$$

If $x \in B_p(f)$, then we can write $x = y + z$, where

$$y_j = \begin{cases} x_j & \text{if } -1 \leq x_j \leq 1 \\ 0 & \text{otherwise} \end{cases}.$$

Note that

$$\sum_{j=1}^{n+1} f(y_j) \leq \sum_{j=1}^{n+1} f(x_j) \leq p$$

and

$$\sum_{j=1}^{n+1} f(z_j) \leq \sum_{j=1}^{n+1} f(x_j) \leq p$$

Therefore, $y \in B_\infty^{n+1} \cap \sqrt{p}B_2^{n+1}$ and $z \in pB_1^{n+1} \cap \mathbb{R}_+^{n+1}$, so $B_p(f) \subset B_\infty^{n+1} \cap \sqrt{p}B_2^{n+1} + pB_1^{n+1} \cap \mathbb{R}_+^{n+1}$.

In preparation for the other inequality, note that for any $a, b \in \mathbb{R}$, either $f(a+b) \leq f(2a)$ or $f(a+b) \leq f(2b)$, depending on whether a or b has the greater absolute value. Thus, $f(a+b) \leq f(2a) + f(2b) \leq 4f(a) + 4f(b)$. Now consider $y \in \frac{1}{8}\left(B_\infty^{n+1} \cap \sqrt{p}B_2^{n+1}\right)$, $z \in \frac{1}{8}\left(pB_1^{n+1} \cap \mathbb{R}_+^{n+1}\right)$. Then, if $x = y + z$,

$$\sum_{j=1}^{n+1} f(x_j) \leq 4\sum_{j=1}^{n+1} f(y_j) + 4\sum_{j=1}^{n+1} f(z_j) \leq 4\frac{p}{64} + 4\frac{p}{8} \leq p.$$

Thus, $\frac{1}{8}\left(B_\infty^{n+1} \cap \sqrt{p}B_2^{n+1} + pB_1^{n+1} \cap \mathbb{R}_+^{n+1}\right) \subset B_p(f)$. Together with the other inequality, this means that

$$\begin{aligned}\mathcal{Z}_p^+\left(\tilde{S}_n\right) &\sim \frac{c_n}{max\{n,p\}} proj_{H_n}\left(B_\infty^{n+1} \cap \sqrt{p}B_2^{n+1} + pB_1^{n+1} \cap \mathbb{R}_+^{n+1}\right) \\ &= \frac{c_n}{max\{n,p\}} proj_{H_n}\left(B_\infty^{n+1} \cap \sqrt{p}B_2^{n+1}\right) + \frac{c_n}{max\{n,p\}} proj_{H_n}\left(pB_1^{n+1} \cap \mathbb{R}_+^{n+1}\right).\end{aligned}$$

Similarly to in the proof of Theorem 3.4,

$$\frac{c_n}{max\{n,p\}} proj_{H_n}\left(pB_1^{n+1} \cap \mathbb{R}_+^{n+1}\right) = \frac{c_n}{max\{n,p\}} proj_{H_n}(conv\{pA_n, 0\}) = \frac{c_n}{max\{n,p\}} conv\{pS_n, 0\}$$
$$= \frac{p}{max\{n,p\}} \tilde{S}_n.$$

The result follows. □

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