FIRST KOSZUL HOMOLOGY OVER A LOCAL ARTINIAN RING

NINA PANDE

Abstract. Let $(R, \mathfrak{m})$ be a local Artinian ring. A conjecture of S. Dutta, M. Hochster, and C. Huneke states that the first Koszul homology module of n elements in $\mathfrak{m}$ with coefficients in R needs at least n generators. We prove this conjecture in the case where R is equicharacteristic and $\mathfrak{m}^3 = 0$.

1. Introduction

We prove the case where R is equicharacteristic and $\mathfrak{m}^3 = 0$ of the following conjecture, which arose in discussions among S. Dutta, M. Hochster and C. Huneke in the early 1980s, but was not published at that time. The conjecture was discussed in a 2004 talk by Hochster [2].

Conjecture 1.1. Let $(R, \mathfrak{m})$ be a local Artinian ring and let $u_1, \ldots, u_n$ be elements of $\mathfrak{m}$. Then the number of generators of $\mathrm{H}_1(u_1, \ldots, u_n; R)$ is at least n.

If true, this conjecture solves an open problem that is part of a family of unsolved problems related to the Buchsbaum-Eisenbud-Horrocks conjecture.

Consequence 1.2. Let $(S, \mathfrak{m})$ be an equicharacteristic regular local ring. Let I, J be $\mathfrak{m}$-primary ideals of S and let n be the Krull dimension of S. Then the number of generators of $\frac{I \cap J}{IJ} \cong \mathrm{Tor}_1\left(\frac{R}{I}, \frac{R}{J}\right)$ is at least n.

There are many open problems concerning lower bounds for $\dim_k k \otimes \mathrm{Tor}_i(M, N)$ when M, N have finite length over a regular local ring of dimension d and additional information such as the numbers of generators of M, N is specified. If $N = k$, the conjecture that $\binom{d}{i}$ is a lower bound on $\dim_k k \otimes \mathrm{Tor}_i(M, N)$ is the Buchsbaum-Eisenbud-Horrocks problem. If $i = 1$ and M, N are cyclic, the conjecture that $\dim_k k \otimes \mathrm{Tor}_i(M, N)$ is bounded below by n is Consequence 1.2. In fact, to prove Consequence 1.2 it suffices to show that Conjecture 1.1 holds when R is equicharacteristic and $u_1, \ldots, u_n$ are purely linear [5], but the general version of the conjecture is an interesting problem in its own right.

The conjecture is easily verified if $n = 1$, and an argument of S. Dutta proves the case where $n = 2$ [5]. Furthermore, A. Zhang has recently shown that the number of generators is at least $n + \mathrm{cid}(R) - \mathrm{cid}\left(\frac{R}{(s_1, \ldots, s_n)R}\right)$, where $\mathrm{cid}(R)$ denotes the complete intersection defect of R [5]. Taking a direct approach, we prove the conjecture in the case where R is equicharacteristic and $\mathfrak{m}^3 = 0$.

In this paper, all rings are taken to be commutative with unity. Section 2 covers some basic concepts in commutative algebra as a starting point for undergraduates reading this paper. Section 3 collects some preliminary lemmas that reduce the problem to the situation where the n elements are either variables or quadratic forms in the remaining variables; these lemmas were known at the outset of this project but not all of them had been previously published. Section 4 contains original results, culminating in the main theorem.

2. Background

Definition 2.1. A ring R is said to be *local* if it has a single maximal ideal. We say $(R, \mathfrak{m})$ is local when it is convenient to specify the maximal ideal $\mathfrak{m}$ of R.

Definition 2.2. A ring R is said to be *Artinian* if it satisfies the descending chain condition, i.e. if any strictly descending chain of ideals of R

$$I_1 \supsetneq I_2 \supsetneq I_3 \supsetneq \cdots$$

is finite.

Note. If $(R, \mathfrak{m})$ is local, then R is Artinian if and only if $\mathfrak{m}^n = 0$ for some $n \in \mathbb{N}$.

Example 2.3. Let $R = \mathbb{Z}/8\mathbb{Z}$. Then R is a local Artinian ring with maximal ideal (2).

Example 2.4. Let $R = k[x, y]/(x, y)^2$, where k is a field. Then R is a local Artinian ring with maximal ideal (x, y).

Example 2.5. Let $R = k[x, y]/(x^2, xy^2, y^3)$, where k is a field. Then R is a local Artinian ring with maximal ideal (x, y).

Definition 2.6. A ring R is said to be *equicharacteristic* if it contains a field. Otherwise, we say R is of *mixed characteristic*.

Definition 2.7. Let $u_1, \ldots, u_n$ be elements of a ring R. A *relation* on $u_1, \ldots, u_n$ is an n-tuple $(r_1, \ldots, r_n)$ such that $r_1u_1 + \cdots + r_nu_n = 0$.

Example 2.8. Let $R = k[x, y]$. Then $(4y, -6)$ is a relation on $3x$ and $2xy$.

Example 2.9. Let $R = k[x, y]/(x^2, xy, y^3)$. Then (x, x), (x, y^2), and $(y, -2x)$ are relations on $2x$ and y.

Definition 2.10. Let $\underline{u} = u_1, \ldots, u_n \in R$. A relation on $\underline{u}$ of the form $(\ldots, 0, u_j, 0, \ldots, 0, -u_i, 0, \ldots)$, where u_j occurs in the ith position and $-u_i$ occurs in the jth position, is called a *Koszul relation* on $\underline{u}$.

Definition 2.11. Let $\underline{u} = u_1, \ldots, u_n \in R$. Then the relations on $\underline{u}$ form a submodule N of R^n and the Koszul relations on $\underline{u}$ generate a submodule of N. The quotient module N/L is called the *first Koszul homology of* $\underline{u}$ *with coefficients in* R and is denoted $\mathrm{H}_1(\underline{u}; R)$.

Throughout the paper we make use of several standard tools in homological algebra, particularly the Koszul complex and the resulting long exact sequence in homology. I. Swanson's online notes [4] are a good introductory reference for these topics. A more advanced treatment of Koszul homology can be found in [1, §1.6] or [3, Ch. IV A].

3. Preliminaries

We first recall some basic facts relating the Koszul homology modules on sets of elements that generate the same ideal in R. These are stated in greater generality in [1, §1.6] and [3, Ch. IV A].

Lemma 3.1. *Let R be a ring. Suppose $\underline{u} = u_1, \ldots, u_n, \underline{v} = v_1, \ldots, v_n \in R$ and there exists an invertible $n \times n$ matrix A with $\underline{v} = \underline{u}A$. Then $\mathrm{H_i}(\underline{v}; R) \cong \mathrm{H_i}(\underline{u}; R)$.*

Lemma 3.2. *Let R be a local ring and let $\underline{u} = u_1, \ldots, u_n, \underline{v} = v_1, \ldots, v_n \in R$ be such that $(\underline{u}) = (\underline{v})$. Then $\mathrm{H_i}(\underline{v}; R) \cong \mathrm{H_i}(\underline{u}; R)$.*

Note that $\underline{u}$ and $\underline{v}$ have the same order in the lemma above; the next few lemmas address the case where their orders differ.

Corollary 3.3. *Let R be a ring and let $v_1, \ldots, v_m \in R$. If $v_m \in (v_1, \ldots, v_{m-1})$, then $\mathrm{H_i}(v_1, \ldots, v_m; R) = \mathrm{H_i}(v_1, \ldots, v_{m-1}, 0; R)$.*

Lemma 3.4. *Let R be a ring and let $\underline{v} = v_1, \ldots, v_m \in R$. Then*

$$\mathrm{H_i}(v_1, \ldots, v_m, 0; R) = \mathrm{H_i}(v_1, \ldots, v_m; R) \oplus \mathrm{H_{i-1}}(v_1, \ldots, v_m; R).$$

Corollary 3.5. *Let $(R, \mathfrak{m})$ be a local ring and let $\underline{v} = v_1, \ldots, v_m \in \mathfrak{m}$. Let $\underline{u} = u_1, \ldots, u_n$ be a set of elements that minimally generate $(\underline{v})$. If the minimal number of generators of $\mathrm{H}_1(\underline{u}; R)$ is at least n, then the minimal number of generators of $\mathrm{H}_1(\underline{v}; R)$ is at least m.*

The next lemma follows from Proposition 1.6.5 of [1].

Lemma 3.6. *Let R be a ring and let $\underline{u} = u_1, \ldots, u_n \in R$. Then $(\underline{u})$ annihilates $\mathrm{H}_1(\underline{u}; R)$. In particular, if $(R, \mathfrak{m})$ is a local ring with $\mathfrak{m} = (x_1, \ldots, x_d)$ and $x_1, \ldots, x_r \in (\underline{u})$, then $\mathfrak{m}\mathrm{H}_1(\underline{u}; R) = (x_{r+1}, \ldots, x_d)\mathrm{H}_1(\underline{u}; R)$. Furthermore, if $\mathrm{H}_1(\underline{u}; R) = N/L$, where N is the module of relations on $\underline{u}$ and L is the module generated by the Koszul relations on $\underline{u}$, then $\mathfrak{m}N + L = (x_{r+1}, \ldots, x_d)N + L$.*

The following lemmas greatly reduce the number of cases that need to be considered to prove the conjecture when $\mathfrak{m}^3 = 0$ and R is equicharacteristic.

Lemma 3.7. *Let $(R, \mathfrak{m}, k)$ be an equicharacteristic local Artinian ring. Then*

$$R \cong \frac{k[[x_1, \dots, x_d]]}{I} \cong \frac{k[x_1, \dots, x_d]}{I}$$

for some $d \in \mathbb{N}$ and $(x_1, \dots, x_d)$-primary ideal I of $k[x_1, \dots, x_d]$. Moreover, we may take $I \subseteq (x_1, \dots, x_d)^2$.

Proof. The first isomorphism follows from the structure theory of complete local rings in the equicharacteristic case (R is complete since it is a local Artinian ring). The second isomorphism holds since I is $\mathfrak{m}$-primary. We may assume $I \subseteq (x_1, \dots, x_d)^2$; otherwise it would be equivalent to decrease d. □

Corollary 3.8. *Let R be a local Artinian ring containing a field. Suppose $\mathfrak{m}^3 = 0$. Then $R \cong S/V$ where $S = k[x_1, \dots, x_d]/\mathfrak{m}^3$ for some $d \in \mathbb{N}$ and $V \subseteq \mathfrak{m}^2/\mathfrak{m}^3$.*

Lemma 3.9. *Let I be a proper ideal of $S = k[x_1, \dots, x_d]/\mathfrak{m}^3$. Let $\underline{v} = v_1, \dots, v_n$ be a minimal generating set of I. Then $\mathrm{H}_1(\underline{v}; S) \cong \mathrm{H}_1(\underline{u}; S)$ for some $\underline{u} = x_1, \dots, x_r, q_{r+1}, \dots, q_n$, where the q_i are quadratic forms in $x_{r+1}, \dots, x_d$ and $x_1, \dots, x_r, q_{r+1}, \dots, q_n$ minimally generate $(\underline{u})$.*

Proof. We may think of S as $k[[x_1, \dots, x_d]]/\mathfrak{m}^3$. Let $r = \dim_k (I + \mathfrak{m}^2)/\mathfrak{m}^2$ and renumber the v_i so that the images of $v_1, \dots, v_r$ form a basis for $(I + \mathfrak{m}^2)/\mathfrak{m}^2$. Subtract scalar linear combinations of $v_1, \dots, v_r$ from $v_{r+1}, \dots, v_n$ to obtain $y_{r+1}, \dots, y_n \in \mathfrak{m}^2$. By Lemma 3.2, $\mathrm{H}_1(\underline{v}; S) \cong \mathrm{H}_1(v_1, \dots, v_r, y_{r+1}, \dots, y_n; S)$. Let $x_i' := v_i$ for $1 \le i \le r$. Extend $x_1', \dots, x_r'$ to a minimal set of generators $x_1', \dots, x_d'$ of $\mathfrak{m}$. By the structure theory of complete regular local rings, there is a k-automorphism $k[[x_1, \dots, x_d]] \to k[[x_1, \dots, x_d]]$ that maps x_i' to x_i for all $i \le d$. This induces a k-automorphism θ of S. Let $w_j := \theta(y_j) \in \mathfrak{m}^2$ for $j > r$. The images of $v_1, \dots, v_r, y_{r+1}, \dots, y_n$ are $x_1, \dots, x_r, w_{r+1}, \dots, w_n$, where the latter are in $\mathfrak{m}^2$. Therefore we have $\mathrm{H}_1(v_1, \dots, v_r, y_{r+1}, \dots, y_n; S) \cong \mathrm{H}_1(x_1, \dots, x_r, w_{r+1}, \dots, w_n; S)$. Subtract linear combinations of $x_1, \dots, x_r$ from $w_{r+1}, \dots, w_n$ to obtain $q_{r+1}, \dots, q_n \in (x_{r+1}, \dots, x_d)^2$. Then by Lemma 3.2,

$$\mathrm{H}_1(x_1, \dots, x_r, q_{r+1}, \dots, q_n; S) \cong \mathrm{H}_1(x_1, \dots, x_r, w_{r+1}, \dots, w_n; S) \cong \mathrm{H}_1(\underline{v}; S).$$ □

We briefly outline the method used to prove the main result, Theorem 4.6. We first find the module $\mathrm{H}_1(\underline{u}; S)$ explicitly in Lemma 4.5. With some manipulations of exact sequences, we use this information to determine $\mathrm{H}_1(\underline{u}; R)$. We then examine

$$\frac{\mathrm{H}_1(\underline{u}; R)}{\mathfrak{m}\mathrm{H}_1(\underline{u}; R)}.$$

Recall that by the local case of Nakayama's lemma, the minimal number of generators of $\mathrm{H}_1(\underline{u};R)$ is equal to the dimension of this module as a vector space over k. In general, the obvious lower bound on the dimension of this module is not quite sufficient to prove the conjecture. Instead, we decompose the module into a direct sum of two submodules, one of which is easy to understand. To handle the other submodule, we construct a surjective map from it onto a completely different module to obtain a lower bound on its minimal number of generators. We show that the sum of this value and the number of generators of the simpler submodule is at least n and the result follows.

4. Results

The first result below is not involved in the proof of the main theorem, but immediately proves the case where $u_1,\dots,u_n$ are quadratic.

Proposition 4.1. *Let $(R,\mathfrak{m})$ be a local Artinian ring such that $\mathfrak{m}$ has $d\geq 1$ generators. Let $\underline{q}=q_1,\dots,q_n\in\mathfrak{m}^2$ be such that $q_1,\dots,q_n$ minimally generate $(\underline{q})$ and $\mathfrak{m}q_i=0$ for all $1\leq i\leq n$. Then the minimal number of generators of $\mathrm{H}_1(\underline{q};R)$ is $nd\geq n$.*

Proof. Let N be the module of all relations on $q_1,\dots,q_n$. Since $q_1,\dots,q_n$ minimally generate $(\underline{q})$, we have that $N\subseteq\mathfrak{m}^{\oplus n}$. Since $\mathfrak{m}$ annihilates $(\underline{q})$, $N=\mathfrak{m}^{\oplus n}$. Let e_i denote the basis element $(\dots,0,1,0,\dots)$ of $R^{\oplus n}$, where the 1 occurs in the ith position. We have $\mathrm{H}_1(\underline{q};R)=N/L$, where L is the module generated by the Koszul relations $q_je_i-q_ie_j$. By Nakayama's lemma, the minimal number of generators of $\mathrm{H}_1(\underline{q};R)$ is equal to the k-vector space dimension of

$$\frac{\mathrm{H}_1(\underline{q};R)}{\mathfrak{m}\mathrm{H}_1(\underline{q};R)}\cong\frac{\mathfrak{m}^{\oplus n}}{(\mathfrak{m}^2)^{\oplus n}}\cong\Big(\frac{\mathfrak{m}}{\mathfrak{m}^2}\Big)^{\oplus n},$$

which is exactly nd. ☐

Corollary 4.2. *Let $S=k[x_1,\dots,x_d]/\mathfrak{m}^3$, where $\mathfrak{m}=(x_1,\dots,x_d)$. Let $V\subseteq\mathfrak{m}^2/\mathfrak{m}^3$ and let $R=S/V$. Let $\underline{q}=q_1,\dots,q_n$ such that $q_i\in\mathfrak{m}^2$ and $q_1,\dots,q_n$ minimally generate $(\underline{q})$. Then the minimal number of generators of $\mathrm{H}_1(\underline{q};R)$ is exactly nd.*

Lemma 4.3. *Let $S=k[x_1,\dots,x_d]/\mathfrak{m}^3$, where $\mathfrak{m}=(x_1,\dots,x_d)$. Let L be the module of Koszul relations on $x_1,\dots,x_r$. Then*

$$\mathrm{H}_1(x_1,\dots,x_r;S)=\frac{S_2^{\oplus r}+L}{L}\cong\frac{S_2^{\oplus r}}{L_2},$$

where $L_2=S_2^{\oplus r}\cap L$. Moreover, $\dim_k L_2=d\binom{r}{2}-\binom{r}{3}$ and $\dim_k\mathrm{H}_1(x_1,\dots,x_r;S)=r\binom{d+1}{2}-d\binom{r}{2}+\binom{r}{3}\geq rd+\binom{r}{3}$.

Proof. Let N be the module of all relations on $x_1, \ldots, x_r$ and let L be the module generated by the Koszul relations on $x_1, \ldots, x_r$. We first show that $N = S_2^{\oplus r} + L$, where L is the module generated by the Koszul relations on $x_1, \ldots, x_r$. Since $S_3 = 0$, it is clear that $S_2^{\oplus r} \subseteq N$. Consider $N/S_2^{\oplus r}$. These may be identified with the relations $(s_1, \ldots, s_r)$ such that the s_i have degree 1. Note that these relations hold in the polynomial ring $T = k[x_1, \ldots, x_d]$. Since $x_1, \ldots, x_r$ form a regular sequence in T, the relations must be in L, so $N = S_2^{\oplus r} + L$. Let $L_2 = S_2^{\oplus r} \cap L$.

Then

$$\mathrm{H}_1(x_1, \ldots, x_r; S) = \frac{S_2^{\oplus r} + L}{L} \cong \frac{S_2^{\oplus r}}{S_2^{\oplus r} \cap L} = \frac{S_2^{\oplus r}}{L_2}.$$

Note that $S_2^{\oplus r}$ is a k-vector space of dimension $r\binom{d+1}{2}$. We claim that L_2 is a k-vector space of dimension $d\binom{r}{2} - \binom{r}{3}$. Take the graded Koszul complex on $x_1, \ldots, x_r$ over S in degree 3. This yields an exact sequence

$$0 \to [\ker d_1]_3 \to [S(-1)^{\oplus r}]_3 \xrightarrow{d_1} [S]_3 \to \left[\frac{S}{(x_1, \ldots, x_r)S}\right]_3 \to 0$$

The dimensions of three of these modules are easy to calculate:

$$\dim_k [S(-1)^{\oplus r}]_3 = \dim_k [S^{\oplus r}]_2 = r\binom{d+1}{2}$$
$$\dim_k [S]_3 = \binom{d+2}{3}$$
$$\dim_k \left[\frac{S}{(x_1, \ldots, x_r)S}\right]_3 = \dim_k k[x_1, \ldots, x_{d-r}]_3 = \binom{d-r+2}{3}$$

Since the alternating sum of the dimensions is zero,

$$\begin{aligned}\dim_k [\ker d_1]_2 &= \dim_k [S(-1)^r]_3 + \dim_k \left[\frac{S}{(x_1, \ldots, x_r)S}\right]_3 - \dim_k [S]_3 \\ &= r\binom{d+1}{2} + \binom{d-r+2}{3} - \binom{d+2}{3} \\ &= d\binom{r}{2} - \binom{r}{3}\end{aligned}$$

Since $\ker d_1 = L$, it follows that $\mathrm{H}_1(x_1, \ldots, x_r; S) = S_2^{\oplus r}/L_2$ has dimension

$$r\binom{d+1}{2} - d\binom{r}{2} + \binom{r}{3} = \frac{rd(d-r+2)}{2} + \binom{r}{3} \geq rd + \binom{r}{3}$$

For heuristic reasons, we give a second, more intuitive argument that the dimension of L_2 is $d\binom{r}{2} - \binom{r}{3}$. First, note that L is a k-vector space spanned by the $\binom{r}{2}$ elements of the form $l_{ij} = x_j e_i - x_i e_j$, where e_i denotes the ith basis element $(\ldots, 0, 1, 0, \ldots)$ of $S^{\oplus r}$. Since $\mathfrak{m}^3 = 0$, every element of L can be written in the form $\sum s_{ij} l_{ij}$ where the s_{ij} have degree at most 1. It follows that L_2 is spanned by the $d\binom{r}{2}$ elements $x_k l_{ij}$ where $1 \leq k \leq d$. There are $\binom{r}{3}$ elements $x_k l_{ij}$ satisfying $1 \leq k < i < j \leq r$, and for any such element we have $x_k l_{ij} = x_i l_{kj} - x_j l_{ki}$. Thus, L_2 is spanned by the $d\binom{r}{2} - \binom{r}{3}$ elements $x_k l_{ij}$ such that $k \geq i$. It remains to show that these are linearly independent over K. Suppose $\sum c_{ijk} x_k l_{ij} = 0$ for some $c_{ijk} \in K$. The qth position of this sum (now excluding the terms where $k < i$) is given by

$$\sum_{k=q}^{d} \sum_{j=q+1}^{n} c_{qjk} x_k x_j - \sum_{i=1}^{q-1} \sum_{k'=i}^{d} c_{iqk'} x_{k'} x_i.$$

The set of $x_k x_j$ from the first double sum and the set of $x_{k'} x_i$ from the second double sum are disjoint, by the following argument. If some $x_k x_j = x_{k'} x_i$, then we must have $k = i$ or $j = i$, but these are both impossible since $q \leq k \leq d$ and $1 \leq i \leq q-1$, and $j > i$ by definition. Finally, since $k' \geq i$, no $x_{k'} x_i$ appears more than once in the second double sum, so each $x_{k'} x_i$ appears exactly once in the entire sum. Since the entire sum is zero, each $c_{iqk} = 0$. But this holds for each q, so every coefficient is zero and the elements are linearly independent. □

The next lemma is superfluous, as it follows from Theorem 4.6 (and the proofs are quite similar), but we include it here for heuristic reasons.

Lemma 4.4. *$S = k[x_1, \ldots, x_d]/\mathfrak{m}^3$ and $V \subseteq \mathfrak{m}^2/\mathfrak{m}^3$, where $\mathfrak{m} = (x_1, \ldots, x_d)$. Let $R = S/V$ and $\underline{x} = x_1, \ldots, x_r$. Then the minimal number of generators of $\mathrm{H}_1(\underline{x}; R)$ is at least r.*

Proof. The short exact sequence $0 \to V \to S \to R \to 0$ yields a long exact sequence in Koszul homology

$$\mathrm{H}_1(\underline{x}; V) \to \mathrm{H}_1(\underline{x}; S) \to \mathrm{H}_1(\underline{x}; R) \to \mathrm{H}_0(\underline{x}; V) \to \mathrm{H}_0(\underline{x}; S) \to \mathrm{H}_0(\underline{x}; R) \to 0$$

which becomes

$$\mathrm{H}_1(\underline{x}; V) \to \mathrm{H}_1(\underline{x}; S) \to \mathrm{H}_1(\underline{x}; R) \to \frac{V}{(\underline{x})V} \to \frac{S}{(\underline{x})S} \to \frac{R}{(\underline{x})R} \to 0$$

Since V is annihilated by $(\underline{x})$, $\mathrm{H}_1(\underline{x}; V) \cong V^{\oplus r}/(0) \cong V^{\oplus r}$ and $V/(\underline{x})V \cong V$. Thus the sequence becomes

$$V^{\oplus r} \xrightarrow{\varphi} \mathrm{H}_1(\underline{x}; S) \xrightarrow{\psi} \mathrm{H}_1(\underline{x}; R) \xrightarrow{\delta} V \xrightarrow{\phi} \frac{S}{(\underline{x})S} \to \frac{R}{(\underline{x})R} \to 0$$

By Lemma 4.3,

$$\mathrm{H}_1(\underline{x}; S) = \frac{S_2^{\oplus r}}{L_2}.$$

Since the map φ is induced by the inclusion $V^{\oplus r} \hookrightarrow S^{\oplus r}$, the image of $V^{\oplus r}$ in $\mathrm{H}_1(\underline{x}; S)$ is

$$\frac{V^{\oplus r} + L_2}{L_2}.$$

By exactness, $\ker\psi = \operatorname{Im}\varphi$, so

$$\operatorname{Im}\psi \cong \frac{\mathrm{H}_1(\underline{x};S)}{\operatorname{Im}\varphi} = \frac{S_2^{\oplus r}}{L_2}\Bigg/\frac{V^{\oplus r}+L_2}{L_2} \cong \frac{S_2^{\oplus r}}{L_2+V^{\oplus r}}.$$

Similarly, $\ker\delta = \operatorname{Im}\varphi$, and $\operatorname{Im}\delta = \ker\phi = V\cap(\underline{x})S$. Thus the short exact sequence

$$0 \to \ker\delta \to \mathrm{H}_1(\underline{x};R) \xrightarrow{\delta} \operatorname{Im}\delta \to 0$$

becomes

$$0 \to \frac{S_2^{\oplus r}}{L_2+V^{\oplus r}} \to \mathrm{H}_1(\underline{x};R) \xrightarrow{\delta} V\cap(\underline{x})S \to 0\ .$$

Let $s = \dim_k V\cap(\underline{x})S$ and let $t = \dim_k V$. Clearly, if $s \geq r$, then we are done. Suppose $s < r$. There is an injection

$$\frac{V}{V\cap(\underline{x})S} \hookrightarrow \frac{S_2}{S_2\cap(\underline{x})S} \cong k[x_{r+1},\dots,x_d]_2,$$

so $t = \dim_k V \leq s + \dim_k k[x_{r+1},\dots,x_d]_2 = s + \binom{d-r+1}{2}$. Because we have an obvious lifting at the chain level, the standard construction of the connecting homomorphism δ yields that δ sends $\sum_{i=1}^r b_ie_i$ to $\sum_{i=1}^r b_ix_i \in (\underline{x})S$. If $\sum_{i=1}^r b_ie_i \in \mathrm{H}_1(\underline{x};R)$, then $\sum_{i=1}^r b_ix_i \in 0_R$, so $\sum_{i=1}^r b_ix_i \in V$. Note that this map sends $[\mathrm{H}_1(\underline{x};R)]_2$ to 0, so $[\mathrm{H}_1(\underline{x};R)]_1 \twoheadrightarrow V\cap(\underline{x})S$. Since $[\ker\delta]_1 = 0$, $\ker\delta\cap \mathrm{H}_1(\underline{x};R)]_1 = 0$, so δ restricted to $[\mathrm{H}_1(\underline{x};R)]_1$ is an isomorphism. It then follows that $[\mathrm{H}_1(\underline{x};R)]_2 = \ker\delta$. By Nakayama's lemma, the minimal number of generators of $\mathrm{H}_1(\underline{x};R)$ is equal to

$$\dim_k \frac{\mathrm{H}_1(\underline{x};R)}{\mathfrak{m}\mathrm{H}_1(\underline{x};R)}.$$

Since $\mathrm{H}_1(\underline{x};R) = [\mathrm{H}_1(\underline{x};R)]_1 \oplus [\mathrm{H}_1(\underline{x};R)]_2$, $\mathfrak{m}[\mathrm{H}_1(\underline{x};R)]_2 = 0$, and $\mathfrak{m}[\mathrm{H}_1(\underline{x};R)]_1 \subseteq [\mathrm{H}_1(\underline{x};R)]_2$,

$$\dim_k \frac{\mathrm{H}_1(\underline{x};R)}{\mathfrak{m}\mathrm{H}_1(\underline{x};R)} = s + \dim_k \frac{[\mathrm{H}_1(\underline{x};R)]_2}{\mathfrak{m}[\mathrm{H}_1(\underline{x};R)]_1}.$$

Let B be the module of degree 1 relations on $x_1,\dots,x_r$. Then

$$[\mathrm{H}_1(\underline{x};R)]_1 = \frac{B}{L_1} \cong V\cap(\underline{x})S$$

and it follows that $\dim_k B/L_1 = s$. We have

$$[\mathrm{H}_1(\underline{x};R)]_2 = \ker\delta = \frac{S_2^{\oplus r}}{L_2+V^{\oplus r}},$$

so

$$\frac{[\mathrm{H}_1(\underline{x};R)]_2}{\mathfrak{m}[\mathrm{H}_1(\underline{x};R)]_1} = \frac{S_2^{\oplus r}}{L_2+V^{\oplus r}}\Bigg/\frac{\mathfrak{m}B+L_2+V^{\oplus r}}{L_2+V^{\oplus r}} = \frac{S_2^{\oplus r}}{\mathfrak{m}B+L_2+V^{\oplus r}} = \frac{S_2^{\oplus r}}{(x_{r+1},\dots,x_d)B+L_2+V^{\oplus r}}$$

(the last equality holds since $x_1, \dots, x_r$ annihilate $\mathrm{H}_1(\underline{x}; R)$; see Lemma 3.6). Let T be the polynomial ring $k[x_1, \dots, x_d]$. Since each element of $S_1 \oplus S_2$ has a unique coset representative in $T_1 \oplus T_2$, there is a natural vector space isomorphism between $S_1 \oplus S_2$ and $T_1 \oplus T_2$. Consider the map

$$\gamma : (S_1 \oplus S_2)^{\oplus r} \to (T_1 \oplus T_2)^{\oplus r} \to T_2 \oplus T_3$$

that sends the ith basis element e_i to x_i. We have $\gamma(S_2^{\oplus r}) = (x_1, \dots, x_r)T_2$ and $\gamma(V^{\oplus r}) = (x_1, \dots, x_r)\tilde{V}$, where $\tilde{V}$ is the isomorphic image of V in T_2^r. Furthermore, $\gamma(L_2) = 0$. Note that γ restricted to $S_1^{\oplus r}$ is the same as the composition obtained by applying δ and then the natural vector space isomorphism from S_2 to T_2, so $\gamma(B) = \tilde{V} \cap (\underline{x})T \cong V \cap (\underline{x})S = \delta(B)$. Thus, γ induces a well-defined surjective map from

$$\frac{[\mathrm{H}_1(\underline{x}; R)]_2}{\mathfrak{m}[\mathrm{H}_1(\underline{x}; R)]_1} \cong \frac{S_2^{\oplus r}}{(x_{r+1}, \dots, x_d)B + L_2 + V^{\oplus r}}$$

to the module

$$\frac{(x_1, \dots, x_r)T_2}{(x_{r+1}, \dots, x_d)(\tilde{V} \cap (\underline{x})T) + (x_1, \dots, x_r)\tilde{V}}.$$

We will show this module has dimension at least $r - s$ and the result will follow. Note that

$$\dim_k (x_1, \dots, x_r)T_2 = \dim_k T_3 - \dim_k (x_{r+1}, \dots, x_d)^3 = \binom{d+2}{3} - \binom{d-r+2}{3}.$$

We have $\dim_k \tilde{V} \cap (\underline{x})T = \dim_k V \cap (\underline{x})S = s$ and

$$\dim_k \tilde{V} = \dim_k V = t \leq s + \binom{d-r+1}{2}.$$

Then the dimension of $(x_{r+1}, \dots, x_d)(\tilde{V} \cap (\underline{x})T) + (x_1, \dots, x_r)\tilde{V}$ is bounded above by

$$(d-r)s + rs + r\binom{d-r+1}{2} = ds + r\binom{d-r+1}{2}.$$

Thus the dimension of the entire module is at least

$$\binom{d+2}{3} - \binom{d-r+2}{3} - ds - r\binom{d-r+1}{2} = \frac{dr^2}{2} + \frac{dr}{2} - ds - \frac{r^3}{3} + \frac{r}{3},$$

which at least $r - s$ for all d, $r \leq d$, and $s < r$. □

Lemma 4.5. *Let $S = k[x_1, \dots, x_d]/\mathfrak{m}^3$ and let $\underline{u} = x_1, \dots, x_r, q_{r+1}, \dots, q_n$, where the q_i are quadratic in $x_{r+1}, \dots, x_d$ and minimally generate $(q_{r+1}, \dots, q_n)$. Let $\underline{x} = x_1, \dots, x_r$. Then*

$$\mathrm{H}_1(\underline{u}; S) \cong \frac{(S_2^{\oplus r} + L) \oplus (S_1 \oplus S_2)^{\oplus n-r}}{L \oplus L' \oplus L''} \cong \frac{S_2^{\oplus r} \oplus (S_1 \oplus S_2)^{\oplus n-r}}{L_2 \oplus L' \oplus L''},$$

where L is generated by the Koszul relations on $x_1, \dots, x_r$, L' is generated by the Koszul relations $q_j e_i - x_i e_j$ where $i \leq r$ and $j \geq r+1$, and L'' is generated by the Koszul relations on $q_{r+1}, \dots, q_n$. Moreover, the minimal number of generators of $\mathrm{H}_1(\underline{u}; S)$ is at least n.

Proof. We first show that

$$\mathrm{H}_1(\underline{u}; S) \cong \frac{(S_2^{\oplus r} + L) \oplus (S_1 \oplus S_2)^{\oplus n-r}}{L \oplus L' \oplus L''}.$$

Note that $L + L' + L''$ encompasses all of the Koszul relations on $x_1, \dots, x_r, q_{r+1}, \dots, q_n$. Clearly, $L \cap L'' = 0$. We have $L \subseteq ((x_1, \dots, x_r)S)^{\oplus r}$, $L'' \subseteq (k[x_{r+1}, \dots, x_d]_2)^{\oplus n-r}$, and $L' \subseteq (k[x_{r+1}, \dots, x_d]_2)^{\oplus r} \oplus ((x_1, \dots, x_r)S)^{\oplus n-r}$, so $(L \oplus L'') \cap L' = 0$ and we can write $L \oplus L' \oplus L''$. Let N be the module of all relations on $x_1, \dots, x_r, q_{r+1}, \dots, q_n$. Since $x_1, \dots, x_r, q_{r+1}, \dots, q_n$ minimally generate $(\underline{u})$, $N \subseteq (S_1 \oplus S_2)^{\oplus n}$. The q_i are annihilated by $S_1 \oplus S_2$, so $(S_1 \oplus S_2)^{\oplus n-r} \subseteq N$, corresponding to $q_{r+1}, \dots, q_n$. Then the rest of N is simply the relations on $x_1, \dots, x_n$, which is $S_2^{\oplus r} + L$ by Lemma 4.3. Simplifying, we have

$$\mathrm{H}_1(\underline{u}; S) \cong \frac{(S_2^{\oplus r} + L) \oplus (S_1 \oplus S_2)^{\oplus n-r}}{L \oplus L' \oplus L''} \cong \frac{S_2^{\oplus r} \oplus (S_1 \oplus S_2)^{\oplus n-r}}{(S_2^{\oplus r} \oplus (S_1 \oplus S_2)^{\oplus n-r}) \cap (L \oplus L' \oplus L'')} \cong \frac{S_2^{\oplus r} \oplus (S_1 \oplus S_2)^{\oplus n-r}}{L_2 \oplus L' \oplus L''}.$$

Then

$$\frac{\mathrm{H}_1(\underline{u}; S)}{\mathfrak{m}\mathrm{H}_1(\underline{u}; S)} \cong \frac{S_2^{\oplus r} \oplus (S_1 \oplus S_2)^{\oplus n-r}}{L_2 \oplus L' \oplus L'' + S_2^{\oplus n-r}} \cong \frac{S_2^{\oplus r} \oplus (S_1 \oplus S_2)^{\oplus n-r}}{L_2 \oplus L' \oplus L''} \cong \frac{S_2^{\oplus r} \oplus S_1^{\oplus n-r}}{L_2 \oplus L' \cap (S_2^{\oplus r} \oplus S_1^{\oplus n-r})}$$

(since $L'' \subseteq S_2^{\oplus n-r}$). We have $L' \cap (S_2^{\oplus r} \oplus S_1^{\oplus n-r}) = \{c(q_j e_i - x_i e_j) \mid c \in k, 1 \leq i \leq r,\ r+1 \leq j \leq n\}$, so modulo $L' \cap (S_2^{\oplus r} \oplus S_1^{\oplus n-r})$, the elements $x_i e_j$ for $1 \leq i \leq r < j \leq n$ are in $S_2^{\oplus r}$. Thus

$$\frac{\mathrm{H}_1(\underline{u}; S)}{\mathfrak{m}\mathrm{H}_1(\underline{u}; S)} \cong \frac{S_2^{\oplus r} \oplus (kx_{r+1} + \cdots + kx_d)^{\oplus n-r}}{L_2}$$

and it follows that

$$\dim_k \frac{\mathrm{H}_1(\underline{u}; S)}{\mathfrak{m}\mathrm{H}_1(\underline{u}; S)} = r\binom{d+1}{2} + (d-r)(n-r) - d\binom{r}{2} + \binom{r}{3}.$$

This value is at least n if and only if

$$\begin{aligned} 0 &\leq \frac{rd^2 + rd}{2} - \frac{dr^2 + dr}{2} + (d-r-1)n - (d-r)r + \binom{r}{3} \\ &= \frac{rd(d-r)}{2} + (d-r-1)n + r^2 + \binom{r}{3}. \end{aligned}$$

Since $d \geq r$, this is certainly non-negative provided $d \geq r+1$. Suppose $d = r$. Then $\underline{u} = x_1, \dots, x_d$, so $n = d = r$ and the above formula becomes $n^2 - n + \binom{n}{3}$, which is at least n for all $n \geq 2$. $\square$

Theorem 4.6. *Let $S = k[x_1, \dots, x_d]/\mathfrak{m}^3$ and $V \subseteq \mathfrak{m}^2/\mathfrak{m}^3$, where $\mathfrak{m} = (x_1, \dots, x_d)$. Let $R = S/V$, $\underline{u} = x_1, \dots, x_r, q_{r+1}, \dots, q_n$, where the q_i are quadratic in $x_{r+1}, \dots, x_d$ and minimally generate $(q_{r+1}, \dots, q_n)$. Then the minimal number of generators of $\mathrm{H}_1(\underline{u}; R)$ is at least n.*

Proof. The short exact sequence $0 \to V \to S \to R \to 0$ yields a long exact sequence in Koszul homology

$$\mathrm{H}_1(\underline{u}; V) \to \mathrm{H}_1(\underline{u}; S) \to \mathrm{H}_1(\underline{u}; R) \to \mathrm{H}_0(\underline{u}; V) \to \mathrm{H}_0(\underline{u}; S) \to \mathrm{H}_0(\underline{u}; R) \to 0$$

which becomes

$$\mathrm{H}_1(\underline{u}; V) \to \mathrm{H}_1(\underline{u}; S) \to \mathrm{H}_1(\underline{u}; R) \to \frac{V}{(\underline{u})V} \to \frac{S}{(\underline{u})S} \to \frac{R}{(\underline{u})R} \to 0$$

Since V is annihilated by the u_i, $\mathrm{H}_1(\underline{u}; V) \cong V^{\oplus n}/(0) \cong V^{\oplus n}$ and $V/(\underline{u})V \cong V$. Therefore the sequence becomes

$$V^{\oplus n} \xrightarrow{\varphi} \mathrm{H}_1(\underline{u}; S) \xrightarrow{\psi} \mathrm{H}_1(\underline{u}; R) \xrightarrow{\delta} V \xrightarrow{\phi} \frac{S}{(\underline{u})S} \to \frac{R}{(\underline{u})R} \to 0$$

By Lemma 4.5,

$$\mathrm{H}_1(\underline{u}; S) \cong \frac{S_2^{\oplus n} \oplus (S_1 \oplus S_2)^{\oplus n-r}}{L_2 \oplus L' \oplus L''}.$$

Since the map φ is induced by the inclusion $V^{\oplus n} \hookrightarrow S_2^{\oplus n}$, the image of $V^{\oplus n}$ in $\mathrm{H}_1(\underline{u}; S)$ is

$$\frac{V^{\oplus n} + L_2 \oplus L' \oplus L''}{L_2 \oplus L' \oplus L''}.$$

By exactness, $\ker \delta \cong \operatorname{Im} \psi$ and $\ker \psi \cong \operatorname{Im} \varphi$, so

$$\ker \delta \cong \operatorname{Im} \psi \cong \frac{\mathrm{H}_1(\underline{x}; S)}{\operatorname{Im} \varphi} \cong \frac{S_2^{\oplus n} \oplus (S_1 \oplus S_2)^{\oplus n-r}}{L_2 \oplus L' \oplus L''} \Bigg/ \frac{V^{\oplus n} + L_2 \oplus L' \oplus L''}{L_2 \oplus L' \oplus L''} \cong \frac{S_2^{\oplus n} \oplus (S_1 \oplus S_2)^{\oplus n-r}}{V^{\oplus n} + L_2 \oplus L' \oplus L''}.$$

Similarly, $\operatorname{Im} \delta \cong \ker \phi \cong V \cap (\underline{u})S$. The connecting homomorphism δ acts as follows. Let a be an element of $\mathrm{H}_1(\underline{u}; R)$ with coset representative $(a_1, \dots, a_n) \in S^{\oplus n}$. Then $\delta(a) = a_1x_1 + \cdots + a_rx_r + a_{r+1}q_{r+1} + \cdots + a_nq_n \in V \cap (\underline{u})S$. By Lemma 3.5, we may assume $x_1, \dots, x_r, q_{r+1}, \dots, q_n$ minimally generate $(\underline{u})$, so that the a_i are in $S_1 \oplus S_2$. Then $a_iq_i = 0$ for all i, so $\operatorname{Im} \delta = V \cap (x_1, \dots, x_r)S$. Thus the short exact sequence

$$0 \to \ker \delta \to \mathrm{H}_1(\underline{u}; R) \to \operatorname{Im} \delta \to 0$$

becomes

$$0 \to \frac{S_2^{\oplus r} \oplus (S_1 \oplus S_2)^{\oplus n-r}}{V^n + L_2 \oplus L' \oplus L''} \to \mathrm{H}_1(\underline{u}; R) \xrightarrow{\delta} V \cap (x_1, \dots, x_r)S \to 0\ .$$

Let $s = \dim_k V \cap (x_1, \dots, x_r)S$ and let $t = \dim_k V$. Clearly, if $s \geq n$, then we are done. Suppose $s < n$. There is an injection

$$\frac{V}{V \cap (x_1, \dots, x_r)S} \hookrightarrow \frac{S_2}{S_2 \cap (x_1, \dots, x_r)S} \cong k[x_{r+1}, \dots, x_d]_2,$$

so $t = \dim_k V \leq s + \dim_k k[x_{r+1}, \dots, x_d]_2 = s + \binom{d-r+1}{2}$. Let $B \subseteq S_1^{\oplus r}$ be the module of degree 1 relations on $x_1, \dots, x_r$. Note that B/L_1 injects into $\mathrm{H}_1(\underline{u}; R)$, so $(x_1, \dots, x_r)B \subseteq (\underline{u})B \subseteq V^{\oplus n} + L_2 \oplus L' \oplus L''$. We claim

$$\frac{B}{L_1} \cong V \cap (x_1, \dots, x_r)S.$$

Let $b_1x_1 + \cdots + b_rx_r \in V \cap (x_1, \dots, x_r)S$. Then $(b_1, \dots, b_r) \in B$ since $b_1x_1 + \cdots + b_rx_r = 0_R$. Note that δ maps $(b_1, \dots, b_r) + L_1$ to $b_1x_1 + \cdots + b_rx_r$, so the composition δ_0 of δ with $B/L_1 \to \mathrm{H}_1(\underline{u}; R)$ is a surjection of B/L_1 onto $V \cap (x_1, \dots, x_r)S$. Suppose $\delta((b_1, \dots, b_r) + L_1) = 0_S$. Then $(b_1, \dots, b_r)$ is a degree 1 relation on $x_1, \dots, x_r$ in S, so it must hold in $T = k[x_1, \dots, x_d]$ and thus is in L_1. Thus δ_0 is the desired isomorphism and it follows that $\dim_k B/L_1 = s$ and

$$\mathrm{H}_1(\underline{u}; R) = \ker \delta \oplus \frac{B}{L_1} = \frac{S_2^{\oplus r} \oplus (S_1 \oplus S_2)^{\oplus n-r}}{V^n + L_2 \oplus L' \oplus L''} \oplus \frac{B}{L_1} \cong \frac{B \oplus S_2^{\oplus r} \oplus (S_1 \oplus S_2)^{\oplus n-r}}{V^n + L_1 \oplus L_2 \oplus L' \oplus L''}$$

since $B \subseteq S_1^{\oplus r}$.

By Nakayama's lemma, the minimal number of generators of $\mathrm{H}_1(\underline{u}; R)$ is precisely

$$\dim_k \frac{\mathrm{H}_1(\underline{u}; R)}{\mathfrak{m}\mathrm{H}_1(\underline{u}; R)}.$$

We have

$$\frac{\mathrm{H}_1(\underline{u}; R)}{\mathfrak{m}\mathrm{H}_1(\underline{u}; R)} = \frac{B \oplus S_2^{\oplus r} \oplus (S_1 \oplus S_2)^{\oplus n-r}}{V^{\oplus n} + L_1 \oplus L_2 \oplus L' \oplus L'' + \mathfrak{m}B + S_2^{\oplus n-r}} \cong \frac{S_2^{\oplus r} \oplus (S_1 \oplus S_2)^{\oplus n-r}}{V^{\oplus n} + L_2 + L' + L'' + \mathfrak{m}B + S_2^{\oplus n-r}} \oplus \frac{B}{L_1}$$

since $\mathfrak{m}$ kills $S_2^{\oplus r} \oplus S_2^{\oplus n-r}$ and $\mathfrak{m}S_1^{\oplus n-r} \subseteq S_2^{\oplus n-r}$. Since $\dim_k B/L_1 = s$, it suffices to show that

$$\frac{S_2^{\oplus r} \oplus (S_1 \oplus S_2)^{\oplus n-r}}{V^{\oplus n} + L_2 \oplus L' \oplus L'' + \mathfrak{m}B + S_2^{\oplus n-r}}$$

has dimension at least $n - s$. Since $L'' \subseteq S_2^{\oplus n-r}$ and $x_1, \dots, x_r$ annihilate $B/L_1 \subseteq \mathrm{H}_1(\underline{u}; R)$, this simplifies to

$$W := \frac{S_2^{\oplus r} \oplus (S_1 \oplus S_2)^{\oplus n-r}}{V^{\oplus n} + L_2 + L' + (x_{r+1}, \dots, x_d)B + S_2^{\oplus n-r}}.$$

Note that $S_2^{\oplus r}$ contains $(x_{r+1}, \dots, x_d)B$ and L_2 and we can write $V^{\oplus n} = V^{\oplus r} \oplus V^{\oplus n-r}$ so that $V^{\oplus r} \subseteq S_2^{\oplus r}$ and $V^{\oplus n-r} \subseteq S_2^{\oplus n-r}$. Furthermore, $L' = (L' \cap (S_2^{\oplus r} \oplus S_1^{n-r})) \oplus \mathfrak{m}L'$, where $L' \cap (S_2^{\oplus r} \oplus S_1^{n-r}) \subseteq S_2^{\oplus r} \oplus S_1^{n-r}$

and $\mathfrak{m}L' \subseteq S_2^{\oplus n-r}$. Putting this all together, we have that

$$\begin{aligned} W =& \frac{S_2^{\oplus r} \oplus (S_1 \oplus S_2)^{\oplus n-r}}{V^{\oplus n} + L_2 + L' + (x_{r+1}, \dots, x_d)B + S_2^{\oplus n-r}} \\ \cong& \frac{S_2^{\oplus r} \oplus (S_1 \oplus S_2)^{\oplus n-r}}{(x_{r+1}, \dots, x_d)B + L_2 + V^{\oplus r} + L' \cap (S_2^{\oplus r} \oplus S_1^{n-r}) + S_2^{\oplus n-r}} \\ \cong& \frac{S_2^{\oplus r} \oplus S_1^{\oplus n-r}}{(x_{r+1}, \dots, x_d)B + L_2 + V^{\oplus r} + L' \cap (S_2^{\oplus r} \oplus S_1^{n-r})}. \end{aligned}$$

Since $L' \cap (S_2^{\oplus r} \oplus S_1^{n-r})$ is spanned by $\{q_j e_i - x_i e_j \mid 1 \leq i \leq r,\ r+1 \leq j \leq n\}$, modulo $L' \cap (S_2^{\oplus r} \oplus S_1^{n-r})$ we have that $x_i e_j \in S_2^{\oplus r}$, so

$$\begin{aligned} \frac{\mathrm{H}_1(\underline{u}; R)}{\mathfrak{m}\mathrm{H}_1(\underline{u}; R)} \cong& \frac{S_2^{\oplus r} \oplus S_1^{\oplus n-r}}{(x_{r+1}, \dots, x_d)B + L_2 + V^{\oplus r} + L' \cap (S_2^{\oplus r} \oplus S_1^{n-r})} \\ \cong& \frac{S_2^{\oplus r} \oplus (kx_{r+1} + \cdots + kx_d)^{\oplus n-r}}{(x_{r+1}, \dots, x_d)B + V^{\oplus r} + L_2} \\ \cong& \frac{S_2^{\oplus r}}{(x_{r+1}, \dots, x_d)B + V^{\oplus r} + L_2} \oplus (kx_{r+1} + \cdots + kx_d)^{\oplus n-r}. \end{aligned}$$

The k-vector space $(kx_{r+1} + \cdots + kx_d)^{\oplus n-r}$ has dimension $(d-r)(n-r)$. If $d = r$, then $\underline{u} = x_1, \dots, x_d$, so $d = r = n$ and $(d-r)(n-r) = n-r = 0$. Moreover, if $d - r \geq 1$, then $(d-r)(n-r) \geq n-r$, so $\dim_k (kx_{r+1} + \cdots + kx_d)^{\oplus n-r} \geq n-r$. If $s \geq r$, then we are done. Suppose $s \leq r$. We will show that

$$\dim_k \frac{S_2^{\oplus r}}{(x_{r+1}, \dots, x_d)B + V^{\oplus r} + L_2} \geq r - s$$

and it will follow that

$$\dim_k \frac{S_2^{\oplus r}}{(x_{r+1}, \dots, x_d)B + V^{\oplus r} + L_2} \oplus (kx_{r+1} + \cdots + kx_d)^{\oplus n-r} \geq n - r + r - s = n - s.$$

Let T be the polynomial ring $k[x_1, \dots, x_d]$. Since each element of $S_1 \oplus S_2$ has a unique coset representative in $T_1 \oplus T_2$, there is a natural vector space isomorphism between $S_1 \oplus S_2$ and $T_1 \oplus T_2$. Consider the map

$$\gamma : (S_1 \oplus S_2)^{\oplus r} \to (T_1 \oplus T_2)^{\oplus r} \to T_2 \oplus T_3$$

that sends the ith basis element e_i to x_i. We have $\gamma(S_2^{\oplus r}) = (x_1, \dots, x_r)T_2$ and $\gamma(V^{\oplus r}) = (x_1, \dots, x_r)\tilde{V}$, where $\tilde{V}$ is the isomorphic image of V in T_2^r. Furthermore, $\gamma(L_2) = 0$. Note that γ restricted to $S_1^{\oplus r}$ is the same as the composition obtained by applying δ and then the natural vector space isomorphism from S_2 to T_2, so $\gamma(B) = \tilde{V} \cap (x_1, \dots, x_r)T \cong V \cap (x_1, \dots, x_r)S = \delta(B)$. Thus, γ induces a well-defined surjective map from

$$\frac{S_2^{\oplus r}}{(x_{r+1}, \dots, x_d)B + V^r + L_2}$$

to the module

$$\frac{(x_1,\ldots,x_r)T_2}{(x_{r+1},\ldots,x_d)(\tilde{V}\cap(x_1,\ldots,x_r)T)+(x_1,\ldots,x_r)\tilde{V}}.$$

We will show this module has dimension at least $r-s$ and the result will follow. Note that

$$\dim_k(x_1,\ldots,x_r)T_2 = \dim_k T_3 - \dim_k(x_{r+1},\ldots,x_d)^3 = \binom{d+2}{3} - \binom{d-r+2}{3}.$$

We have $\dim_k \tilde{V}\cap(x_1,\ldots,x_r)T = \dim_k V\cap(x_1,\ldots,x_r)S = s$ and

$$\dim_k \tilde{V} = \dim_k V = t \leq s + \binom{d-r+1}{2}.$$

Then the dimension of $(x_{r+1},\ldots,x_d)(\tilde{V}\cap(\underline{x})T)+(x_1,\ldots,x_r)\tilde{V}$ is bounded above by

$$(d-r)s + rs + r\binom{d-r+1}{2} = ds + r\binom{d-r+1}{2}.$$

Thus the dimension of the entire module is at least

$$\binom{d+2}{3} - \binom{d-r+2}{3} - ds - r\binom{d-r+1}{2} = \frac{dr^2}{2} + \frac{dr}{2} - ds - \frac{r^3}{3} + \frac{r}{3},$$

which at least $r-s$ for all d, $r \leq d$, and $s < r$. □

By Lemma 3.9, we have the following corollary.

Corollary 4.7. *Let $S = k[x_1,\ldots,x_d]/\mathfrak{m}^3$ and $V \subseteq \mathfrak{m}^2/\mathfrak{m}^3$, where $\mathfrak{m} = (x_1,\ldots,x_d)$. Let $R = S/V$ and I be a proper ideal of R minimally generated by $\underline{u} = u_1,\ldots,u_n$. Then the minimal number of generators of $\mathrm{H}_1(\underline{u};R)$ is at least n.*

Thus by Corollaries 3.5 and 3.7, we obtain the following result.

Corollary 4.8. *Let $(R,\mathfrak{m})$ be an equicharacteristic local Artinian ring such that $\mathfrak{m}^3 = 0$ and let $u_1,\ldots,u_n$ be elements of $\mathfrak{m}$. Then the minimal number of generators of $\mathrm{H}_1(u_1,\ldots,u_n;R)$ is at least n.*

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